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INVARIANCE OF RELATIVE INVERSE FUNCTION ORDERINGS UNDER COMPOSITIONS OF DISTRIBUTIONS

Abstract. Bartoszewicz and Benduch (2009) applied an idea of Lehmann and Rojo (1992) to a new setting and used the GTTT transform to define invariance properties and distances of some stochastic orders. In this paper Lehmann and Rojo's idea is applied to the class of models which is based on distributions which are compositions of distribution functions on $[0,1]$ with underlying distributions. Some stochastic orders are invariant with respect to these models.

1. Preliminaries. Let X and Y be two random variables, F and G their respective probability distribution functions and f and g their density functions, if they exist. Denote by $\overline{F} = 1 - F$ the tail (or survival function) of F, by $F^{-1}(u) = \inf\{x : F(x) \geq u\}$, $u \in (0,1)$, the quantile function and by $F^{-1}(0)$ and $F^{-1}(1)$ the lower and upper bounds of the support of F respectively. We identify the distribution functions F and G with the respective probability distributions and denote their supports by S_F, S_G respectively. We denote by fF^{-1} the composition of f and F^{-1} and by $G^{-1}F$ the composition of G^{-1} and F (similarly for other densities, distributions and quantile functions). We use *increasing* in place of *nondecreasing* and *decreasing* in place of *nonincreasing*.

We deal with some stochastic orders. Let us recall their definitions and some properties for completeness. For more details we refer to Shaked and Shanthikumar (2007).

We say that *F is smaller than G in the likelihood ratio order* $(F \leq_{\mathrm{lr}} G)$ if $g(x)/f(x)$ is increasing. We say that *F is smaller than G in the hazard rate order* $(F \leq_{\mathrm{hr}} G)$ if $\overline{G}(x)/\overline{F}(x)$ is increasing, or $r_F(x) \geq r_G(x)$ for every x if

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F and G are absolutely continuous, where $r_F(x) = f(x)/\overline{F}(x)$ is the *hazard rate function* of F. We say that F *is smaller than G in the reversed hazard rate order* ($F \leq_{\text{rh}} G$) if $G(x)/F(x)$ is increasing, or $\breve{r}_F(x) \leq \breve{r}_G(x)$ for every x if F and G are absolutely continuous, where $\breve{r}_F(x) = f(x)/F(x)$ is the *reversed hazard rate function* of F. We say that F is *stochastically smaller than G* ($F \leq_{\text{st}} G$) if $F(x) \geq G(x)$ for every x, or equivalently $\overline{F}(x) \leq \overline{G}(x)$ for every x.

Lehmann and Rojo (1992) noticed that these stochastic orders may be defined by properties of the function $k(u) = GF^{-1}(u)$, $u \in (0,1)$: $F \leq_{\text{lr}} G$ $\Leftrightarrow k(u)$ is convex; $F \leq_{\text{hr}} G \Leftrightarrow 1 - k^{-1}(1-u)$ is star-shaped; $F \leq_{\text{rh}} G \Leftrightarrow k(u)$ is star-shaped; $F \leq_{\text{st}} G \Leftrightarrow k(u) \leq u$ for all $u \in (0,1)$.

It is also well known that

$$F \leq_{\text{lr}} G \;\Rightarrow\; F \leq_{\text{hr}} G$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$F \leq_{\text{rh}} G \;\Rightarrow\; F \leq_{\text{st}} G$$

We shall also consider so called *relative inverse function orderings*; see Müller and Stoyan (2002) and van Zwet (1964). They are defined by properties of the function $G^{-1}F$. For continuous distributions the graph of the relative inverse distribution function is identical with the so-called Q-Q plot.

We say that F *is smaller than G in the dispersive order* ($F \leq_{\text{disp}} G$) if $G^{-1}F$ is *dispersive*, i.e. $G^{-1}F(x) - x$ is increasing.

Let now X and Y be positive random variables. We say that F *is smaller than G in the convex transform order* ($F \leq_{\text{c}} G$) if $G^{-1}F$ is convex on S_F. We say that F *is smaller than G in the star order* ($F \leq_* G$) if $G^{-1}F$ is star-shaped on S_F, i.e. $G^{-1}F(x)/x$ is increasing in $x > 0$. We say that F *is smaller than G in the superadditive order* ($F \leq_{\text{su}} G$) if $G^{-1}F$ is superadditive, i.e. $G^{-1}F(x+y) \geq G^{-1}F(x) + G^{-1}F(y)$, $x \geq 0$ for $y \geq 0$.

It is well known (see Shaked and Shanthikumar, 2007) that

$$F \leq_{\text{c}} G \;\Rightarrow\; F \leq_* G \;\Rightarrow\; F \leq_{\text{su}} G.$$

If $-\infty < F^{-1}(0) = G^{-1}(0)$, then

$$F \leq_{\text{disp}} G \;\Rightarrow\; F \leq_{\text{st}} G.$$

Lehmann and Rojo (1992) characterized stochastic orders defined by properties of GF^{-1} in terms of invariance under monotone transformations. Bartoszewicz and Benduch (2009) gave a characterization of relative inverse function orders in terms of invariance under the GTTT transform with respect to a class of positive continuous functions on $(0,1)$. In this note, Lehmann and Rojo's idea is applied to a new setting. Compositions of some distributions on $[0,1]$ with underlying distributions are used to define invariance properties and distances of some stochastic orders.

2. Results

2.1. Invariant preorderings. In this section we consider the convex, star-shaped, dispersive, superadditive and common stochastic orders.

Let $\mathcal{F}$ denote the class of continuous distributions on $[0, \infty)$ and let $\boldsymbol{\Psi}$ be the class of continuous strictly increasing distribution functions on $[0, 1]$. It is easy to see that $\boldsymbol{\Psi}$ is a group with respect to composition. Also, for every $F \in \mathcal{F}$ and every $\psi \in \boldsymbol{\Psi}$, $\psi F \in \mathcal{F}$.

The distribution functions ψF with ψ increasing, not necessarily strictly increasing, $\psi(0) = 0$ and $\psi(1) = 1$ are called *distorted distributions* and have been studied in different areas. Wang and Young (1998) connected these types of distributions with stochastic orders. Recently, Sordo and Suarez-Llorens (2008) and Khaledi and Shaked (2010) obtained some new results about stochastic orderings of distorted distributions. Navarro and Rychlik (2010) obtained some bounds and showed that the distributions of coherent systems with independent and identically distributed components are also distorted distributions obtained from the common component distribution. There are many examples of stochastic models where the probability distributions under consideration are of the form $\psi F \in \mathcal{F}$, $\psi \in \boldsymbol{\Psi}$. Here we list some of them.

EXAMPLE 1 (Generalized order statistics). Kamps (1995) introduced uniform generalized order statistics. Let $n \in \mathbb{N}$, $k \geq 1$, $m_1, \ldots, m_{n-1} \in \mathbb{R}$, $M_r = m_r + \cdots + m_{n-1}$, be parameters such that $\gamma_r = k + n - r + M_r \geq 1$ for all $r \in \{1, \ldots, n-1\}$ and let $\tilde{m} = (m_1, \ldots, m_{n-1})$ if $n \geq 2$ ($m \in \mathbb{R}$ arbitrary if $n = 1$). If random variables $U(r, n, \tilde{m}, k)$, $r = 1, \ldots, n$, possess a joint density function of the form

$$(1) \qquad f^{U(1,n,\tilde{m},k),\ldots,U(n,n,\tilde{m},k)}(u_1, \ldots, u_n)$$
$$= k \Big[\prod_{j=1}^{n-1} \gamma_j \Big] \Big[\prod_{i=1}^{n-1} (1 - u_i)^{m_i} \Big] (1 - u_n)^{k-1}$$

on the cone $\{(u_1, \ldots, u_n) : 0 \leq u_1 \leq \cdots \leq u_n \leq 1\} \subset \mathbb{R}^n$, then they are called *uniform generalized order statistics*.

Let F be a distribution function. The random variables

$$X(r, n, \tilde{m}, k) = F^{-1}(U(r, n, \tilde{m}, k)), \qquad r = 1, \ldots, n,$$

are called *generalized order statistics* based on the distribution function F.

It is clear that the distribution function of $\tilde{X} = X(r, n, \tilde{m}, k)$ is of the form

$$F_{\tilde{X}}(x) = \psi_{r,n,\tilde{m},k} F(x),$$

where $\psi_{r,n,\tilde{m},k} \in \boldsymbol{\Psi}$ is the distribution function of $U(r, n, \tilde{m}, k)$.

Choosing appropriate parameters in (1) we obtain among others the following models: ordinary order statistics of a sample of size n, order statistics with non-integral sample size, sequential order statistics, record values, kth record values, Pfeifer's record values (see Kamps, 1995). We present here two particular important cases.

1. *Order statistics.* Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be order statistics of a sample of size n from the distribution F. It is well known that the distribution function of $X_{i:n}$, $i = 1, \ldots, n$, is of the form

$$F_{i:n}(x) = B_{i:n}F(x),$$

where $B_{i:n}$ is the distribution function of the beta distribution $B(i, n-i+1)$. Obviously, $B_{i:n} \in \mathbf{\Psi}$.

Extensions of order statistics to those with non-integral sample size are due to Stiegler (1977). Rohatgi and Saleh (1988) provide another example of models with distributions from $\mathbf{\Psi}$.

2. *Record values.* Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of iid random variables with an absolutely continuous distribution F and density f. The random variables $L(n)$, $n \in \mathbb{N}$, defined by $L(1) = 1$ and $L(n+1) = \min\{j > L(n);\ X_j > X_{L(n)}\}$, are called *record times* and $X_{L(n)}$, $n \in \mathbb{N}$, are called *upper record values*. The marginal density of $L(r)$ is of the form

$$f_{X_{L(r)}}(x) = \frac{1}{(r-1)!}\left[\log\frac{1}{1-F(x)}\right]^{r-1}f(x)$$

and the marginal distribution function is given by

$$F_{X_{L(r)}}(x) = 1 - [1 - F(x)]\sum_{j=0}^{r-1}\frac{1}{j!}\left[\log\frac{1}{1-F(x)}\right]^j,$$

i.e.

$$F_{X_{L(r)}}(x) = \psi_r F(x),$$

where

$$\psi_r(t) = 1 - (1-t)\sum_{j=0}^{r-1}\frac{1}{j!}\left[\log\frac{1}{1-t}\right]^j, \quad t \in [0,1].$$

Similarly, the distribution of the kth record value (see Dziubdziela and Kopociński (1976) or Kamps (1995) for the definition) is of the form

$$F_{X_{L(r)_k}}(x) = \psi_{r,k}F(x),$$

where

$$\psi_{r,k}(t) = 1 - (1-t)^k\sum_{j=0}^{r-1}\frac{1}{j!}\left[k\log\frac{1}{1-t}\right]^j, \quad t \in [0,1].$$

Obviously, ψ_r and $\psi_{r,k}$ are distribution functions from $\mathbf{\Psi}$.

EXAMPLE 2 (Models in survival analysis). Three models are of great importance in survival analysis (we give the corresponding probability distribution functions):

1. *Proportional hazard model*

$$\psi_\alpha F(x) = 1 - [1 - F(x)]^\alpha, \quad x \in \mathbb{R}, \alpha > 0,$$

i.e.

$$\psi_\alpha(t) = 1 - (1 - t)^\alpha, \quad t \in [0, 1].$$

2. *Proportional reversed hazard model*

$$\psi_\alpha F(x) = [F(x)]^\alpha, \quad x \in \mathbb{R}, \alpha > 0,$$

i.e.

$$\psi_\alpha(t) = t^\alpha, \quad t \in [0, 1].$$

3. *Proportional odds model*

$$\psi_\alpha F(x) = 1 - \frac{\alpha(1 - F(x))}{1 - (1 - \alpha)(1 - F(x))}, \quad x \in \mathbb{R}, \alpha > 0,$$

i.e.

$$\psi_\alpha(t) = 1 - \frac{\alpha(1 - t)}{1 - (1 - \alpha)(1 - t)}, \quad t \in [0, 1].$$

One can also consider the two-parameter proportional odds model with

$$\psi_{\alpha,c}(t) = 1 - \left[\frac{\alpha(1 - t)^c}{1 - (1 - \alpha)(1 - t)^c}\right]^{1/c}, \quad t \in [0, 1], \alpha > 0, c > 0.$$

Of some importance are exponential mixtures of distributions in the proportional hazard model and the proportional reversed hazard model (see Marshall and Olkin, 1997). If the parameter α has the exponential distribution $K(\alpha) = 1 - e^{-\theta\alpha}$, $\alpha > 0$, $\theta > 0$, then we obtain the following distributions from $\boldsymbol{\Psi}$:

$$\psi_\theta(t) = \frac{\theta}{\theta - \log t}, \quad t \in (0, 1),$$

$$\psi_\theta(t) = 1 - \frac{\theta}{\theta - \log(1 - t)}, \quad t \in (0, 1),$$

for the proportional hazard model and the proportional reversed hazard model respectively.

The proportional hazard model is well known, it was introduced and studied by Cox (1972). The proportional reversed hazard model was extensively studied by Di Crescenzo (2000). Kirmani and Gupta (2001) considered the proportional odds model; see also Marshall and Olkin (1997) and Benduch-Frąszczak (2010). The two-parameter proportional odds model was studied by Dąbrowska and Doksum (1988).

Define a preorder in $\mathcal{F}$ to be any set $\mathcal{S}$ of ordered pairs (F, G) in $\mathcal{F} \times \mathcal{F}$ satisfying

$$(F, F) \in \mathcal{S} \quad \text{for all } F \in \mathcal{F},$$
$$(F, G) \in \mathcal{S} \text{ and } (G, H) \in \mathcal{S} \quad \text{implies} \quad (F, H) \in \mathcal{S}.$$

When $(F, G) \in \mathcal{S}$, we write $F \leq_S G$.

DEFINITION 1. The preorder $\mathcal{S}$ is *invariant* under the class $\boldsymbol{\Psi}$ if

(2) $$F \leq_S G \;\Rightarrow\; \psi F \leq_S \psi G \text{ for all } \psi \in \boldsymbol{\Psi}.$$

Similarly to Lehmann and Rojo (1992) we define orbits of pairs (F, G) under transformations ψ from $\boldsymbol{\Psi}$ in the space $\mathcal{F} \times \mathcal{F}$,

(3) $$\mathcal{O}(F, G) = \{(\psi F, \psi G) : \psi \in \boldsymbol{\Psi}\}.$$

Thus $(F, G) \in \mathcal{S}$ if and only if the orbit $\mathcal{O}(F, G)$ is contained in $\mathcal{S}$. Also as in Lehmann and Rojo (1992) we label the orbits by a maximal invariant under the group $\boldsymbol{\Psi}$. We prove the following:

THEOREM 1. *The function*

(4) $$\lambda(u) = G^{-1}F(t), \quad t > 0,$$

is a maximal invariant under the group $\boldsymbol{\Psi}$.

Proof. It is easy to notice that $(\psi G)^{-1}\psi F = G^{-1}F$ for every $\psi \in \boldsymbol{\Psi}$, i.e. $G^{-1}F$ is invariant under the group $\boldsymbol{\Psi}$.

Let (F_1, G_1) and (F_2, G_2) be two pairs of distributions from $\mathcal{F}$ and let

(5) $$G_1^{-1}F_1(t) = G_2^{-1}F_2(t), \quad t > 0.$$

We should prove that (F_1, G_1) and (F_2, G_2) are in the same orbit, i.e. there exists a function $\psi_0 \in \boldsymbol{\Psi}$ such that $F_2(t) = \psi_0 F_1(t)$, for G_1 and G_2 analogously. Since (5) holds, we have also

$$G_2 G_1^{-1}(t) = F_2 F_1^{-1}(t), \quad t > 0,$$

and then we put

$$\psi_0(t) = G_2 G_1^{-1}(t) = F_2 F_1^{-1}(t), \quad t > 0.$$

Thus we have $\psi_0 F_1(t) = F_2 F_1^{-1} F_1(t) = F_2(t)$ and $\psi_0 G_1(t) = G_2 G_1^{-1} G_1(t) = G_2(t)$. ∎

From Theorem 1 and the definitions of the relevant stochastic orders we obtain

COROLLARY 1. *The convex, star, superadditive, dispersive and common stochastic orders are invariant under the group* $\boldsymbol{\Psi}$.

Denote by $\boldsymbol{\Lambda}_S$ the class of functions of the form (4) which are maximal invariants for the preorder S under the group $\boldsymbol{\Psi}$. From Corollary 1 it follows that $\boldsymbol{\Lambda}_S$ is the class of positive increasing convex functions if S is the convex

transform order; the class of positive increasing star-shaped functions if S is the star order; the class of dispersive functions if S is the dispersive order, etc. It follows from transitivity and symmetry of the order that $\mathbf{\Lambda}_S$ is closed under composition and contains the identity function.

2.2. Ordering the orbits and distances between ordered distributions. The following idea of Lehmann and Rojo (1992) (see also Bartoszewicz and Benduch, 2009) may also be applied to some relative inverse function orderings. They considered quadruples $(F_1, G_1; F_2, G_2)$ and used invariant preorderings to define the concept of G_2 being further to the right from F_2 than G_1 is of F_1. Let us repeat their considerations in our setting.

Consider a quadruple $(F_1, G_1; F_2, G_2)$ such that $F_1, G_1 \in \mathcal{F}$, $F_2 = \psi F_1$ and $G_2 = \psi G_1$ for some $\psi \in \mathbf{\Psi}$. Since (F_1, G_1) and (F_2, G_2) are in the same orbit, we say that F_1 and G_1 are at the same distance from each other as F_2 and G_2, i.e. $d(F_1, G_1) = d(F_2, G_2)$, where d is a function which measures the distance if such a function exists. On the other hand, if $F_1 \leq_S G_1 \leq_S G_2$, we say that G_2 is further to the right from F_1 than G_1 is.

One now asks when for any quadruple $(F_1, G_1; F_2, G_2)$ the distribution G_2 will be further to the right from F_2 than G_1 is of F_1 in the preorder S. Let $F_1, G_1 \in \mathcal{F}$ and $F_i \leq_S G_i$, $i = 1, 2$. Notice that the function $\psi_0 = F_2 F_1^{-1} \in \mathbf{\Psi}$ has the property that $F_2 = \psi_0 F_1$. Therefore $\psi_0 G_1 = F_2 F_1^{-1} G_1$ and G_2 is further to the right from F_2 than G_1 is of F_1 in the preorder S if

$$\psi_0(G_1) \leq_S G_2,$$

i.e.

$$G_2^{-1} \psi_0 G_1 = G_2^{-1} F_2 F_1^{-1} G_1 = G_2^{-1} F_2 (G_1^{-1} F_1)^{-1} \in \mathbf{\Lambda}_S.$$

Denote $\lambda_i = G_i^{-1} F_i$, $i = 1, 2$. We may formulate the following definition.

DEFINITION 2. Let $F_1 \leq_S G_1$, $F_2 \leq_S G_2$. Then G_2 is said to be *further to the right from F_2 than G_1 is of F_1* if

(6)
$$\lambda_2 \lambda_1^{-1} \in \mathbf{\Lambda}_S.$$

The following examples illustrate this concept.

EXAMPLE 3. Consider the scale family $F(t/\theta)$, $\theta > 0$, where $F \in \mathcal{F}$ is fixed, and define $G_1(t) = F(t/\theta_1)$ and $G_2(t) = F(t/\theta_2)$ with $\theta < \theta_1 < \theta_2$. Obviously, $F \leq_{\mathrm{disp}} G_1$ and $F \leq_{\mathrm{disp}} G_2$. Let $\lambda_i = G_i^{-1} F$, $i = 1, 2$. It is easy to see that

$$\lambda_2 \lambda_1^{-1}(t) = G_2^{-1} F F^{-1} G_1(t) = \frac{\theta_2}{\theta_1} t$$

is dispersive. According to Definition 2, G_2 is further to the right from F than G_1 is in dispersive ordering.

EXAMPLE 4. Let F_α be the beta distribution with density

$$f_\alpha(t) = \alpha(1-t)^{\alpha-1}, \quad \alpha > 0, \, 0 < t < 1.$$

Then $F_\alpha^{-1}(t) = 1-(1-t)^{1/\alpha}$. Notice that $F_\beta^{-1}F_\alpha(t) = 1-(1-t)^{\alpha/\beta}$ is convex if $\alpha < \beta$. Therefore $F_\alpha \leq_c F_\beta$ and $F_\delta \leq_c F_\gamma$ if $0 \leq \alpha < \beta$ and $0 \leq \delta < \gamma$. Then $F_\gamma^{-1}F_\delta(F_\beta^{-1}F_\alpha)^{-1}(t) = 1-(1-t)^{\beta\delta/\alpha\gamma}$ is convex if $\alpha\gamma/\beta\delta \geq 1$. Hence if $\alpha/\beta \geq \delta/\gamma$, then F_γ is further to the right from F_δ than F_β is of F_α in the convex ordering.

EXAMPLE 5. Consider the distribution functions

$$F_i(t) = 1 - e^{-t^{\beta_i}}, \quad \beta_i > 0, \, i = 1, 2, \quad \text{and} \quad G(t) = 1 - e^{-t}, \quad t \geq 0.$$

with the quantile functions of the respective forms

$$F_i^{-1}(t) = -\log(1-t)^{1/\beta_i}, \quad G^{-1}(t) = -\log(1-t).$$

It is well known that for $\beta_i \geq 1$, $i = 1, 2$, the distributions F_1 and F_2 are IFR and $F_i \leq_c G$, $i = 1, 2$. We compute

$$(7) \qquad\qquad F_1^{-1}F_2(t) = t^{\beta_2/\beta_1}.$$

The function (7) is convex if $\beta_2/\beta_1 \geq 1$. Thus if $\beta_2 \geq \beta_1$, then G is further to the right from F_2 than F_1 is in the convex transform ordering as well as in the star ordering.

One may ask whether there exists a metric $d(F, G)$ which is consistent with the above comparison of distributions. Since λ-functions are invariant under the function $\psi \in \mathbf{\Psi}$, the distance $d(F, G)$ must also be invariant under these transforms. Therefore such a metric must satisfy the following two conditions:

(A) $d(F, G) = d(\psi F, \psi G)$ for all $\psi \in \mathbf{\Psi}$;
(B) $d(F_1, G_1) \leq d(F_2, G_2)$ if (7) holds.

THEOREM 2. Let F and G be distributions from $\mathcal{F}$ such that $F^{-1}(0) = G^{-1}(0) = 0$. If S is the dispersive order, then the distance

$$(8) \qquad d_*(F, G) = \sup_{t \in (0,1)} |G^{-1}(t) - F^{-1}(t)| = \sup_x |G^{-1}F(x) - x|$$

satisfies conditions (A) and (B).

Proof. It is evident that d_* satisfies (A).

The condition (8) is equivalent to the fact that $G_2^{-1}F_2F_1^{-1}G_1$ is dispersive, that is, $G_2^{-1}F_2F_1^{-1}G_1(x) - x$ is increasing. From the assumption we have $G_2^{-1}F_2F_1^{-1}G_1(x) - x \geq 0$ and then $G_2^{-1}F_2(x) - x \geq G_1^{-1}F_1(x) - x \geq 0$, $x \geq 0$. Hence

$$d_*(F_1, G_1) \leq d_*(F_2, G_2). \quad \blacksquare$$

EXAMPLE 6. Let $\overline{F}(x) = \frac{1}{2}(e^{-\alpha x} + e^{-x})$, $\alpha > 1$, and $\overline{G}(x) = e^{-x}$. Then

$$G^{-1}F(x) - x = \log\left(\frac{2}{e^{-\alpha x} + e^{-x}}\right) - x = \log\left(\frac{2e^x}{e^{-(\alpha+1)x} + 1}\right) - \log e^x$$

$$= \log\left(\frac{2}{e^{-(\alpha+1)x} + 1}\right)$$

is increasing and hence $F \leq_{\text{disp}} G$. Therefore we obtain the distance d_* between the distributions F and G,

$$d_*(F, G) = \sup_x |G^{-1}F(x) - x| = \sup_x \left|\log\left(\frac{2}{e^{-(\alpha+1)x} + 1}\right)\right| = \log 2.$$

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