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A MODEL OF COMPETITION

Dedicated to Professor Janusz Matkowski
on the occasion of his 70th birthday

Abstract. A competition model is described by a nonlinear first-order *differential* equation (of Riccati type). Its solution is then used to construct a *functional* equation in two variables (admitting essentially the same solution) and several iterative functional equations; their continuous solutions are presented in various forms (closed form, power series, integral representation, asymptotic expansion, continued fraction). A constant $C = 0.917\dots$ (inherent in the model) is shown to be a transcendental number.

1. Introduction. A motivation for the present investigation is to build a bulk model for a competition process in cloud physics. Cloud seeding with AgI (silver iodide) is performed to suppress hail by *competition*: silver iodide particles, in their crystalline structure, are very similar to tiny ice crystals; artificially increasing the number of silver iodide particles causes a larger number of smaller hailstones (*soft hail*, less dangerous than fewer but larger hailstones, cf. [6]). A similar model holds for the number of ions in air as a result of ionisation and recombination (cf. [3]). Bulk models of competition processes can give qualitative insight into certain mechanisms. It appears plausible that competition models are also applicable to some marketing problems.

We start with a differential equation and solve it; then we turn to functional equations having essentially the same solution, but providing additional representations and interpretations.

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2. Model equation: a nonlinear differential equation. We use the following model equation of competition:

$$\frac{df(x)}{dx} = \frac{1 - bf(x)^2}{1 + ax^2}$$

with domain $x \geq 0$, range $f(x) \geq 0$, and real constants (model parameters) $a, b \geq 0$. The initial condition is $f(x_0) = f_0$ with real constants $x_0, f_0 \geq 0$. Physical interpretation of (properly scaled) variables: x time, f amount of soft hail.

This simple model equation is a nonlinear first-order differential equation of Riccati type. Interpretation of a, b: the campaign parameter a is a measure of *intensity and duration* of external activity (cloud seeding), and the saturation parameter b is a measure for the internal effect of *self-limitation* by interactions within the soft hail produced.

The parameters a and b characterize a hierarchy of competition models. If $a, b \in \{0, 1\}$, we have four models.

MODEL 1. $a = 0$, $b = 0$:

$$\frac{df(x)}{dx} = 1 \quad \text{with solution} \quad f(x) - f_0 = x - x_0,$$

meaning that the soft hail production is going on unlimited. (Useful as a crude short-time approximation, but unrealistic for longer times.)

MODEL 2. $a = 1$, $b = 0$:

$$\frac{df(x)}{dx} = \frac{1}{1 + x^2} \quad \text{with solution} \quad f(x) - f_0 = \tan^{-1}\left(\frac{x - x_0}{1 + xx_0}\right),$$

[$\tan^{-1}$ denoting the principal value of the inverse of tan], showing that the soft hail production is limited by the campaign resources. Here, with finite campaign resources, but (unrealistically) without saturation, we would achieve 57% more soft hail than in the next model 3 (the case with saturation): $\tan^{-1}(\infty) = \pi/2 = 1.57\ldots$ [using for simplicity $x_0 = f_0 = 0$].

MODEL 3. $a = 0$, $b = 1$:

$$\frac{df(x)}{dx} = 1 - f(x)^2 \quad \text{with solution} \quad \frac{f(x) - f_0}{1 - f(x)f_0} = \tanh(x - x_0),$$

describing that the soft hail production is limited by a saturation effect (more and more soft hail particles get into each other's way and make the formation of new ones difficult). Yet the unlimited campaign resources (intensity and duration) provide 100% soft hail in the long run: $\tanh(\infty) = 1$ [using for simplicity $x_0 = f_0 = 0$].

MODEL 4. $a = 1$, $b = 1$:

$$\frac{df(x)}{dx} = \frac{1 - f(x)^2}{1 + x^2} \quad \text{with solution} \quad \frac{f(x) - f_0}{1 - f(x) f_0} = \tanh\left(\tan^{-1}\left(\frac{x - x_0}{1 + x x_0}\right)\right),$$

implying that the soft hail production is limited both by the saturation effect and by the finite campaign resources. Yet in the long run, we obtain 92% of the soft hail of case 3 (where the campaign resources are unlimited): $\tanh(\tan^{-1}(\infty)) = 0.917\ldots$ [using for simplicity $x_0 = f_0 = 0$].

Competition model 4 is treated in the present paper. In the next section, the (physically motivated) restriction on x and $f(x)$ to be nonnegative will be dropped: x and $f(x)$ (also x_0 and f_0) may take arbitrary real values.

3. The competition function g**.** According to Section 2, competition model 4 is characterized by the nonlinear differential equation of Riccati type

$$\frac{df(x)}{dx} = \frac{1 - f(x)^2}{1 + x^2} \tag{1}$$

with initial condition

$$f(x_0) = f_0. \tag{2}$$

PROPOSITION 3.1. *The analytic solution* f *of* (1) *with condition* (2) *is*

$$f(x) = \frac{f_0 + g\left(\frac{x - x_0}{1 + x x_0}\right)}{1 + f_0 g\left(\frac{x - x_0}{1 + x x_0}\right)}, \tag{3}$$

where the function g *is defined by the composition* $g = \tanh \circ \tan^{-1}$, *i.e.*

$$g(x) = \tanh(\tan^{-1}(x)), \tag{4}$$

with $\tan^{-1}$ *denoting the principal value of the inverse of* $\tan$.

Proof. In Section 2, Model 4, we obtained the solution in implicit form,

$$\frac{f(x) - f_0}{1 - f(x) f_0} = g\left(\frac{x - x_0}{1 + x x_0}\right),$$

which implies the explicit version (3). ■

We note some interesting *particular* cases of the solution (3).

I. The initial condition $x_0 = 0$ leads to the solution

$$f(x) = \frac{f_0 + g(x)}{1 + f_0 g(x)}. \tag{5}$$

Special subcases include:

$$(6) \qquad x_0 = 0 \text{ and } f_0 = 0, \text{ giving the solution } f(x) = g(x),$$

$$(7) \qquad x_0 = 0 \text{ and } f_0 \to \infty, \text{ yielding the solution } f(x) = 1/g(x),$$

$$(8) \qquad x_0 = 0 \text{ and } f_0 = 1, \text{ giving the solution } f(x) = 1,$$

$$(9) \qquad x_0 = 0 \text{ and } f_0 = -1, \text{ yielding the solution } f(x) = -1.$$

II. The initial condition $f_0 = 0$ leads to the solution

$$f(x) = g\left(\frac{x - x_0}{1 + x x_0}\right). \tag{10}$$

Special subcases include:

$$(11) \qquad x_0 = 0 \text{ and } f_0 = 0, \text{ giving the solution } f(x) = g(x),$$

$$(12) \qquad x_0 \to \infty \text{ and } f_0 = 0, \text{ yielding the solution } f(x) = g(-1/x),$$

$$(13) \qquad x_0 = 1 \text{ and } f_0 = 0, \text{ giving the solution } f(x) = g\left(\frac{x-1}{x+1}\right),$$

$$(14) \qquad x_0 = -1 \text{ and } f_0 = 0, \text{ yielding the solution } f(x) = g\left(\frac{1+x}{1-x}\right).$$

REMARK 3.2. We note that the zero function $f = 0$ is *not* a solution of (1).

3.1. A power series expansion for the g function

PROPOSITION 3.3. *The g function has the following power series expansion at $x = 0$:*

$$g(x) = x - \frac{2}{3}x^3 + \frac{2}{3}x^5 - \frac{46}{63}x^7 + \cdots, \qquad |x| < 1. \tag{15}$$

Proof. Writing the well-known power series expansion of tanh in this way:

$$\tanh(x) = T_1 x + T_3 x^3 + T_5 x^5 + T_7 x^7 + \cdots, \qquad |x| < \pi/2,$$

(with known coefficients T_n), we see that the function $g = \tanh \circ \tan^{-1}$ can be represented as an arctan series:

$$g(x) = T_1 \tan^{-1}(x) + T_3 \tan^{-1}(x)^3 + T_5 \tan^{-1}(x)^5 + T_7 \tan^{-1}(x)^7 + \cdots, \ |x| < 1.$$

In order to obtain a *power* series expansion of g,

$$g(x) = G_1 x + G_3 x^3 + G_5 x^5 + G_7 x^7 + \ldots, \qquad |x| < 1,$$

(with coefficients G_n to be determined), we substitute the well-known power series for $\tan^{-1}$ into the above arctan expansion of g and compare the coefficients of equal powers of x. This leads to the following scheme:

$$\begin{aligned}
&G_1 = T_1,\\
&G_3 = T_3 - (1/3)T_1,\\
&G_5 = T_5 - (3/3)T_3 + (1/5)T_1,\\
&G_7 = T_7 - (5/3)T_5 + (14/15)T_3 - (1/7)T_1, \quad \ldots
\end{aligned}$$

The coefficients T_n of the tanh power series are well known; they can be expressed in terms of Bernoulli numbers:

$$T_{2n-1} = [2^{2n}(2^{2n} - 1]/(2n)!]B_{2n}, \qquad n = 1, 2, \ldots.$$

With $B_2 = 1/6, B_4 = -1/30, B_6 = 1/42, B_8 = -1/30, \ldots$, we finally obtain the coefficients G_n as linear combinations of Bernoulli numbers:

$$\begin{aligned}
&G_1 = 6B_2 = 1,\\
&G_3 = 10B_4 - 1/3 = -2/3,\\
&G_5 = (28/5)B_6 - 10B_4 + 1/5 = 2/3,\\
&G_7 = (34/21)B_8 - (28/3)B_6 + (28/3)B_4 - 1/7 = -46/63, \quad \ldots. \ \blacksquare
\end{aligned}$$

PROPOSITION 3.4. *The g function has an asymptotic value of $g(\infty) = C = 0.9171523\ldots$. This value is a transcendental number.*

Proof. Using the definition $g(x) = \tanh(\tan^{-1}(x))$ and letting x grow indefinitely, the function $\tan^{-1}(x)$ (principal value) goes asymptotically towards $\pi/2$, so $g(x)$ goes towards $C = \tanh(\pi/2) = 0.917\ldots$

Since Gelfond's constant $G_0 = \exp(\pi)$ is known to be transcendental (cf. [2]), it is evident that $C = \tanh(\pi/2) = [\exp(\pi) - 1]/[\exp(\pi) + 1] = (G_0 - 1)/(G_0 + 1)$ is transcendental. ■

PROPOSITION 3.5. *The transcendental constant C is algebraically independent of π.*

Proof. Using Lambert's well-known continued fraction expansion of $\tanh(x)$ (cf. [5]), we have for $C = \tanh(\pi/2)$ the representation

$$C = \cfrac{1}{1\frac{2}{\pi} + \cfrac{1}{3\frac{2}{\pi} + \cfrac{1}{5\frac{2}{\pi} + \cfrac{1}{7\frac{2}{\pi} + \ldots}}}} \equiv \frac{1}{1\frac{2}{\pi}+}\ \frac{1}{3\frac{2}{\pi}+}\ \frac{1}{5\frac{2}{\pi}+}\ \frac{1}{7\frac{2}{\pi}+}\cdots$$

(where the continued fraction involves all positive odd numbers, each multiplied by the transcendental number $2/\pi$). This is not a rational or algebraically irrational function of π, therefore C is algebraically independent of π. ■

REMARK 3.6. The g function has the following integral representation (well suited for numerical evaluation):

$$g(x) = \int_0^{2\tan^{-1}(x)} \frac{1}{1+\cosh(t)}\,dt, \qquad x \in \mathbb{R}.$$

For $x \to \infty$, this yields the representation $C = \int_0^\pi [1/(1+\cosh(t))]\,dt$.

REMARK 3.7. The proximity of $C = 0.917\ldots$ to Catalan's constant $G = 0.915\ldots$ is coincidental.

3.2. A functional equation for the g function

PROPOSITION 3.8. *Employing the constant $C = g(\infty) = \tanh(\pi/2)$, the g function satisfies the following (iterative) functional equation:*

$$g(x) = \frac{C\,\mathrm{signum}(x) - g(1/x)}{1 - C\,\mathrm{signum}(x) g(1/x)}, \quad x \in \mathbb{R}. \tag{16}$$

Proof. Using the identity $\tan^{-1}(x) = \cot^{-1}(1/x) = \pi/2 - \tan^{-1}(1/x)$ for $x \geq 0$, we have

$$g(x) = \tanh(\tan^{-1}(x)) = \tanh(\pi/2 - \tan^{-1}(1/x))$$

(for $x \geq 0$), which is

$$g(x) = \frac{\tanh(\pi/2) - \tanh(\tan^{-1}(1/x))}{1 - \tanh(\pi/2)\tanh(\tan^{-1}(1/x))} = \frac{C - g(1/x)}{1 - Cg(1/x)}, \qquad x \geq 0.$$

The case $x < 0$ is taken into account by the factor $\mathrm{signum}(x)$ (for the sign of C), where $\mathrm{signum}(x) = -1$ for $x < 0$, and $+1$ for $x \geq 0$. ∎

REMARK 3.9. The functional equation (16) may be applied for analytic continuation of g. Using for instance the power series (15) with its rather limited interval of convergence ($|x| < 1$), we get $g(x)$ for $x \in \mathbb{R}$ [namely $g(x)$ for $|x| < 1$ by (15) itself, and $g(x)$ for $|x| > 1$ by (16) with $g(1/x)$ via (15)].

3.3. An asymptotic expansion for the g function

PROPOSITION 3.10. *Introducing coefficients K_n $(n = 0, 1, 2, \ldots)$ as polynomial functions (of degree $n+1$) of $C = g(\infty) = 0.917\ldots$,*

$$K_0 = C, \qquad K_4 = \frac{4}{3}C - \frac{7}{3}C^3 + C^5,$$

$$K_1 = -1 + C^2, \qquad K_5 = -\frac{2}{3} + \frac{8}{3}C^2 - 3C^4 + C^6,$$

$$K_2 = -C + C^3, \qquad K_6 = -\frac{16}{9}C + \frac{40}{9}C^3 - \frac{11}{3}C^5 + C^7,$$

$$K_3 = \frac{2}{3} - \frac{5}{3}C^2 + C^4, \qquad K_7 = \frac{46}{63} - \frac{256}{63}C^2 + \frac{20}{3}C^4 - \frac{13}{3}C^6 + C^8, \ \ldots,$$

the g function admits the following asymptotic expansion, valid for $x \to \infty$ (*numerically useful already for* $x > 0$)*:*

$$g(x) = \sum_{n=0}^{\infty} K_n x^{-n} \equiv K_0 + K_1 x^{-1} + K_2 x^{-2} + K_3 x^{-3} + \cdots . \tag{17}$$

Proof. Utilizing the functional equation (16) in its simpler form for $x>0$,

$$g(x) = \frac{C - g(1/x)}{1 - Cg(1/x)}, \quad x > 0,$$

and expanding $g(1/x)$ into a power series of $1/x$, we obtain the K_n. ∎

3.4. A continued fraction expansion for the g function

PROPOSITION 3.11. *The g function admits the following continued fraction expansion* (*valid for* $|x| < \pi/2$):

$$g(x) = \cfrac{1}{\frac{1}{x} + \cfrac{2}{\frac{3}{x} + \cfrac{5}{\frac{5}{x} + \cfrac{10}{\frac{7}{x} + \cfrac{17}{\frac{9}{x} + \ldots}}}}} \equiv \frac{1+0^2}{\frac{1}{x}+} \, \frac{1+1^2}{\frac{3}{x}+} \, \frac{1+2^2}{\frac{5}{x}+} \, \frac{1+3^2}{\frac{7}{x}+} \, \frac{1+4^2}{\frac{9}{x}+} \cdots \tag{18}$$

The formation law for the coefficients is interesting: the top numerator is unity, the numerators to the left comprise all positive odd numbers, and the numerators to the right are each the sum of the preceding two numbers.

Proof. Developing the continued fraction (18) into a power series, we get full agreement with the former power series expansion (15) up to any order. For instance, up to $O(x^9)$: $g(x) = x-(2/3)x^3+(2/3)x^5-(46/63)x^7+O(x^9)$. ∎

REMARK 3.12. By an obvious equivalence transformation, the continued fraction (18) takes the following form (valid for $|x| < \pi/2$):

$$g(x) = \cfrac{x}{1 + \cfrac{2x^2}{3 + \cfrac{5x^2}{5 + \cfrac{10x^2}{7 + \ldots}}}} \equiv \frac{(1+0^2)x}{1+} \, \frac{(1+1^2)x^2}{3+} \, \frac{(1+2^2)x^2}{5+} \, \frac{(1+3^2)x^2}{7+} \cdots \tag{19}$$

4. Functional equations with solutions expressible by g

4.1. A functional equation in two variables

PROPOSITION 4.1. *The functional equation in two real variables* x, y

$$f\left(\frac{x+y}{1-xy}\right) = \frac{f(x) + f(y)}{1 + f(x)f(y)} \quad \textit{for } xy < 1 \tag{20}$$

with side condition (*involving two real constants* x_0, f_0)

$$f(x_0) = f_0 \tag{21}$$

has the following continuous solution:

$$f(x) = \tanh\left(\frac{\tanh^{-1}(f_0)}{\tan^{-1}(x_0)}\tan^{-1}(x)\right), \qquad x \in \mathbb{R}. \tag{22}$$

Equivalently, with a real constant c defined by

$$c := \tanh^{-1}(f_0)/\tan^{-1}(x_0) \text{ for } x_0 \neq 0, \qquad c := 1 \text{ for } x_0 = f_0 = 0,$$

the latter solution reads

$$f(x) = \tanh(c\tan^{-1}(x)), \qquad x \in \mathbb{R}. \tag{22$'$}$$

It is convenient to distinguish as standard solution $g(x)$ *the case $c = 1$ [corresponding to $f(0) = 0$, i.e. $x_0 = f_0 = 0$]:*

$$(c = 1{:}) \quad f(x) = g(x) := \tanh(\tan^{-1}(x)), \quad x \in \mathbb{R}. \tag{23a}$$

Other particular cases include:

$$(c = -1{:}) \quad f(x) = -g(x),\ x \in \mathbb{R} \quad \text{(negative standard solution)}, \tag{23b}$$

$$(c = 0{:}) \quad f(x) = 0,\ x \in \mathbb{R} \quad \text{(trivial solution)}, \tag{23c}$$

$$(c \to \pm\infty{:}) \quad f(x) = \pm 1,\ x \in \mathbb{R} \quad \text{(non-trivial constant solutions)}. \tag{23d}$$

Proof. Let c be a real constant. Using the addition theorem for tanh,

$$\tanh(x+y) = [\tanh(x) + \tanh(y)]/[1 + \tanh(x)\tanh(y)],$$

and substituting $c\tan^{-1}(x)$ for x and $c\tan^{-1}(y)$ for y, we obtain

$$\tanh(c\tan^{-1}(x) + c\tan^{-1}(y)) = \frac{\tanh(c\tan^{-1}(x)) + \tanh(c\tan^{-1}(y))}{1 + \tanh(c\tan^{-1}(x))\tanh(c\tan^{-1}(y))}.$$

Since $\tan^{-1}(x) + \tan^{-1}(y) = \tan^{-1}[(x+y)/(1-xy)]$ for $xy < 1$, we get functional equation (20) with continuous solution $f(x) = \tanh(c\tan^{-1}(x))$, $x \in \mathbb{R}$. ∎

Remark 4.2. Equation (20), with the condition "$xy < 1$", is an example of a *conditional* functional equation (functional equation on a *restricted domain*) (cf. [1]).

Remark 4.3. Unlike the differential equation (1), the functional equation (20) admits a trivial solution (23c). In search for constant solutions, putting in (20) $f = k$ (where k is a real constant) produces a cubic equation for k, namely $(k^2 - 1)k = 0$, with roots $k = 1, -1, 0$ [results (23d) and (23c)]. Moreover, if $f(x)$ is a solution of (20), then $-f(x)$ is also a solution of (20) [cf. e.g. (23a) and (23b), but also (23d)].

4.2. An iterative functional equation

Proposition 4.4. *The iterative functional equation*

$$f\left(\frac{2x}{1-x^2}\right) = \frac{2f(x)}{1+f(x)^2} \qquad \text{for } |x| < 1 \tag{24}$$

with side condition (involving two real constants x_0, f_0)

$$f(x_0) = f_0 \tag{25}$$

has the following analytic solution:

$$f(x) = \tanh\left(\frac{\tanh^{-1}(f_0)}{\tan^{-1}(x_0)}\tan^{-1}(x)\right), \qquad x \in \mathbb{R}. \tag{26}$$

Equivalently, with a real constant c (defined as in Subsection 4.1), the latter solution reads

$$f(x) = \tanh(c\tan^{-1}(x)), \qquad x \in \mathbb{R}. \tag{26$'$}$$

As before in Subsection 4.1, it is convenient to distinguish as standard solution $g(x)$ *the case $c = 1$ [corresponding to $f(0) = 0$]:*

$$(c = 1:) \quad f(x) = g(x) := \tanh(\tan^{-1}(x)), \qquad x \in \mathbb{R}. \tag{27}$$

(Other particular cases are as in Subsection 4.1.)

Proof. Putting $y = x$ in equation (20) and applying Proposition 4.1, we get the result. ∎

REMARK 4.5. Unlike the differential equation (1), the functional equation (24) admits a trivial solution.

REMARK 4.6. Choosing as side condition (25) in particular $f(0) = 0$, we get a functional equation for g itself:

$$g\left(\frac{2x}{1-x^2}\right) = \frac{2g(x)}{1+g(x)^2} \qquad \text{for } |x| < 1. \tag{28}$$

4.3. Generation of g by iteration

LEMMA 4.7. *If $f : \mathbb{R} \to \mathbb{R}$ satisfies equation (24), is analytic at 0, and $f(t_0) = 0$ for some $t_0 \neq 0$, then $f(x) = 0$ for all $x \in \mathbb{R}$.*

Proof. The function $x \mapsto 2x/(1-x^2)$ for $x \in (-1,1)$ is continuous, strictly increasing, and maps the interval $(-1,1)$ onto $\mathbb{R}$. Let α be its inverse. Then

$$\alpha(t) = t/[1 + (t^2+1)^{1/2}], \quad t \in \mathbb{R},$$

is continuous, strictly increasing, mapping $\mathbb{R}$ onto the interval $(-1,1)$ and

$$0 < \alpha(t) < t \quad \text{for } t > 0; \qquad -t < \alpha(t) < 0 \quad \text{for } t < 0.$$

It follows that

$$\lim_{n\to\infty} \alpha^n(t) = 0, \qquad t \in \mathbb{R},\, t \neq 0,$$

where $\alpha^n(t)$ denotes the nth iterate of α. Setting $x = \alpha(t)$ in (24) we get

$$f(t) = 2f(\alpha(t))/[1 + f(\alpha(t))^2], \qquad t \in \mathbb{R}.$$

By assumption, $f(t_0) = 0$. It follows that $f(\alpha(t_0)) = 0$ and $\alpha(t_0) \neq 0$. Hence, by induction, $f(\alpha^n(t_0)) = 0$ and $\alpha^n(t_0) \neq 0$ for all $n \in \mathbb{N}$, which proves that

0 is an accumulation point of zeros of f. Now the analyticity of f implies that f is the zero function. ∎

In view of Lemma 4.7, we will now deal with solutions f of equation (24) such that $f(x) \neq 0$ for $x \neq 0$. The main result reads as follows.

THEOREM 4.8. *Suppose that a function* $f : \mathbb{R} \to \mathbb{R}$ *is such that* $f'(0) = 1$ *and the function*

$$(\mathbb{R}\backslash\{0\}) \ni x \mapsto \frac{f(x) - f'(0)x}{x^2}$$

is bounded in the vicinity of 0. *If* f *satisfies the functional equation*

$$f\left(\frac{2x}{1-x^2}\right) = \frac{2f(x)}{1+f(x)^2} \quad \textit{for } |x| < 1, \tag{24$'$}$$

then $f = g := \tanh \circ \tan^{-1}$.

Proof. Of course, the function $g = \tanh \circ \tan^{-1}$ satisfies the functional equation (24′), $g(0) = 0$, $g'(0) = 1$, and the function

$$(\mathbb{R}\backslash\{0\}) \ni x \mapsto \frac{g(x) - g'(0)x}{x^2}$$

is bounded in the vicinity of 0.

Suppose that $f : \mathbb{R} \to \mathbb{R}$ fulfills the stated conditions and satisfies equation (24′). Then, of course, $f(0) = 0$. The function $\alpha : \mathbb{R} \to (-1, 1)$ defined by

$$\alpha(t) = t/[1 + (t^2 + 1)^{1/2}], \quad t \in \mathbb{R},$$

is the inverse of $(-1, 1) \ni x \mapsto 2x/(1 - x^2)$. Therefore, setting $x = \alpha(t)$ in (24′) we obtain

$$f(t) = \frac{2f(\alpha(t))}{1 + f(\alpha(t))^2}, \quad t \in \mathbb{R},$$

whence, putting

$$P(y) := \frac{2y}{1 + y^2}, \quad y \in \mathbb{R},$$

we see that f satisfies the iterative functional equation

$$f(t) = P[f(\alpha(t))], \quad t \in \mathbb{R}. \tag{29}$$

Since $|\alpha(t)| < |t|$ for $t \neq 0$, $\alpha'(0) = 1/2$, and P is differentiable with $P'(0) = 2$, we have

$$[\alpha'(0)]^2 P'(0) = 1/2 < 1.$$

It follows that there exist numbers $\delta > 0$, $c > 0$, and $s[1/2, 1)$, $l \geq 2$, such that $s^2 l < 1$ and

$$|\alpha(t)| \leq s|t| \text{ when } t \in (-\delta, \delta), \quad |P(y) - P(\overline{y})| \leq l|y - \overline{y}| \text{ when } y, \overline{y} \in [-c, c].$$

Since $f(0) = g(0) = 0$ and $f'(0) = g'(0) = 1$, by a uniqueness theorem of J. Matkowski ([4, Theorem 1]) we conclude that $f = g$. ∎

As an immediate consequence of this result, we obtain the following characterization of the function $g = \tanh \circ \tan^{-1}$.

THEOREM 4.9. *Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable at the point 0 and $f'(0) = 1$. If*

$$f\left(\frac{x+y}{1-xy}\right) = \frac{f(x)+f(y)}{1+f(x)f(y)} \quad \text{for } xy < 1, \tag{20'}$$

then $f = g := \tanh \circ \tan^{-1}$.

REMARK 4.10. The functional equation (29) may serve to generate the function $g(x)$, $x \in \mathbb{R}$, by iteration *analytically* and/or *numerically*. Sketch of algorithm: with two auxiliary functions $P : \mathbb{R} \to (-1, 1)$ and $\alpha : \mathbb{R} \to (-1, 1)$,

$$P(t) = \frac{2t}{1+t^2}, \quad t \in \mathbb{R}, \quad \text{and} \quad \alpha(x) = \frac{x}{1+(1+x^2)^{1/2}}, \quad x \in \mathbb{R},$$

and with identity as an iteration start, the first iteration steps (indexed by $n = 0, 1, \ldots$) are

$$g_0(x) = x, \quad g_1(x) = P(g_0(\alpha(x))) = P(\alpha(x)),$$
$$g_2(x) = P(g_1(\alpha(x))) = P^2(\alpha^2(x)), \quad g_3(x) = P(g_2(\alpha(x))) = P^3(\alpha^3(x)), \ldots$$

Iterating without limit ($n \to \infty$) would lead to $g_n(x) \to g(x)$ exactly. Stopping at a finite n (e.g. $n = 4$) yields a useful approximation of $g(x)$.

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