APPLICATIONES MATHEMATICAE
41,2-3 (2014), pp. 131–140

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EXISTENCE OF A RENORMALIZED SOLUTION OF NONLINEAR DEGENERATE ELLIPTIC PROBLEMS

Abstract. We study a general class of nonlinear elliptic problems associated with the differential inclusion $\beta(u) - \operatorname{div}(a(x, Du) + F(u)) \ni f$ in Ω where $f \in L^\infty(\Omega)$. The vector field $a(\cdot,\cdot)$ is a Carathéodory function. Using truncation techniques and the generalized monotonicity method in function spaces we prove existence of renormalized solutions for general L^∞-data.

1. Introduction. Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$ with Lipschitz boundary if $N \geq 2$, let p be a real number such that $1 < p < \infty$ and $w = \{w_i(x), 0 \leq i \leq N\}$ be a vector of weight functions on Ω (i.e., every component $w_i(x)$ is a measurable function which is positive a.e. in Ω). Let $W_0^{1,p}(\Omega, w)$ be the weighted Sobolev space associated with the vector w. Our aim is to show existence of renormalized solutions to the nonlinear elliptic equation

$$(E, f) \quad \begin{cases} \beta(u) - \operatorname{div}(a(x, Du) + F(u)) \ni f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

with right-hand side $f \in L^\infty(\Omega)$. Furthermore, F and β are functions satisfying the following assumption:

(A_0) $F\colon \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ is locally Lipschitz continuous and $\beta\colon \mathbb{R} \to 2^{\mathbb{R}}$ a set valued, maximal monotone mapping such that $0 \in \beta(0)$. Moreover,

$$\beta^0(l) \in L^1(\Omega) \tag{1.1}$$

for each $l \in \mathbb{R}$, where β^0 denotes the minimal selection of the graph of β, that is, $\beta_0(l) = \inf\{|r| \mid r \in \mathbb{R} \text{ and } r \in \beta(l)\}$.

Moreover, $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following assumptions:

2010 *Mathematics Subject Classification*: Primary 35J70; Secondary 35J85.
Key words and phrases: weighted Sobolev spaces, Hardy inequality, degenerate elliptic.

DOI: 10.4064/am41-2-1

(A_1) There exists a positive constant λ such that

$$a(x,\xi)\cdot\xi \geq \lambda \sum_{i=1}^{N} w_i|\xi_i|^p$$

for all $\xi \in \mathbb{R}^N$ and almost every $x \in \Omega$.

(A_2) $|a_i(x,\xi)| \leq \alpha w_i^{1/p}(x)[k(x) + \sum_{j=1}^{N} w_j^{1/p'}(x)|\xi_j|^{p-1}]$ for almost every $x \in \Omega$, all $i = 1,\dots,N$, and every $\xi \in \mathbb{R}^N$, where $k(\cdot)$ is a non-negative function in $L^{p'}(\Omega)$, $p' = p/(p-1)$, and $\alpha > 0$.

(A_3) $(a(x,\xi) - a(x,\eta))\cdot(\xi-\eta) \geq 0$ for almost every $x \in \Omega$ and every $\xi, \eta \in \mathbb{R}^N$.

Note that in the case with variable exponents and Orlicz spaces the problem was studied by Wittbold et al. [9, 12]. Other work in this direction can be found in [2, 5, 6].

2. Preliminaries. Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$, let p be a real number such that $1 < p < \infty$, and let $w = \{w_i(x), 0 \leq i \leq N\}$ be a vector of weight functions, i.e., every component $w_i(x)$ is a measurable function which is positive a.e. in Ω. Further, we suppose in all our considerations that

$$w_i \in L^1_{\mathrm{loc}}(\Omega), \tag{2.1}$$

$$w_i^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega), \tag{2.2}$$

for any $0 \leq i \leq N$. We denote by $W^{1,p}(\Omega, w)$ the space of all real-valued functions $u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions fulfill $\partial u/\partial x_i \in L^p(\Omega, w_i)$ for $i = 1,\dots,N$, which is a Banach space under the norm

$$\|u\|_{1,p,w} = \Bigg[\int_\Omega |u(x)|^p w_0(x)\,dx + \sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u(x)}{\partial x_i}\right|^p w_i(x)\,dx\Bigg]^{1/p}. \tag{2.3}$$

The condition (2.1) implies that $C_0^\infty(\Omega)$ is a subspace of $W^{1,p}(\Omega,w)$, and consequently we can define the subspace $X = W_0^{1,p}(\Omega,w)$ of $W^{1,p}(\Omega,w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.3). Moreover, condition (2.2) implies that $W^{1,p}(\Omega,w)$ as well as $W_0^{1,p}(\Omega,w)$ are reflexive Banach spaces. We recall that the dual space of $W_0^{1,p}(\Omega,w)$ is equivalent to $W^{-1,p'}(\Omega,w^*)$, where $w^* = \{w_i^* = w_i^{1-p'}, i = 0,\dots,N\}$ and where p' is the conjugate of p, i.e. $p' = p/(p-1)$ (for more details we refer to [10]).

Assumption (H1). The expression

$$|\!|\!|u|\!|\!|_X = \Bigg(\sum_{i=1}^{N}\int_\Omega \left|\frac{\partial u(x)}{\partial x_i}\right|^p w_i(x)\,dx\Bigg)^{1/p}$$

is a norm defined on X and is equivalent to the norm (2.3). There exist a weight function σ on Ω and a parameter q, $1 < q < \infty$, such that

$$\sigma^{1-q'} \in L^1(\Omega), \tag{2.4}$$

with $q' = q/(q-1)$. The Hardy inequality,

$$\Big(\int_\Omega |u(x)|^q \sigma\, dx\Big)^{1/q} \le c\Big(\sum_{i=1}^N \int_\Omega \Big|\frac{\partial u(x)}{\partial x_i}\Big|^p w_i(x)\, dx\Big)^{1/p}, \tag{2.5}$$

holds for every $u \in X$ with a constant $c > 0$ independent of u, and moreover the imbedding

$$X \hookrightarrow\hookrightarrow L^q(\Omega, \sigma), \tag{2.6}$$

expressed by the inequality (2.5), is compact. Note that $(X, \|\cdot\|_X)$ is a uniformly convex (and thus reflexive) Banach space.

3. Notion of solutions and existence results

Definition 3.1. A *renormalized solution* to (E, f) is a pair of functions (u, b) satisfying the following conditions:

(R1) $u : \Omega \to \mathbb{R}$ is measurable, $b \in L^1(\Omega)$, $u(x) \in \mathcal{D}(\beta(x))$ and $b(x) \in \beta(u(x))$ for a.e. $x \in \Omega$.

(R2) For each $k > 0$, $T_k(u) \in W_0^{1,p}(\Omega, w)$ and

$$\int_\Omega b \cdot h(u)\varphi + \int_\Omega (a(x, Du) + F(u)) \cdot D(h(u)\varphi) = \int_\Omega f h(u)\varphi \tag{3.1}$$

for all $h \in C_c^1(\mathbb{R})$ and all $\varphi \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$, where $T_k(\cdot)$ is truncation at height k.

(R3) $\int_{\{k \le |u| \le k+1\}} a(x, Du) \cdot Du \to 0 \quad$ as $k \to \infty$.

Theorem 3.2. *Under assumptions* (H_1), (A_0)–(A_3) *and* $f \in L^\infty(\Omega)$ *there exists at least one renormalized solution* (u, b) *to* (E, f).

Proof. Step 1: *Approximate problem.* First we approximate (E, f) for $f \in L^\infty(\Omega)$ by problems for which existence can be proved by standard variational arguments. For $0 < \varepsilon \le 1$, let $\beta_\varepsilon : \mathbb{R} \to \mathbb{R}$ be the Yosida approximation of β (see [7]). We introduce the operators

$$A_{1,\varepsilon} : W_0^{1,p}(\Omega, w) \to W^{-1,p'}(\Omega, w^*), \qquad u \mapsto \beta_\varepsilon(T_{1/\varepsilon}(u)) - \operatorname{div} a(x, Du),$$

$$A_{2,\varepsilon} : W_0^{1,p}(\Omega, w) \to W^{-1,p'}(\Omega, w^*), \qquad u \mapsto -\operatorname{div} F(T_{1/\varepsilon}(u)).$$

Because of (A_2) and (A_3), $A_{1,\varepsilon}$ is well-defined and monotone (see [11, p. 157]). Since $\beta_\varepsilon \circ T_{1/\varepsilon}$ is bounded and continuous and thanks to the growth condition (A_2) on a, it follows that $A_{1,\varepsilon}$ is hemicontinuous (see [11, p. 157]). From the continuity and boundedness of $F \circ T_{1/\varepsilon}$ it follows that $A_{2,\varepsilon}$ is strongly continuous. Therefore $A_\varepsilon := A_{1,\varepsilon} + A_{2,\varepsilon}$ is pseudomonotone. Using the monotonicity of β_ε, the Gauss–Green Theorem for Sobolev functions and the boundary condition on the convection term $\int_\Omega F(T_{1/\varepsilon}(u)) \cdot Du$, we show by similar arguments to [5] that A_ε is coercive and bounded. Then it follows from [11, Theorem 2.7] that A_ε is surjective, i.e., for each $0 < \varepsilon \le 1$ and $f \in W^{-1,p'}(\Omega, w^*)$ there exists a solution $u_\varepsilon \in W_0^{1,p}(\Omega, w)$ to the problem

$$(E_\varepsilon, f) \quad \begin{cases} \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) - \operatorname{div}\big(a(x, Du_\varepsilon) + F(T_{1/\varepsilon}(u_\varepsilon))\big) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

such that

$$\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\varphi + \int_\Omega \big(a(x, Du_\varepsilon) + F(T_{1/\varepsilon}(u_\varepsilon))\big) \cdot D\varphi = \langle f, \varphi \rangle \tag{3.2}$$

for all $\varphi \in W_0^{1,p}(\Omega, w)$.

STEP 2: *A priori estimates*

LEMMA 3.3. *For $0 < \varepsilon \le 1$ and $f \in L^\infty(\Omega)$ let $u_\varepsilon \in W_0^{1,p}(\Omega, w)$ be a solution of (E_ε, f). Then:*

(i) *There exists a constant $C_1 = C_1(\|f\|_\infty, \lambda, p, N) > 0$, not depending on ε, such that*

$$\|u_\varepsilon\| \le C_1. \tag{3.3}$$

(ii)

$$\|\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\|_\infty \le \|f\|_\infty \tag{3.4}$$

(iii) *For all $l, k > 0$ we have*

$$\int_{\{l \le |u_\varepsilon| \le l+k\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon \le k \int_{\{|u_\varepsilon| > l\}} |f|. \tag{3.5}$$

Proof. (i) Taking u_ε as a test function in (3.2) we obtain

$$\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))u_\varepsilon \, dx + \int_\Omega a(x, Du_\varepsilon) \cdot Du_\varepsilon \, dx + \int_\Omega F(T_{1/\varepsilon}(u_\varepsilon))) \cdot Du_\varepsilon \, dx = \int_\Omega f u_\varepsilon \, dx$$

As the first term on the left-hand side is nonnegative and the integral over the convection term vanishes, by (A_1) we have

$$\lambda \sum_{i=1}^N \int_\Omega \left| \frac{\partial u_\varepsilon}{\partial x_i} \right|^p w_i(x) \, dx \le \sum_{i=1}^N \int_\Omega a_i(x, Du_\varepsilon) \cdot \frac{\partial u_\varepsilon}{\partial x_i} \, dx \le \int_\Omega f u_\varepsilon \, dx$$
$$\le C\|f\|_\infty \bigg(\sum_{i=1}^N \int_\Omega \left| \frac{\partial u_\varepsilon(x)}{\partial x_i} \right|^p w_i(x) \, dx \bigg)^{1/p} \Big(\int_\Omega \sigma^{1-q'} dx \Big)^{1/q'}$$

(due to (2.5)). Thus $\|u_\varepsilon\|^p \le C_2 \|u_\varepsilon\|$ where C_2 is a positive constant. Hence we can deduce that u_ε remains bounded in $W_0^{1,p}(\Omega, w)$, i.e., $\|u_\varepsilon\| \le C_1$.

(ii) Taking $\frac{1}{\delta}[T_{k+\delta}(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))) - T_k(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))]$ as a test function in (3.2), letting $\delta \to 0$ and choosing $k > \|f\|_\infty$ we obtain (ii).

(iii) For $k, l > 0$ fixed we take $T_k(u_\varepsilon - T_l(u_\varepsilon))$ as a test function in (3.2). Using $\int_\Omega a(x, Du_\varepsilon) \cdot DT_k(u_\varepsilon - T_l(u_\varepsilon))\,dx = \int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon\,dx$, and as the first term on the left-hand side is nonnegative and the convection term vanishes, we get

$$\int\limits_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon\,dx \le \int\limits_\Omega f T_k(u_\varepsilon - T_l(u_\varepsilon))\,dx \le k \int\limits_{\{|u_\varepsilon|>l\}} |f|\,dx.$$

REMARK 3.4. For $k > 0$, since

$$|\{|u_\varepsilon| > l\}| \le \frac{C_2}{l^{1-1/p}}, \tag{3.6}$$

from Lemma 3.3(iii) we deduce that

$$\int\limits_{\{l\le|u_\varepsilon|\le l+k\}} a(x, Du_\varepsilon) \cdot Du_\varepsilon \le k\|f\|_\infty |\{|u_\varepsilon| > l\}| \le \frac{C_2(k)}{l^{1-1/p}}. \tag{3.7}$$

STEP 3: *Basic convergence results*

LEMMA 3.5. *For $0 < \varepsilon \le 1$ and $f \in L^\infty(\Omega)$ let $u_\varepsilon \in W_0^{1,p}(\Omega, w)$ be the solution of (E_ε, f). There exist $u \in W_0^{1,p}(\Omega, w)$ and $b \in L^\infty(\Omega)$ such that for a not relabeled subsequence of $(u_\varepsilon)_{0<\varepsilon\le 1}$ as $\varepsilon \downarrow 0$:*

$$u_\varepsilon \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega, w) \text{ and a.e. in } \Omega, \tag{3.8}$$

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \quad \text{in } W_0^{1,p}(\Omega, w) \text{ and strongly in } L^q(\Omega, \sigma), \tag{3.9}$$

$$\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \rightharpoonup b \quad \text{in } L^\infty(\Omega), \tag{3.10}$$

Moreover, for any $k > 0$

$$DT_k(u_\varepsilon) \rightharpoonup DT_k(u) \quad \text{in } \prod_{i=1}^N L^p(\Omega, w_i), \tag{3.11}$$

$$a(x, DT_k(u_\varepsilon)) \rightharpoonup a(x, DT_k(u)) \quad \text{in } \prod_{i=1}^N L^{p'}(\Omega, w_i^*). \tag{3.12}$$

Proof. (3.10) follows directly from Lemma 3.3 and Remark 3.4. From (3.6), (3.3) and (2.6) we deduce with a classical argument (see, e.g., [1]) that for a subsequence still indexed by ε, (3.8)–(3.9) and (3.11) hold as ε tend to 0, where u is a measurable function defined on Ω.

It is left to prove (3.12). For this, by (A_2) and (3.3) it follows that given any subsequence of $(a(x, DT_k(u_\varepsilon))_\varepsilon)$, there exists a subsequence, still denoted by $(a(x, DT_k(u_\varepsilon)))_\varepsilon$, such that $a(x, DT_k(u_\varepsilon)) \rightharpoonup \Phi_k$ in $\prod_{i=1}^N L^{p'}(\Omega, w_i^*)$. We will prove that $\Phi_k = a(x, DT_k(u))$ a.e. on Ω. The proof consists of three steps.

Step i: For every $h \in W^{1,\infty}(\mathbb{R})$, $h \geq 0$ and $\operatorname{supp}(h)$ compact, we will prove that

$$\limsup_{\varepsilon\to 0} \int_\Omega a(x, DT_k(u_\varepsilon)) \cdot D[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))]\,dx \leq 0. \tag{3.13}$$

Taking $h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))$ as a test function in (3.2) we have

(3.14)

$$\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) + \int_\Omega a(x, Du_\varepsilon) \cdot D[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))]$$
$$+ \int_\Omega F(T_{1/\varepsilon}(u_\varepsilon)) \cdot D[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))] = \int_\Omega f h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)).$$

Using $|h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))| \leq 2k\|h\|_\infty$, by Lebesgue's dominated convergence theorem we find that $\lim_{\varepsilon\to 0}\int_\Omega f h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) = 0$ and $\lim_{\varepsilon\to 0}\int_\Omega F(T_{1/\varepsilon}(u_\varepsilon))D[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))] = 0$. By using the same arguments as in [4] we can prove that

$$\limsup_{\varepsilon\to 0} \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \cdot [h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))]\,dx \geq 0.$$

Passing to the limit in (3.14) and using the above results we obtain (3.13).

Step ii: We now prove that for every $k > 0$,

$$\limsup_{\varepsilon\to 0} \int_\Omega a(x, DT_k(u_\varepsilon)) \cdot [DT_k(u_\varepsilon) - DT_k(u)]\,dx \leq 0. \tag{3.15}$$

Indeed, for $k > l$, take $h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))$ as a test function in (3.2). Letting $\varepsilon \downarrow 0$ and then $l \to \infty$ we obtain

$$\begin{aligned}
\int_\Omega a(x, DT_k(u_\varepsilon)) \cdot D[h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))]\,dx \\
&= \int_{[|u_\varepsilon|\leq k]} h_l(u_\varepsilon)a(x, DT_k(u_\varepsilon)) \cdot [DT_k(u_\varepsilon) - DT_k(u)]\,dx \\
&\quad + \int_{[|u_\varepsilon|> k]} h_l(u_\varepsilon)a(x, DT_k(u_\varepsilon)) \cdot (-DT_k(u))\,dx \\
&\quad + \int_\Omega h_l'(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))a(x, DT_k(u_\varepsilon)) \cdot Du_\varepsilon\,dx \\
&= E_1 + E_2 + E_3.
\end{aligned}$$

Since $l > k$, on the set $[|u_\varepsilon| \leq k]$ we have $h_l(u_\varepsilon) = 1$ so that we can write

$$\limsup_{\varepsilon\to 0} E_1 = \limsup_{\varepsilon\to 0} \int_\Omega a(x, DT_k(u_\varepsilon)) \cdot (DT_k(u_\varepsilon) - DT_k(u))\,dx.$$

For E_2, using Lebesgue's dominated convergence theorem we get

$$\lim_{\varepsilon\to 0} E_2 = -\int_{[|u|>k]} h_l(u)\Phi_{l+1}\cdot DT_k(u)\,dx = 0.$$

For E_3, we have

$$-\int_\Omega h_l'(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))a(x,DT_k(u_\varepsilon))Du_\varepsilon\,dx \le 2k\int_{[l<|u_\varepsilon|\le l+1]} a(x,Du_\varepsilon)Du_\varepsilon\,dx.$$

Using (3.7) we deduce that

$$\limsup_{l\to\infty}\limsup_{\varepsilon\to 0}\Big(-\int_\Omega h_l'(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))a(x,DT_k(u_\varepsilon))\cdot Du_\varepsilon\,dx\Big)\le 0.$$

Applying (3.13) with h replaced by h_l, $l>k$, we get

$$\limsup_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))\cdot(DT_k(u_\varepsilon)-DT_k(u))\,dx \le \limsup_{\varepsilon\to 0}\Big(-\int_\Omega h_l'(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))a(x,DT_k(u_\varepsilon))\cdot Du_\varepsilon\,dx\Big).$$

Now letting $l\to\infty$ yields (3.15).

Step iii: In this step we prove by monotonicity arguments that for $k>0$, $\Phi_k = a(x,DT_k(u))$ for almost every $x\in\Omega$. Let $\varphi\in\mathcal{D}(\Omega)$ and $\tilde\alpha\in\mathbb{R}$. Using (3.15), we have

$$\tilde\alpha\lim_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))\cdot D\varphi\,dx \ge \tilde\alpha\int_\Omega a(x,D(T_k(u)-\tilde\alpha\varphi))\cdot D\varphi\,dx.$$

Dividing by $\tilde\alpha>0$ and by $\tilde\alpha<0$ and letting $\tilde\alpha\to 0$ we obtain

$$\lim_{\varepsilon\to 0}\int_\Omega a(x,DT_k(u_\varepsilon))\cdot D\varphi\,dx = \int_\Omega a(x,DT_k(u))\cdot D\varphi\,dx.$$

This means that for all $k>0$, $\int_\Omega \Phi_k\cdot D\varphi dx = \int_\Omega a(x,DT_k(u))\cdot D\varphi\,dx$ and so $\Phi_k = a(x,DT_k(u))$ in $\mathcal{D}'(\Omega)$ for all $k>0$. Hence $\Phi_k = a(x,DT_k(u))$ a.e. in Ω and so $a(x,DT_k(u_\varepsilon))\rightharpoonup a(x,DT_k(u))$ weakly in $\prod_{i=1}^N L^{p'}(\Omega,w_i^*)$.

STEP 4: *Proof of existence.* Let $h\in C_c^1(\mathbb{R})$ and $\phi\in W_0^{1,p}(\Omega,w)\cap L^\infty(\Omega)$. Taking $h_l(u_\varepsilon)h(u)\phi$ as a test function in (3.2), we obtain

$$I^1_{\varepsilon,l}+I^2_{\varepsilon,l}+I^3_{\varepsilon,l}=I^4_{\varepsilon,l} \tag{3.16}$$

where

$$I^1_{\varepsilon,l} = \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))h_l(u_\varepsilon)h(u)\phi,$$
$$I^2_{\varepsilon,l} = \int_\Omega a(x, Du_\varepsilon) \cdot D(h_l(u_\varepsilon)h(u)\phi),$$
$$I^3_{\varepsilon,l} = \int_\Omega F(T_{1/\varepsilon}(u_\varepsilon)) \cdot D(h_l(u_\varepsilon)h(u)\phi),$$
$$I^4_{\varepsilon,l} = \int_\Omega f h_l(u_\varepsilon)h(u)\phi.$$

Step i: Letting $\varepsilon \downarrow 0$ using the convergence results (3.8), (3.10) from Lemma 3.5 we can immediately calculate the following limits:

$$\lim_{\varepsilon\to 0} I^1_{\varepsilon,l} = \int_\Omega b h_l(u)h(u)\phi, \tag{3.17}$$

$$\lim_{\varepsilon\to 0} I^4_{\varepsilon,l} = \int_\Omega f h_l(u)h(u)\phi. \tag{3.18}$$

We write $I^2_{\varepsilon,l} = I^{2,1}_{\varepsilon,l} + I^{2,2}_{\varepsilon,l}$ where

$$I^{2,1}_{\varepsilon,l} = \int_\Omega h'_l(u_\varepsilon)a(x, Du_\varepsilon) \cdot Du_\varepsilon h(u)\phi, \qquad I^{2,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)a(x, Du_\varepsilon) \cdot D(h(u)\phi).$$

Using (3.7) we get the estimate

$$\Big|\lim_{\varepsilon\to 0} I^{2,1}_{\varepsilon,l}\Big| \le \|h\|_\infty \|\phi\|_\infty \cdot C_2 l^{-(1-1/p)}. \tag{3.19}$$

By Lebesgue's dominated convergence theorem it follows that for any $i \in \{1, \dots, N\}$ we have

$$h_l(u_\varepsilon)\frac{\partial}{\partial x_i}(h(u)\phi) \to h_l(u)\frac{\partial}{\partial x_i}(h(u)\phi) \quad \text{ in } L^p(\Omega, \sigma) \text{ as } \varepsilon \downarrow 0.$$

Keeping in mind that $I^{2,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)a(x, DT_{l+1}(u_\varepsilon)) \cdot D(h(u)\phi)$, by (3.12), we get

$$\lim_{\varepsilon\to 0} I^{2,2}_{\varepsilon,l} = \int_\Omega h_l(u)a(x, DT_{l+1}(u)) \cdot D(h(u)\phi). \tag{3.20}$$

Let us write $I^3_{\varepsilon,l} = I^{3,1}_{\varepsilon,l} + I^{3,2}_{\varepsilon,l}$, where

$$I^{3,1}_{\varepsilon,l} = \int_\Omega h'_l(u_\varepsilon)F(T_{1/\varepsilon}(u_\varepsilon)) \cdot Du_\varepsilon h(u)\phi,$$
$$I^{3,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)F(T_{1/\varepsilon}(u_\varepsilon)) \cdot D(h(u)\phi).$$

For any $l \in \mathbb{N}$, there exists $\varepsilon_0(l)$ such that for all $\varepsilon < \varepsilon_0(l)$,

$$I_{\varepsilon,l}^{3,1} = \int_\Omega h_l'(T_{l+1}(u_\varepsilon))F(T_{l+1}(u_\varepsilon)) \cdot DT_{l+1}(u_\varepsilon)h(u)\phi. \tag{3.21}$$

Using the Gauss–Green Theorem for Sobolev functions in (3.21) we get

$$I_{\varepsilon,l}^{3,1} = -\int_\Omega \int_0^{T_{l+1}(u_\varepsilon)} h_l'(r)F(r)\,dr \cdot D(h(u)\phi). \tag{3.22}$$

Now, using (3.8) and the Gauss–Green Theorem, after letting $\varepsilon \downarrow 0$ we get

$$\lim_{\varepsilon\to 0} I_{\varepsilon,l}^{3,1} = \int_\Omega h_l'(u)F(u) \cdot Du\, h(u)\phi. \tag{3.23}$$

Choosing ε small enough, we can write

$$I_{\varepsilon,l}^{3,2} = \int_\Omega h_l(u_\varepsilon)F(T_{l+1}(u_\varepsilon)) \cdot D(h(u)\phi) \tag{3.24}$$

and conclude

$$\lim_{\varepsilon\to 0} I_{\varepsilon,l}^{3,2} = \int_\Omega h_l(u)F(u) \cdot D(h(u)\phi). \tag{3.25}$$

Step ii: We let $l \to \infty$. Combining (3.16) and (3.17)–(3.25) we find

$$I_l^1 + I_l^2 + I_l^3 + I_l^4 + I_l^5 = I_l^6 \tag{3.26}$$

where

$$I_l^1 = \int_\Omega bh_l(u)h(u)\phi, \qquad I_l^2 = \int_\Omega h_l(u)a(x, DT_{l+1}(u)) \cdot D(h(u)\phi),$$
$$|I_l^3| \le C_2 l^{-(1-1/p)}\|h\|_\infty\|\phi\|_\infty, \qquad I_l^4 = \int_\Omega h_l(u)F(u) \cdot D(h(u)\phi),$$
$$I_l^5 = \int_\Omega h_l'(u)F(u) \cdot Duh(u)\phi, \qquad I_l^6 = \int_\Omega fh_l(u)h(u)\phi.$$

Obviously, we have

$$\lim_{l\to\infty} I_l^3 = 0. \tag{3.27}$$

Choosing $m > 0$ such that $\operatorname{supp} h \subset [-m, m]$, we can replace u by $T_m(u)$ in $I_l^1, I_l^2, \ldots, I_l^6$, and

$$h_l'(u) = h_l'(T_m(u)) = 0 \text{ if } l+1 > m, \qquad h_l(u) = h_l(T_m(u)) = 1 \text{ if } l > m.$$

Therefore, letting $l \to \infty$ and combining (3.26) with (3.27) we obtain

$$\int_\Omega bh(u)\phi + \int_\Omega (a(x, Du) + F(u)) \cdot D(h(u)\phi) = \int_\Omega fh(u)\phi \tag{3.28}$$

for all $h \in C_c^1(\mathbb{R})$ and all $\phi \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$.

Step iii: Subdifferential argument. It is left to prove that $u(x) \in \mathcal{D}(\beta(x))$ and $b(x) \in \beta(u(x))$ for almost all $x \in \Omega$. Since β is a maximal monotone graph, there exists a convex, l.s.c., proper function $j : \mathbb{R} \to [0, \infty]$ such that $\beta(r) = \partial j(r)$ for all $r \in \mathbb{R}$. According to [7], for $0 < \varepsilon \leq 1$, $j_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by $j_\varepsilon(r) = \int_0^r \beta_\varepsilon(s)\, ds$ has the properties as in [12]. Using the same argument as in [12] we can prove that for all $r \in \mathbb{R}$ and almost every $x \in \Omega$, $u \in \mathcal{D}(\beta)$ and $b \in \beta(u)$ almost everywhere in Ω. With this last step the proof of Theorem 3.2 is completed.

References

[1] L. Aharouch, E. Azroul and A. Benkirane, *Quasilinear degenerated equations with L^1 datum and without coercivity in perturbation terms*, Electron. J. Qualit. Theory Differential Equations 2006, no. 19, 18 pp.

[2] L. Aharouch, A. Benkirane, J. Bennouna and A. Touzani, *Existence and uniqueness of solutions of some nonlinear equations in Orlicz spaces and weighted Sobolev spaces*, in: Recent Development in Nonlinear Analysis, World Sci., 2010, 170–180.

[3] Y. Akdim, E. Azroul and A. Benkirane, *Existence of solutions for quasi-linear degenerated elliptic equations*, Electron. J. Differential Equations 2001, no. 71, 19 pp.

[4] F. Andreu, N. Igbida, J. M. Mazón and J. Toledo, *L^1 existence and uniqueness results for quasi-linear elliptic equations with nonlinear boundary conditions*, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (2007), 61–89.

[5] M. Bendahmane and P. Wittbold, *Renormalized solutions for nonlinear elliptic equations with variable exponents and L^1-data*, Nonlinear Anal. 70 (2009), 567–583.

[6] A. Benkirane and J. Bennouna, *Existence of solutions for nonlinear elliptic degenerate equations*, Nonlinear Anal. 54 (2003), 9–37.

[7] H. Brézis, *Opérateurs Maximaux Monotones*, North-Holland, Amsterdam, 1973.

[8] R.-J. DiPerna and P.-L. Lions, *On the Cauchy problem for Boltzmann equations: Global existence and weak stability*, Ann. of Math. 130 (1989), 321–366.

[9] P. Gwiazda, P. Wittbold, A. Wróblewska and A. Zimmermann, *Renormalized solutions of nonlinear elliptic problems in generalized Orlicz spaces*, J. Differential Equations 253 (2012), 635–666.

[10] A. Kufner, *Weighted Sobolev Spaces*, Wiley, 1985.

[11] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.

[12] P. Wittbold and A. Zimmermann, *Existence and uniqueness of renormalized solutions to nonlinear elliptic equations with variable exponents and L^1-data*, Nonlinear Anal. 72 (2010), 2990–3008.

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Received on 25.2.2014;
revised version on 18.7.2014 (2233)