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EXISTENCE AND UNIQUENESS RESULT FOR A CLASS OF NONLINEAR PARABOLIC EQUATIONS WITH L^1 DATA

Abstract. We prove the existence and uniqueness of a renormalized solution for a class of nonlinear parabolic equations with no growth assumption on the nonlinearities.

1. Introduction. This paper is concerned with the initial-boundary value problem

$$(P)\qquad \begin{cases} \frac{\partial b(x,u)}{\partial t} - \operatorname{div}(A(x,t)Du + \Phi(u)) + f(x,t,u) = 0 \quad \text{in } Q,\\ b(x,u)(t=0) = b(x,u_0) \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^N$ $(N \geq 1)$, T is a positive real number, $Q \equiv \Omega \times (0,T)$, while the data $b(x,u_0)$ is in $L^1(\Omega)$. The matrix $A(x,t)$ is a bounded symmetric and coercive matrix; $b(x,s)$ is a strictly increasing C^1-function of s (for every $x \in \Omega$) but which is not restricted by any growth condition with respect to s (see assumptions (2.1) and (2.2) of Section 2); f is a Carathéodory function in $Q \times \mathbb{R}$ and not controlled with respect to s. The function Φ is just assumed to be continuous in $\mathbb{R}$.

Note that a large number of papers have been devoted to the study of the existence and uniqueness of solutions of parabolic problems under various assumptions and in different contexts; for classical results see e.g. [8], [9], [10], [1]–[5], [12], [19], [20].

2. Assumptions on the data and definition of a renormalized solution. Let Ω be a bounded open set in $\mathbb{R}^N$ $(N \geq 1)$, $T > 0$ and $Q =$

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$\Omega \times (0,T)$. Assume that:

(2.1) $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for every $x \in \Omega$, $b(x,s)$ is a strictly increasing C^1-function of s, with $b(x,0) = 0$;

(2.2) for any $K > 0$, there exists $\lambda_K > 0$, a function A_K in $L^\infty(\Omega)$ and a function B_K in $L^2(\Omega)$ such that

$$\lambda_K \le \frac{\partial b(x,s)}{\partial s} \le A_K(x) \quad \text{and} \quad \left| \nabla_x \left(\frac{\partial b(x,s)}{\partial s} \right) \right| \le B_K(x)$$

for almost every $x \in \Omega$ and every s such that $|s| \le K$;

(2.3) $A(x,t)$ is a symmetric coercive matrix field with coefficients lying in $L^\infty(Q)$, i.e. $A(x,t) = (a_{ij}(x,t))_{1\le i,\, j\le N}$ with $a_{ij}(\cdot,\cdot) \in L^\infty(Q)$ and $a_{ij}(x,t) = a_{ji}(x,t)$ a.e. in Q, for all $i,\ j$, and there exists $\alpha > 0$ such that $A(x,t)\xi \cdot \xi \ge \alpha|\xi|^2$ a.e. (x,t) in Q, $\forall \xi \in \mathbb{R}^N$;

(2.4) $\Phi : \mathbb{R} \to \mathbb{R}^N$ is a continuous function;

(2.5) $f : Q \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function;

(2.6) for almost every $(x,t) \in Q$, and every $s \in \mathbb{R}$,

$$\operatorname{sign}(s) f(x,t,s) \ge 0 \quad \text{and} \quad f(x,t,0) = 0;$$

(2.7) $\max_{\{|s|\le K\}} |f(x,t,s)| \in L^1(Q)$ for any $K > 0$;

(2.8) u_0 is a measurable function such that $b(\cdot,u_0) \in L^1(\Omega)$.

Remark 2.1. As already mentioned in the introduction Problem (P) does not admit a weak solution under assumptions (2.1)–(2.8) since the growth of $b(x,u)$, $\Phi(u)$ and $f(x,t,u)$ is not controlled with respect to u.

Throughout, for any nonnegative real number K we denote by $T_K(r) = \min(K, \max(r,-K))$ the truncation function at height K. The definition of a renormalized solution for Problem (P) can be stated as follows.

Definition 2.2. A measurable function u defined on Q is a *renormalized solution* of Problem (P) if

$$T_K(u) \in L^2(0,T; H_0^1(\Omega))\ \forall K \ge 0 \text{ and } b(x,u) \in L^\infty(0,T; L^1(\Omega)), \tag{2.9}$$

$$\int\limits_{\{(t,x)\in Q;\, n\le |u(x,t)|\le n+1\}} A(x,t) Du Du \, dx\, dt \to 0 \quad \text{as } n \to \infty, \tag{2.10}$$

and if for every function S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support,

$$\begin{aligned} \frac{\partial b_S(x,u)}{\partial t} &- \operatorname{div}(S'(u)A(x,t)Du) + S''(u)A(x,t)Du \cdot Du \\ &- \operatorname{div}(S'(u)\Phi(u)) + S''(u)\Phi(u)Du + f(x,t,u)S'(u) = 0 \text{ in } \mathcal{D}'(Q), \end{aligned} \tag{2.11}$$

and

$$b_S(x,u)(t=0) = b_S(x,u_0) \ \text{ in } \Omega, \tag{2.12}$$

where $b_S(x,r) = \int_0^r \frac{\partial b(x,s)}{\partial s} S'(s)\, ds$.

REMARK 2.3. Note that due to (2.9) each term in (2.11) has a meaning in $L^1(Q) + L^2(0,T;H^{-1}(\Omega))$, and $\partial b_S(x,u)/\partial t$ belongs to $L^1(Q) + L^2(0,T;H^{-1}(\Omega))$. Due to the properties of S and (2.2), we see that $b_S(x,u)$ belongs to $L^2(0,T;H_0^1(\Omega))$, which implies that $b_S(x,u) \in C^0([0,T];L^1(\Omega))$ (for a proof of this trace result see [18]), so that the initial condition (2.12) makes sense.

3. Existence result. This section is devoted to establishing the following existence theorem.

THEOREM 3.1. *Under assumptions* (2.4)–(2.8) *there exists a renormalized solution u of Problem* (P).

Proof. The proof is divided into three steps.

STEP 1: *Approximate problem.* Let us introduce the following regularization of the data: for $\varepsilon > 0$ fixed

(3.1) $b_\varepsilon(x,s) = b(x,T_{1/\varepsilon}(s)))$ a.e. in Ω, $\forall s \in \mathbb{R}$.

(3.2) Φ_ε is a Lipschitz-continuous bounded function from $\mathbb{R}$ into $\mathbb{R}^N$ such that Φ_ε uniformly converges to Φ on any compact subset of $\mathbb{R}$ as $\varepsilon \to 0$.

(3.3) $f^\varepsilon(x,t,s) = f(x,t,T_{1/\varepsilon}(s))$ a.e. in Q, $\forall s \in \mathbb{R}$.

(3.4) $u_0^\varepsilon \in C_0^\infty(\Omega)$ and $b_\varepsilon(x,u_0^\varepsilon) \to b(x,u_0)$ in $L^1(\Omega)$ as $\varepsilon \to 0$.

Let us now consider the regularized problem

$$(P^\varepsilon) \qquad \begin{cases} \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial t} - \operatorname{div}(A(x,t)Du^\varepsilon + \Phi_\varepsilon(u^\varepsilon)Du^\varepsilon) + f^\varepsilon(x,t,u^\varepsilon) = 0 & \text{in } Q, \\ b_\varepsilon(x,u^\varepsilon)(t=0) = b_\varepsilon(x,u_0^\varepsilon) & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega \times (0,T), \end{cases}$$

Proving existence of a weak solution $u^\varepsilon \in L^2(0,T;H_0^1(\Omega))$ of (P^ε) is an easy task (see e.g. [15]).

STEP 2: *A priori estimates.* The estimates derived in this step rely on usual techniques for problems of type (P^ε) and we just sketch their proof (the reader is referred to [7], [10], [8] or [17] for elliptic versions of (P)). Using $T_K(u^\varepsilon)$ as a test function in (P), we deduce that

$$T_K(u^\varepsilon) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)) \tag{3.5}$$

independently of ε for any $K \ge 0$. Proceeding as in [7], [10] for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' is compact ($\operatorname{supp} S' \subset [-K,K]$) we find that

$$b_S(x,u^\varepsilon) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)), \tag{3.6}$$

$$\partial b_S(x,u^\varepsilon)/\partial t \text{ is bounded in } L^1(Q) + L^2(0,T;H^{-1}(\Omega)) \tag{3.7}$$

independently of ε.

For any integer $n \geq 1$, using the admissible test function $\theta_n(u^\varepsilon) = T_{n+1}(u^\varepsilon) - T_n(u^\varepsilon)$ in (P^ε), we obtain for almost $t \in (0,T)$,

$$\int_0^t \int_\Omega A(x,t) Du^\varepsilon \cdot D\theta_n(u^\varepsilon)\,dx\,ds \leq \int_\Omega b_{\varepsilon,n}(x,u_0^\varepsilon)\,dx. \tag{3.8}$$

Again as in [7], [8] and [10], estimates (3.6) and (3.7) imply that, for a subsequence still indexed by ε,

$$u^\varepsilon \to u \quad \text{almost everywhere in } Q, \tag{3.9}$$

$$T_K(u^\varepsilon) \to T_K(u) \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)), \tag{3.10}$$

$$b_\varepsilon(x,u^\varepsilon) \to b(x,u) \quad \text{strongly in } L^1(Q), \tag{3.11}$$

$$\theta_n(u^\varepsilon) \rightharpoonup \theta_n(u) \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)), \tag{3.12}$$

as $\varepsilon \to 0$ for any $K > 0$ and any $n \geq 1$. We conclude that

$$b(x,u) \in L^\infty(0,T;L^1(\Omega)), \tag{3.13}$$

and

$$\lim_{n\to\infty} \limsup_{\varepsilon\to 0} \int_{\{n\leq|u^\varepsilon|\leq n+1\}} A(x,t) DT_{n+1}(u^\varepsilon) \cdot DT_{n+1}(u^\varepsilon)\,dx\,dt = 0. \tag{3.14}$$

Step 3: *Time regularization.* In this step we introduce, for $K \geq 0$ fixed, a time regularization of the function $T_K(u)$. This specific time regularization is defined as follows. Let $(v_0^\mu)_\mu$ be a sequence of functions defined on Ω such that for all $\mu > 0$,

$$v_0^\mu \in L^\infty(\Omega) \cap H_0^1(\Omega),\ \|v_0^\mu\|_{L^\infty(\Omega)} \leq K,\ v_0^\mu \to T_K(u_0),\ \lim_{\mu\to\infty} \frac{1}{\mu}\|v_0^\mu\|_{L^2(\Omega)} = 0.$$

Let us consider the unique solution $T_K(u)_\mu \in L^\infty(Q) \cap L^2(0,T;H_0^1(\Omega))$ of

$$\frac{\partial T_K(u)_\mu}{\partial t} + \mu(T_K(u)_\mu - T_K(u)) = 0 \quad \text{in } \mathcal{D}'(Q), \tag{3.15}$$

$$T_K(u)_\mu(t=0) = v_0^\mu \quad \text{in } \Omega. \tag{3.16}$$

We just recall here that (3.15)–(3.16) imply that $T_K(u)_\mu \to T_K(u)$ a.e. in Q and weakly-$\star$ in $L^\infty(Q)$ and strongly in $L^2(0,T;H_0^1(\Omega))$ as $\mu \to \infty$. Let $h \in W^{1,\infty}(\mathbb{R})$, $h \geq 0$, with $\operatorname{supp} h$ compact. The main estimate is

Lemma 3.2 (see [19]).

$$\limsup_{\mu\to\infty} \lim_{\varepsilon\to 0} \int_0^T \int_0^s \left\langle \frac{\partial b_\varepsilon(x,u^\varepsilon)}{\partial t}, h(u^\varepsilon)(T_K(u^\varepsilon) - (T_K(u))_\mu) \right\rangle dt\,ds \geq 0$$

where $\langle\ ,\ \rangle$ denotes the duality pairing between $L^1(\Omega) + H^{-1}(\Omega)$ and $L^\infty(\Omega) \cap H_0^1(\Omega)$.

Lemma 3.3 (see [19]). *A subsequence of u^ε defined in Step 3 satisfies, for any $K \ge 0$,*

$$\liminf_{\varepsilon\to 0} \int_0^T \int_0^t \int_\Omega A(x,t)[DT_K(u^\varepsilon) - DT_K(u)] \cdot [DT_K(u^\varepsilon) - DT_K(u)]\, dx\, dt\, ds = 0,$$

and

$$T_K(u^\varepsilon) \to T_K(u) \quad \textit{strongly in } L^2(0,T;H_0^1(\Omega)) \textit{ as } \varepsilon \to 0.$$

Now, let S in $W^{2,\infty}(\mathbb{R})$ be such that S' has a compact support, say $\operatorname{supp} S' \subset [-K,K]$. Pointwise multiplication of the approximate equation (P^ε) by $S'(u^\varepsilon)$ leads to

$$\frac{\partial b_S^\varepsilon(x,u^\varepsilon)}{\partial t} - \operatorname{div}(S'(u^\varepsilon)A(x,t)Du^\varepsilon) + S''(u^\varepsilon)A(x,t)Du^\varepsilon \cdot Du^\varepsilon - \operatorname{div}(S'(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)) + S''(u^\varepsilon)\Phi_\varepsilon(u^\varepsilon)Du^\varepsilon + f^\varepsilon(x,t,u^\varepsilon)S'(u^\varepsilon) = 0 \text{ in } \mathcal{D}'(Q), \tag{3.17}$$

where

$$b_S^\varepsilon(x,r) = \int_0^r \frac{\partial b_\varepsilon(x,s)}{\partial s} S'(s)\, ds.$$

Letting $\varepsilon \to 0$ in each term of (3.17), we conclude that u satisfies (2.11).

As a consequence, an Aubin type lemma (see e.g. [21, Corollary 4]) implies that $b_S(x,u^\varepsilon)$ lies in a compact subset of $C^0([0,T];W^{-1,s}(\Omega))$ for any $s < \inf(2, N/(N-1))$. It follows that, on one hand, $b_S(x,u^\varepsilon)(t=0) = b_S(x,u_0^\varepsilon)$ converges to $b_S(x,u)(t=0)$ strongly in $W^{-1,s}(\Omega)$. On the other hand, (3.4) and the smoothness of S imply that $b_S(x,u_0^\varepsilon)$ converges to $b_S(x,u)(t=0)$ strongly in $L^q(\Omega)$ for all $q < \infty$. Then we conclude that $b_S(x,u)(t=0) = b_S(x,u_0)$ in Ω. By Steps 1–3, the proof of Theorem 3.1 is complete.

4. Comparison principle and uniqueness result. This section is concerned with a comparison principle (and a uniqueness result) for renormalized solutions in the case where $f(x,t,u)$ is independent of u. We establish the following theorem.

Theorem 4.1. *Assume that assumptions* (2.1)–(2.4) *and* (2.8) *hold true and moreover that:*

(4.1) *for any $K > 0$, there exists $\beta_K > 0$ such that*

$$\left| \frac{\partial b(x,z_1)}{\partial s} - \frac{\partial b(x,z_2)}{\partial s} \right| \le \beta_K |z_1 - z_2|$$

for almost every x in Ω, and all z_1 and z_2 such that $|z_1|, |z_2| \le K$,

(4.2) *Φ is a locally Lipschitz-continuous function on $\mathbb{R}$,*

(4.3) *$f_1, f_2 \in L^1(Q)$.*

Let u_1 and u_2 be renormalized solutions corresponding to the data (f_1, u_0^1) and (f_2, u_0^2) for the problem $(i = 1, 2)$

$$(P_i)\qquad \begin{cases} \frac{\partial b(x,u_i)}{\partial t} - \operatorname{div}(A(x,t)Du_i + \Phi(u_i)) = f_i(x,t) \quad \textit{in } Q, \\ b(x,u_i)(t=0) = b(x,u_0^i) \quad \textit{in } \Omega, \\ u_i = 0 \quad \textit{on } \partial\Omega \times (0,T). \end{cases}$$

If $f_1 \le f_2$ and $u_0^1 \le u_0^2$ a.e., then $u_1 \le u_2$ a.e. in Q.

Sketch of the proof. Here we just give an idea of how $u_1 \le u_2$ can be obtained following the outline of [20]. Let us introduce a specific S in (2.11). For all $n > 0$, let $S_n \in C^1(\mathbb{R})$ be defined by $S_n'(r) = 1$ for $|r| \le n$, $S_n'(r) = n + 1 - |r|$ for $n \le |r| \le n+1$ and $S_n'(r) = 0$ for $|r| \ge n+1$. Taking $S = S_n$ in (2.11) yields

$$(4.4)\qquad \frac{\partial b_{S_n}(x,u_i)}{\partial t} - \operatorname{div}(S_n'(u_i)A(x,t)Du_i) + S''(u_i)A(x,t)Du_iDu_i - \operatorname{div}(\Phi_{S_n}(u_i)) = f_iS_n'(u_i) \quad \text{in } \mathcal{D}'(Q)$$

for $i = 1, 2$ with $b_{S_n}(x,r) = \int_0^r \frac{\partial b(x,s)}{\partial s} S_n'(s)\,ds$.

We use $\frac{1}{\sigma}T_\sigma^+(b_{S_n}(x,u_1) - b_{S_n}(x,u_2))$ as a test function in the difference of equations (4.4) for u_1 and u_2 to get

$$(4.5)\qquad \frac{1}{\sigma}\int_0^T\int_0^t \left\langle \frac{\partial(b_{S_n}(x,u_1) - b_{S_n}(x,u_2))}{\partial t}, T_\sigma^+(b_{S_n}(x,u_1) - b_{S_n}(x,u_2)) \right\rangle ds\,dt + A_n^\sigma = B_n^\sigma + C_n^\sigma + D_n^\sigma$$

for any $\sigma > 0$, $n > 0$, where

$$A_n^\sigma = \frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega [S_n'(u_1)A(t,x)Du_1 - S_n'(u_2)A(t,x)Du_2] \cdot DT_\sigma^+(b_{S_n}(x,u_1) - b_{S_n}(x,u_2))\,dx\,ds\,dt,$$

$$B_n^\sigma = \frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega S_n''(u_1)A(x,t)Du_1Du_1T_\sigma^+(b_{S_n}(x,u_1) - b_{S_n}(x,u_2))\,dx\,ds\,dt - \frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega S_n''(u_2)A(x,t)Du_2Du_2T_\sigma^+(b_{S_n}(x,u_1) - b_{S_n}(x,u_2))\,dx\,ds\,dt,$$

$$C_n^\sigma = \frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega [\Phi_{S_n}(u_1) - \Phi_{S_n}(u_2)]DT_\sigma^+(b_{S_n}(x,u_1) - b_{S_n}(x,u_2))\,dx\,ds\,dt,$$

$$D_n^\sigma = \frac{1}{\sigma}\int_0^T\int_0^t\int_\Omega [f_1S_n'(u_1) - f_2S_n'(u_2)]T_\sigma^+(b_{S_n}(x,u_1) - b_{S_n}(x,u_2))\,dx\,ds\,dt.$$

We will pass to the limit in (4.5) as $\sigma \to 0$ and then $n \to \infty$. Upon application of Lemma 2.4 of [11], the first term on the right hand side of (4.5) is

$$\frac{1}{\sigma}\int_0^T\int_0^t \left\langle \frac{\partial(b_{S_n}(x,u_1)-b_{S_n}(x,u_2))}{\partial t}, T_\sigma^+(b_{S_n}(x,u_1)-b_{S_n}(x,u_2))\right\rangle ds\,dt \tag{4.6}$$

$$= \frac{1}{\sigma}\int_Q \tilde{T}_\sigma^+(b_{S_n}(x,u_1)-b_{S_n}(x,u_2))\,dx\,dt - \frac{T}{\sigma}\int_\Omega \tilde{T}_\sigma^+(b_{S_n}(x,u_0^1)-b_{S_n}(x,u_0^2))\,dx$$

where $\tilde{T}_\sigma^+(t) = \int_0^t T_\sigma^+(s)\,ds$. Due to the assumption $u_0^1 \le u_0^2$ a.e. in Ω and the monotone character of $b_{S_n}(x,\cdot)$ and $T_\sigma(\cdot)$, we have

$$\int_\Omega \tilde{T}_\sigma^+(b_{S_n}(x,u_0^1)-b_{S_n}(x,u_0^2))\,dx = 0. \tag{4.7}$$

It follows from (4.5)–(4.7) that

$$\frac{1}{\sigma}\int_Q \tilde{T}_\sigma^+(b_{S_n}(x,u_1)-b_{S_n}(x,u_2))\,dx\,dt + A_n^\sigma = B_n^\sigma + C_n^\sigma + D_n^\sigma \tag{4.8}$$

for any $\sigma > 0$ and any $n > 0$. We need the following lemma (see [20])

Lemma 4.2. *We have*

$$\begin{aligned} &\liminf_{n\to\infty}\liminf_{\sigma\to 0} A_n^\sigma \ge 0, && \liminf_{n\to\infty}\liminf_{\sigma\to 0} B_n^\sigma = 0,\\ &\liminf_{\sigma\to 0} C_n^\sigma = 0, && \liminf_{n\to\infty}\limsup_{\sigma\to 0} D_n^\sigma \le 0. \end{aligned} \tag{4.9}$$

In view of (4.7)–(4.9) we have $\int_Q (b(x,u_1)-b(x,u_2))^+\,dx\,dt \le 0$, so that $b(x,u_1) \le b(x,u_2)$ a.e. in Q, which in turn implies that $u_1 \le u_2$ a.e. in Q, and Theorem 4.1 is established.

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