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ON A NECESSARY CONDITION IN THE CALCULUS OF VARIATIONS IN SOBOLEV SPACES WITH VARIABLE EXPONENT

Abstract. We prove an approximation theorem in generalized Sobolev spaces with variable exponent $W^{1,p(\cdot)}(\Omega)$ and we give an application of this approximation result to a necessary condition in the calculus of variations.

1. Introduction. In the present work, our first main goal is to prove an approximation theorem in the general setting of Sobolev spaces with variable exponent, and the second main goal is to give an application of this approximation result to a necessary condition in the calculus of variations in the same functional framework of $W^{1,p(\cdot)}(\Omega)$. The theory of Sobolev spaces with variable exponent has experienced a revival of interest, shown in a substantial amount of publications over the past few years. An extensive list of references concerning the recent advances and open problems can be found in Diening and al. [DHN].

We consider functionals of the kind $J(u) = \int_\Omega f(x, u, \nabla u)\, dx$ for functions u in some Sobolev space with variable exponent $W^{1,p(\cdot)}(\Omega)$, with Ω a bounded domain of $\mathbb{R}^N$. In the case of a constant exponent $p(\cdot) \equiv p$, $1 < p < \infty$, sufficient conditions for those functionals to attain an extreme value were studied in [D]. The most important problem is to verify the weak lower semicontinuity of those functionals with respect to the space involved. In [L] Landes has studied the reverse problem at a fixed level set and in many situations he has showed that this hypothesis implies the following alternative: *Either this particular level is an extreme value of the functional J or the defining function f is convex in the gradient.* Recently, the results

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achieved in [L] and [MS] have been generalised in the work of E. Azroul and A. Benkirane [AA] where the classical setting of Sobolev spaces $W^{1,p}(\Omega)$ is replaced by the more general one of Orlicz–Sobolev spaces $W^1L_M(\Omega)$ corresponding to an N-function M, which is considered as a relaxation of the constant p. Our third objective in the present work is to generalize the results of [L] to the general setting of Sobolev spaces with variable exponent.

2. Background material. Let Ω be a bounded open domain in $\mathbb{R}^N$, $N \geq 2$. We denote $\mathcal{C}_+(\overline{\Omega}) = \{p \in \mathcal{C}(\overline{\Omega}) : p(x) > 1 \text{ for all } x \in \overline{\Omega}\}$. For every $p \in \mathcal{C}_+(\overline{\Omega})$ we define $p^+ = \sup_{x\in\Omega} p(x)$ and $p^- = \inf_{x\in\Omega} p(x)$. The *variable exponent Lebesgue space* is defined as

$$L^{p(\cdot)}(\Omega) = \Big\{u : u \text{ is a measurable real-valued function},\ \exists\lambda > 0 : \int_\Omega |u(x)/\lambda|^{p(x)}\,dx < \infty\Big\},$$

normed by the so-called *Luxemburg* norm,

$$\|u\|_{p(\cdot)} = \inf\Big\{\lambda > 0 : \int_\Omega |u(x)/\lambda|^{p(x)}\,dx \leq 1\Big\}.$$

For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the *Hölder inequality*

$$\Big|\int_\Omega u(x)v(x)\,dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{p^+}\Big)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}$$

holds true. An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the modular of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)}(u) : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by $\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\,dx$, for all $u \in L^{p(\cdot)}(\Omega)$. We define the *generalized Lebesgue–Sobolev space* $W^{1,p(\cdot)}(\Omega)$ by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

which is endowed with the norm $\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$. We define

$$W_0^{1,p(\cdot)}(\Omega) = \overline{\mathcal{C}_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}.$$

Then the dual space of $W_0^{1,p(\cdot)}(\Omega)$ can be identified with $W^{-1,p'(\cdot)}(\Omega)$.

PROPOSITION 2.1 ([KJ]).

(1) *$W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are Banach spaces, which are separable if $p \in L^\infty(\Omega)$ and reflexive if $1 < p^- < p^+ < \infty$.*

(2) *If $q \in \mathcal{C}_+(\overline{\Omega})$ with $q(x) < p^*(x)$ then we have the compact embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ for all $p(x) < N$. Since $p(x) < p^*(x)$ for all $x \in \Omega$, in particular we have*

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{p(\cdot)}(\Omega). \tag{2.1}$$

(3) *There exists a constant $c > 0$ with $\|u\|_{p(\cdot)} \le c\|\nabla u\|_{p(\cdot)}$ for all $u \in W_0^{1,p(\cdot)}(\Omega)$, hence $\|\nabla u\|_{p(\cdot)}$ and $\|u\|_{1,p(\cdot)}$ are equivalent norms on $W_0^{1,p(\cdot)}(\Omega)$.*

3. Approximation result

THEOREM 3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$. If u is a function in $W^{1,p(\cdot)}(\Omega)$, then for almost every $x_0 \in \Omega$, there exists a ball $B(x_0,\alpha)$, $\alpha > 0$, a constant $C(\alpha, x_0)$ and a function $u_\alpha \in W^{1,p(\cdot)}(\Omega)$ satisfying:*

(i) *$u_\alpha \to u$ in $W^{1,p(\cdot)}(\Omega)$ as $\alpha \to 0$.*
(ii) *$u_\alpha \equiv C(\alpha, x_0)$ in $B(x_0,\alpha)$.*

Proof. Let ψ_α be a regular function with support in $B(0, 2\alpha)$ such that

$$\psi_\alpha \equiv 1 \quad \text{in } B(0,\alpha) \quad \text{and} \quad |\nabla \psi_\alpha| \le 2/\alpha, \tag{3.1}$$

and let x_0 be a Lebesgue point of u in Ω, hence, we can take $C(\alpha, x_0) = u(x_0)$. We define in Ω the function u_α by

$$u_\alpha(x) = u(x)(1 - \psi_\alpha(x - x_0)) + u(x_0)\psi_\alpha(x - x_0). \tag{3.2}$$

By using the Lebesgue theorem, we can write

$$u_\alpha \to u \quad \text{in } L^{p(\cdot)}(\Omega) \text{ as } \alpha \to 0. \tag{3.3}$$

We have

$$\frac{\partial}{\partial x_i}(u(x) - u_\alpha(x)) = \frac{\partial u(x)}{\partial x_i}\psi_\alpha(x - x_0) + \frac{\partial}{\partial x_i}\psi_\alpha(x - x_0)(u(x) - u(x_0)),$$

and the convexity of $\rho_{p(\cdot)}(\cdot)$ yields

$$\begin{aligned}
\rho_{p(\cdot)}\left(\lambda\left(\frac{\partial u}{\partial x_i} - \frac{\partial u_\alpha}{\partial x_i}\right)\right) &= \int_\Omega \left|\lambda\left(\frac{\partial u(x)}{\partial x_i} - \frac{\partial u_\alpha(x)}{\partial x_i}\right)\right|^{p(x)} dx \\
&\le \frac{1}{2}\int_\Omega \left(2\lambda\left|\frac{\partial u(x)}{\partial x_i}\psi_\alpha(x - x_0)\right|\right)^{p(x)} dx \\
&\quad + \frac{1}{2}\int_\Omega 2\lambda\left|\frac{\partial}{\partial x_i}\psi_\alpha(x - x_0)\right|(u(x) - u(x_0))^{p(x)}\, dx \\
&= \frac{1}{2}I_\alpha^1 + \frac{1}{2}I_\alpha^2.
\end{aligned}$$

By using the Lebesgue theorem, we have $\lim_{\alpha\to 0} I_\alpha^1 = 0$. And $I_\alpha^2 \to 0$ as $\alpha \to 0$ is a direct consequence of the lemma below. ∎

LEMMA 3.2. *For almost every $x_0 \in \Omega$, there exists a sequence $\alpha_n \to 0$ as $n \to \infty$ such that*

$$\int_{B(x_0,2\alpha_n)} \left(\frac{\lambda|u(x) - u(x_0)|}{\alpha_n}\right)^{p(x)} dx \to 0 \qquad \text{as } n \to \infty.$$

Proof. Let $x_0 \in \Omega$. For each $t>0$, we define $\Omega_t=\{x\in\Omega : \operatorname{dist}(x,\partial\Omega)>t\}$. Let $\alpha_0 > 0$. For $\alpha < \alpha_0$, we consider the function $\phi_\alpha : \Omega_{2\alpha_0} \to \mathbb{R}$ defined by

$$\phi_\alpha(y) = \int\limits_{B(y,2\alpha)} \left(\frac{\lambda|u(x)-u(y)|}{\alpha}\right)^{p(x)} dx.$$

We can also write

$$\phi_\alpha(y) = \int\limits_{\Omega} \left(\frac{\lambda|u(x)-u(y)|}{\alpha}\right)^{p(x)} \chi_{B(0,2\alpha)}\, dx,$$

where χ_E denotes the characteristic function of the set E. The function ϕ_α is measurable, and for all $\alpha_0 > 0$, we shall show that

$$|\phi_\alpha(y)| \to 0 \quad \text{in } L^1(\Omega_{2\alpha_0}) \text{ as } \alpha \to 0, \text{ for all } \alpha < \alpha_0. \tag{3.4}$$

Indeed, let $u_\epsilon = u * \varphi_\epsilon$ be the mollification of u, where $\varphi_\epsilon \in \mathcal{D}(\mathbb{R}^N)$, such that

$$\varphi_\epsilon \equiv 1 \quad \text{for } |x| \geq \epsilon, \qquad \varphi_\epsilon \geq 0, \qquad \int\limits_{\mathbb{R}^N} \varphi_\epsilon(x)\, dx = 1.$$

Therefore, φ_ϵ is well defined in $\Omega_{2\alpha_0}$ for all $\epsilon < \alpha_0$ and we have

$$\begin{aligned}\int\limits_{\Omega_{2\alpha_0}} |\phi_\alpha(y)|\, dy &= \int\limits_{\Omega_{2\alpha_0}} \int\limits_{B(y,2\alpha)} \left(\frac{\lambda|u(x)-u(y)|}{\alpha}\right)^{p(x)} dx\, dy \\ &\leq \lim_{\epsilon\to 0} \int\limits_{\Omega_{2\alpha_0}} \int\limits_{B(0,2\alpha)} \left(\frac{\lambda|u_\epsilon(y-x)-u_\epsilon(y)|}{\alpha}\right)^{p(x)} dx\, dy = \lim_{\epsilon\to 0} I_\alpha.\end{aligned}$$

Since u_α is continuously differentiable, we have

$$\begin{aligned}I_\alpha &\leq \int\limits_{\Omega_{2\alpha_0}} \int\limits_{B(0,2\alpha)} \left(\lambda\int_0^1 \frac{|\nabla u_\epsilon(y-tx)|\,|x|}{\alpha}\, dt\right)^{p(x)} dx\, dy \\ &\leq \int\limits_{\Omega_{2\alpha_0}} \int\limits_{B(0,2\alpha)} \left(\lambda\int_0^1 2|\nabla u_\epsilon(y-tx)|\, dt\right)^{p(x)} dx\, dy.\end{aligned}$$

Then, it follows by Jensen's inequality that

$$\begin{aligned}I_\alpha &\leq \int\limits_{\Omega_{2\alpha_0}} \int_0^1 \int\limits_{B(0,2\alpha)} (2\lambda|\nabla u_\epsilon(y-tx)|)^{p(x)}\, dx\, dt\, dy \\ &\overset{(*)}{=} \int_0^1 \int\limits_{\Omega_{2\alpha_0}} \int\limits_{B(0,2\alpha)} \left(2\lambda\Big|\int\limits_{B(0,\epsilon)} \nabla u(y-tx-z)\varphi_\epsilon(z)\, dz\Big|\right)^{p(x)} dx\, dy\, dt \\ &\leq C_2 \int_0^1 \int\limits_{B(0,2\alpha)} \int\limits_{B(0,\epsilon)} \|\lambda C_1 \nabla u\|_{ps}^{ps}\, dz\, dx\, dt \leq C_3 \|\lambda C_1\nabla u\|_{ps}^{ps} \left(\frac{\delta_N}{N}\right)^2 (2\alpha)^N,\end{aligned}$$

for some positive constants C_1, C_2 and C_3, where δ_N denotes the measure of the unit sphere in $\mathbb{R}^N$ and p_s is defined by

$$p_s = \begin{cases} p^- & \text{if } |\lambda K_1 \nabla u(\cdot)| \le 1, \\ p^+ & \text{if } |\lambda K_1 \nabla u(\cdot)| > 1. \end{cases}$$

To justify $(*)$ we recall that in $\Omega_{2\alpha_0}$ differentiation and mollification commute for $\delta < \alpha_0$. Therefore, $\lim_{\alpha\to 0} I_\alpha = 0$. So, we conclude immediately that $\int_{\Omega_{2\alpha_0}} |\phi_\alpha(y)|\, dy \to 0$ as $\alpha \to 0$. Consequently, for almost every $x_0 \in \Omega$, we get $\phi_{\alpha_n}(x_0) \to 0$ as $n \to \infty$, for a subsequence $(\alpha_n)_n$ with $\alpha_n \to 0$ as $n \to \infty$. ■

4. Functional dependence on x and ∇u. Let Ω be a bounded domain of $\mathbb{R}^N$ and $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ a Carathéodory function satisfying

$$|f(x,\xi)| \le \ell(x) k(|\xi|) \tag{4.1}$$

for some nondecreasing function $k : \mathbb{R} \to \mathbb{R}$ and some $\ell(\cdot) \in L^1(\Omega)$. Now, we consider the continuous functional $J : W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega f(x, \nabla u(x))\, dx. \tag{4.2}$$

Definition 4.1. For each real μ, we define $\mathcal{L}_\mu$ as the level set of J, i.e., $\mathcal{L}_\mu = \{u \in W^{1,p(\cdot)}(\Omega) : J(u) = \mu\}$, and $\overline{\mathcal{L}}_\mu^\omega$ for the closure of $\mathcal{L}_\mu$ in $W^{1,p(\cdot)}(\Omega)$.

Definition 4.2. A functional $J : W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ is called *weakly lower semicontinuous* at a level set $\mathcal{L}_\mu$ if $J(u) \le \mu$ for all $u \in \overline{\mathcal{L}}_\mu^\omega$.

Theorem 4.3. *Let $J : W^{1,p(\cdot)}(\Omega) \to \mathbb{R}$ be a continuous functional defined as in* (4.2) *with the Carathéodory function f satisfying* (4.1). *If J is weakly lower semicontinuous at a non-void level set $\mathcal{L}_\mu$ and if μ is not an extreme value of J, then $f(x,\xi)$ is convex in ξ for almost all $x \in \Omega$.*

Proof. Assume that the real μ is not an extreme value of J. We shall show that

$$f(x, \lambda\xi + (1-\lambda)\xi^*) \le \lambda f(x,\xi) + (1-\lambda) f(x,\xi^*)$$

for all $\lambda \in [0,1]$, all $\xi, \xi^* \in \mathbb{R}^N$ and almost every $x \in \Omega$. We can assume that $\mu = 0$ and that in $W^{1,p(\cdot)}(\Omega)$ there are two functions $\widehat{a}_1$ and $\widehat{a}_2$ such that, $J(\widehat{a}_1) < -\delta_0$ and $J(\widehat{a}_2) > \delta_0$, for some $\delta_0 > 0$. Let x_0 be a Lebesgue point of $f(x,\xi)$ for all $\xi \in \mathbb{Q}^N$. We can assume that $x_0 = 0$. Using the continuity of the functional J and Theorem 3.1, there is a ball $B(0,R_0) \subset \Omega$ and there are functions $\overline{u}$, $\overline{u}_1$ and $\overline{u}_2$ (see [L]) such that

$$\nabla\overline{u} = \nabla\overline{u}_1 = \nabla\overline{u}_2 \equiv 0 \quad \text{on } B(0,R_0), \tag{4.3}$$

$$J(\overline{u}_1) < \frac{7}{8}\delta_0, \quad J(\overline{u}_2) > \frac{7}{8}\delta_0 \quad \text{and} \quad |J(\overline{u})| < \frac{1}{8}\delta_0. \tag{4.4}$$

Furthermore, for each function $\overline{a}$ satisfying $|J(\overline{a})| < \frac{7}{8}\delta_0$, there is a number $t_i \in [0,1]$ with $i = i(\overline{a}) \in \{1,2\}$ such that the function $\overline{c} = \overline{a} + t_i(\overline{u}_i - \overline{a})$ lies in the level set $\mathcal{L}_0$, i.e., $J(\overline{c}) = \mu = 0$. We fix $\lambda \in [0,1] \cap \mathbb{Q}$ and ξ, $\xi^* \in \mathbb{Q}^N$. As in [L], we recall that

$$\begin{aligned} g_n(x) = g_\lambda(nx_1) &\to \lambda && \text{in } L^\infty(\Omega) \text{ weak-star,} \\ 1 - g_n(x) &\to 1 - \lambda && \text{in } L^\infty(\Omega) \text{ weak-star,} \end{aligned}$$

where

$$g_\lambda(x) = \begin{cases} 1 & \text{if } 0 < x < \lambda, \\ 0 & \text{if } \lambda < x < 1. \end{cases}$$

We also define the sequence of functions $\widehat{\omega}_n(x) = \xi^* \cdot x + \int_0^{(\xi-\xi^*)\cdot x} g_\lambda(nt)\,dt$. This sequence has the properties

$$\begin{aligned} &\nabla\widehat{\omega}_n(x) = \xi^* + (\xi - \xi^*) g_\lambda(n(\xi - \xi^*)\cdot x), \\ &\widehat{\omega}_n \to \widehat{\omega}_0 \quad \text{in } W^{1,p(\cdot)}(\Omega). \end{aligned} \tag{4.5}$$

Indeed,

$$\begin{aligned} \rho_{p(\cdot)}(\omega_n - \widehat{\omega}_0) &= \int_\Omega (\omega_n(x) - \widehat{\omega}_0(x))^{p(x)}\,dx \\ &\le \int_\Omega \Big(\int_0^{(\xi-\xi^*)\cdot x} g_\lambda(nt)\,dt - \lambda(\xi - \xi^*)\cdot x \Big)^{p_s} dx, \end{aligned}$$

where

$$p_s = \begin{cases} p^+ & \text{if } |\int_0^{(\xi-\xi^*)\cdot x} g_\lambda(nt)\,dt - \lambda(\xi-\xi^*)\cdot x| > 1, \\ p^- & \text{if } |\int_0^{(\xi-\xi^*)\cdot x} g_\lambda(nt)\,dt - \lambda(\xi-\xi^*)\cdot x| \le 1. \end{cases}$$

Therefore,

$$\lim_{n\to\infty} \rho_{p(\cdot)}(\widehat{\omega}_n - \widehat{\omega}_0) \le \lim_{n\to\infty} \int_\Omega \Big(\int_0^{(\xi-\xi^*)\cdot x} g_\lambda(nt)\,dt - \lambda(\xi - \xi^*)\cdot x \Big)^{p_s} dx \le 0.$$

In the same manner and by using the dominated convergence theorem, we obtain $\lim \rho_{p(\cdot)}(\nabla\widehat{\omega}_n - \nabla\widehat{\omega}_0) = 0$. Finally, we conclude that $\widehat{\omega}_n \to \widehat{\omega}_0$ in $W^{1,p(\cdot)}(\Omega)$.

Now, let $\psi : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$-function with support in the interval $]-1,1[$ and $\psi(t) = 1$ for $|t| < 1/2$. Defining $\widetilde{\omega}_R(x) = \psi(|x|/R)\widehat{\omega}_0(x)$ for $R > 0$, we have

$$\nabla\widetilde{\omega}_R(x) = \psi'\Big(\frac{|x|}{R}\Big)\frac{1}{R}\operatorname{sign}(x)\widehat{\omega}_0(x) + \psi\Big(\frac{|x|}{R}\Big)\nabla\widehat{\omega}_0(x).$$

Moreover (see [L, Proposition 3.1]),

$$|\nabla\widetilde{\omega}_R(x)| \le C \quad \text{in } \Omega, \tag{4.6}$$

$$\int_{B(0,R)} f(x, \nabla\widetilde{\omega}_R(x))\,dx \to 0 \quad \text{as } R \to 0. \tag{4.7}$$

Note that (4.1) is used to prove (4.7). Next, we consider the sequence $\widehat{\omega}_n(\cdot)$ in a ball $B(0,r)$, say. We shall show that it is possible to alter each element of this sequence in such a manner that it coincides with the limit $\widehat{\omega}_0(\cdot)$ at the boundary. The proposition below is a generalization of Proposition 3.2 in [L] to the case of Sobolev spaces with variable exponent. ■

PROPOSITION 4.4. *There is a sequence $(a_n)_n$ in $W^{1,p(\cdot)}(\Omega)$ satisfying;*

(a) $a_n(x) = \widehat{\omega}_0(x) = (\lambda\xi + (1-\lambda)\xi^*)\cdot x$ *in* $\partial B(0,r)$,
(b) $a_n - \widehat{\omega}_n \to 0$ *in* $W^{1,p(\cdot)}(\Omega)$ *as* $n \to 0$,
(c) $a_n \to \widehat{\omega}_0$ *in* $W^{1,p(\cdot)}(\Omega)$,
(d) $\|\nabla a_n\|_\infty + \|\nabla \widehat{\omega}_n\|_\infty \le C$,
(e) $|\int_{B(0,r)} f(x,\nabla\widehat{\omega}_n(x))\,dx - \int_{B(0,r)} f(x,\nabla a_n(x))\,dx| \to 0$ *as* $n\to\infty$,
(f) $\int_{B(0,r)} f(x,\nabla a_n(x))\,dx \to 0$ *uniformly with respect to* n.

Proof. Let ϖ_ϵ be a $\mathcal{C}^\infty$-function with support in $[-1,1]$ such that $\varpi_\epsilon(t) = 1$ for all $|t| < 1-\epsilon$ and $|\varpi'_\epsilon(t)| < 2/\epsilon$ for all t. Define $\omega_\epsilon(x) = \varpi_\epsilon(|x|/r)$ and $a_{n,\epsilon}(x) = \widehat{c}_0(x)+\omega_\epsilon(x)(\widehat{c}_n(x)-\widehat{c}_0(x))$. We have the following inequalities (see [L, proof of Proposition 3.2]):

$$|\nabla(\widehat{c}_n(x)-\widehat{c}_0(x))|(1-\omega_\epsilon(x)) \le C_1 r(|\xi^*|+|\xi|)(1-\omega_\epsilon(x)), \tag{4.8}$$

$$|(\widehat{c}_n(x)-\widehat{c}_0(x))|\,|\nabla\omega_\epsilon(x)| \le \mathcal{O}(n^{-1})\frac{1}{\epsilon}\chi_{\mathrm{supp}(\nabla\omega_\epsilon)}, \tag{4.9}$$

with C_1 is a positive constant. We have

$$\begin{aligned}
&\rho_{p(\cdot)}(|a_{n,\epsilon}-\widehat{c}_n|) + \rho_{p(\cdot)}(|\nabla(a_{n,\epsilon}-\widehat{c}_n)|) \\
&= \int\limits_{\Omega\setminus\overline{B}(0,r)} |a_{n,\epsilon}(x)-\widehat{c}_n(x)|^{p(x)}\,dx + \int\limits_{\overline{B}(0,r)} |a_{n,\epsilon}(x)-\widehat{c}_n(x)|^{p(x)}\,dx \\
&\quad + \int\limits_{\Omega\setminus\overline{B}(0,r)} |\nabla(a_{n,\epsilon}(x)-\widehat{c}_n(x))|^{p(x)}\,dx + \int\limits_{\overline{B}(0,r)} |\nabla(a_{n,\epsilon}(x)-\widehat{c}_n(x))|^{p(x)}\,dx \\
&= \int\limits_{\Omega\setminus\overline{B}(0,r)} |\widehat{c}_n(x)-\widehat{c}_0(x)|^{p(x)}\,dx + \int\limits_{\overline{B}(0,r)} |(1-\omega_\epsilon(x))(\widehat{c}_n(x)-\widehat{c}_0(x))|^{p(x)}\,dx \\
&\quad + \int\limits_{\Omega\setminus\overline{B}(0,r)} |\nabla(\widehat{c}_n(x)-\widehat{c}_0(x))|^{p(x)}\,dx \\
&\quad + \int\limits_{\overline{B}(0,r)} |\nabla((1-\omega_\epsilon(x))(\widehat{c}_n(x)-\widehat{c}_0(x)))|^{p(x)}\,dx.
\end{aligned}$$

Since $1-\omega_\epsilon(x)\to 0$ a.e. $x\in\overline{B}(0,r)$, and $\widehat{c}_n \to \widehat{c}_0$ in $W^{1,p(\cdot)}(\Omega)$, one has

$$\int_{\Omega\setminus\overline{B}(0,r)} |\widehat{c}_n(x) - \widehat{c}_0(x)|^{p(x)}\,dx + \int_{\Omega\setminus\overline{B}(0,r)} |\nabla(\widehat{c}_n(x) - \widehat{c}_0(x))|^{p(x)}\,dx \to 0 \quad \text{as } n \to \infty.$$

So, we conclude that

$$\rho_{p(\cdot)}(|a_{n,\epsilon} - \widehat{c}_n|) + \rho_{p(\cdot)}(|\nabla(a_{n,\epsilon} - \widehat{c}_n)|) \leq \mathcal{O}(\epsilon) + C_2 \int_{B(0,r)} (|\nabla(\widehat{c}_n - \widehat{c}_0)(1-\omega_\epsilon)|)^{p(\cdot)}\,dx \tag{4.10}$$

for some positive constant C_2. Now, by the definition of ω_ϵ, we obtain

$$\omega_\epsilon(x) = \begin{cases} 0 & \text{in } \Omega \setminus \overline{B}(0,r), \\ 1 & \text{in } B(0,(1-\epsilon)r), \\ \varpi_\epsilon(|x|/r) & \text{in } \overline{B}(0,r) \setminus B(0,(1-\epsilon)r), \end{cases}$$

which implies that

$$a_{n,\epsilon} - \widehat{c}_n(x) = \begin{cases} \widehat{c}_0(x) - \widehat{c}_n(x) & \text{in } \Omega \setminus \overline{B}(0,r), \\ 0 & \text{in } B(0,(1-\epsilon)r), \\ (1-\varpi_\epsilon(|x|/r))(\widehat{c}_0(x) - \widehat{c}_n(x)) & \text{in } \overline{B}(0,r) \setminus B(0,(1-\epsilon)r) \end{cases}$$

and

$$\nabla(a_{n,\epsilon}(x) - \widehat{c}_n(x)) = \begin{cases} \nabla(\widehat{c}_0(x) - \widehat{c}_n(x)) & \text{in } \Omega \setminus \overline{B}(0,r), \\ 0 & \text{in } B(0,(1-\epsilon)r), \\ \nabla(\varpi_\epsilon(|x|/r))(\widehat{c}_0(x) - \widehat{c}_n(x)) & \\ \quad + (1-\varpi_\epsilon(|x|/r))\nabla(\widehat{c}_0(x) - \widehat{c}_n(x)) & \text{in } \overline{B}(0,r) \setminus B(0,(1-\epsilon)r). \end{cases}$$

Hence, for n large enough, we have the estimate

$$\begin{aligned} &\rho_{p(\cdot)}(|a_{n,\epsilon} - \widehat{c}_n|) + \rho_{p(\cdot)}(|\nabla(a_{n,\epsilon} - \widehat{c}_n)|) \\ &\quad\leq \mathcal{O}(\epsilon) + C_2 \int_{B(0,r)\setminus B(0,(1-\epsilon)r)} \big(|\nabla((\widehat{c}_n(x) - \widehat{c}_0(x))(1-\omega_\epsilon(x)))|\big)^{p(x)}\,dx \\ &\quad\leq \mathcal{O}(\epsilon) + C_2(\mathcal{O}(C_3 n^{-1})/\epsilon)^{p^-} |\overline{B}(0,r) \setminus B(0,(1-\epsilon)r)|, \end{aligned}$$

with C_3 a positive constant. Selecting ϵ_n such that $\mathcal{O}(n^{-1})/\epsilon_n = 1$, this implies that $\mathcal{O}(\epsilon_n) = \mathcal{O}(n^{-1})$ and $\epsilon_n \to 0$ as $n \to \infty$. Then we conclude

$$\begin{aligned} &\rho_{p(\cdot)}(|a_{n,\epsilon} - \widehat{c}_n(x)|) + \rho_{p(\cdot)}(|\nabla(a_{n,\epsilon} - \widehat{c}_n(x))|) \\ &\qquad\leq \mathcal{O}(n^{-1}) + C_2(C_3\mathcal{O}(n^{-1})/\epsilon)^{p^-} |\overline{B}(0,r) \setminus B(0,(1-\epsilon)r)|, \end{aligned}$$

which converges to 0 as $n \to \infty$. We define $a_n = a_{n,\epsilon}$, and we get $a_{n,\epsilon} - \widehat{c}_n \to 0$ in $W^{1,p(\cdot)}(\Omega)$ as $n \to \infty$, which gives (b), and consequently, we get

$a_n - \widehat{c}_0 \to 0$ in $W^{1,p(\cdot)}(\Omega)$ as $n \to \infty$. And by the definition of a_n we get easily the properties (a), (d) and (f). Now, we are able to complete the proof of Theorem 4.1. For $R \le R_0$ and $r = R/2$, we define the sequence

$$\widehat{u}_n(x) = \begin{cases} \overline{u}(x) & \text{for } x \in \Omega \setminus B(0,R), \\ \overline{u}(x) + c_R(x) & \text{for } x \in B(0,R) \setminus B(0,r), \\ \overline{u}(x) + a_n(x) & \text{for } x \in B(0,r), \end{cases}$$

which converges in $W^{1,p(\cdot)}(\Omega)$ to

$$u_0(x) = \begin{cases} \overline{u}(x) & \text{for } x \in \Omega \setminus B(0,R), \\ \overline{u}(x) + c_R(x) & \text{for } x \in B(0,R). \end{cases}$$

Indeed, for $x \in B(0,r)$ we have

$$\rho_{p(\cdot)}(\widehat{u}_n - (\overline{u} + c_R)) = \int_{B(0,r)} (a_n(x) - \overline{c}_R(x))^{p(x)}\, dx = \int_{B(0,r)} (a_n(x) - \overline{c}_0(x))^{p(x)}\, dx.$$

Since $a_n \to \widehat{c}_0$ in $W^{1,p(\cdot)}(\Omega)$, we conclude that $\lim_{n\to\infty} \rho_{p(\cdot)}(\widehat{u}_n - \overline{c}_R) = 0$. Therefore, $\widehat{u}_n \to u_0$ in $W^{1,p(\cdot)}(\Omega)$. Due to (4.6), (4.7) and Proposition 4.4 (see also [L]), we have for $R > 0$ small enough $|J(\overline{u}_n)| < \frac{7}{8}\varepsilon_0$ for all $n \in \mathbb{N}$. Hence, for any n, we find $t_n \in [0,1]$ and $i_n \in \{1,2\}$ such that for $u_n := \widehat{u}_n + t_n(\widehat{u}_{i_n} - \widehat{u}_n)$, we have $J(u_n) = 0$. Choosing a subsequence such that $t_n \to t_0$ and $i_n = i \in \{1,2\}$, we have $u_n \to u_0$ in $W^{1,p(\cdot)}(\Omega)$. The lower semicontinuity of J at the level set $\mathcal{L}_a$ gives $J(u_0) \le 0$. By using the continuity of J with respect to the strong topology of $W^{1,p(\cdot)}(\Omega)$, we get

$$\lim_{n\to\infty} J(\overline{u} + t_n(\overline{u}_{i_n} - \overline{u})) = J(\overline{u} + t_0(\overline{u}_i - \overline{u})).$$

And by construction, one has $f(x, \nabla(\overline{u} + t_n(\overline{u}_i - \overline{u}))) = f(x,0)$ in $B(0,R)$, yielding

$$\lim_{n\to\infty} \int_{B(0,R)} f(x, \nabla u_n(x))\, dx \ge \int_{B(0,R)} f(x, \nabla u_0(x))\, dx.$$

Since $u_n = u_0$ in $B(0,R) \setminus B(0,r)$ and $r = R/2$, we finally get

$$\begin{aligned} \int_{B(0,r)} f(x, \lambda\xi + (1-\lambda)\xi^*)\, dx &= \int_{B(0,r)} f(x, \nabla u_0(x))\, dx \\ &\le \lim_{n\to\infty} \int_{B(0,r)} f(x, \nabla u_n(x))\, dx \\ &= \lim_{n\to\infty} \int_{B(0,R)} f(x, \nabla a_n(x))\, dx \\ &= \lambda \int_{B(0,r)} f(x,\xi)\, dx + (1-\lambda) \int_{B(0,r)} f(x,\xi^*)\, dx. \end{aligned}$$

Since the above inequality can be obtained for all balls $B(0, r)$ with radius $r < R/2$, we conclude that $f(x_0, \lambda\xi + (1-\lambda)\xi^*) \leq \lambda f(x_0, \xi) + (1-\lambda) f(x_0, \xi^*)$ for all $\lambda \in [0,1] \cap \mathbb{Q}$ and all $\xi, \xi^* \in \mathbb{Q}^N$. It then follows by the continuity of $f(x, \xi)$ with respect to ξ that the above inequality holds for all $\lambda \in [0,1]$ and all $\xi, \xi^* \in \mathbb{R}^N$. ∎

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