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Abdelmoujib Benkirane (Fez)
Fatimazahra Blali (Fez)
Mohamed Sidi El Vally (Abha)

AN EXISTENCE THEOREM FOR A STRONGLY NONLINEAR ELLIPTIC PROBLEM IN MUSIELAK–ORLICZ SPACES

Abstract. We prove an existence result for some class of strongly nonlinear elliptic problems in the Musielak–Orlicz spaces $W^1L_\varphi(\Omega)$, under the assumption that the conjugate function of φ satisfies the Δ_2-condition.

1. Introduction. Let Ω be an open subset of $\mathbb{R}^n$. This paper is concerned with the existence of solutions for strongly nonlinear elliptic problems of the form

$$A(u) + g(x,u,\nabla u) = f \quad \text{in } \Omega, \tag{1.1}$$

where A is a Leray–Lions operator: $A(u) = -\operatorname{div} a(x,u,\nabla u)$.

A. Benkirane and A. Elmahi [BE1] have proved the existence of a solution for problem (1.1) in the Orlicz–Sobolev space $W^1L_M(\Omega)$, assuming a sign condition and a natural growth condition on g.

A. Elmahi and D. Meskine [EM] have proved an existence theorem for problem (1.1) without assuming the Δ_2-condition on M and its conjugate function.

In the main result of [BE1], M is supposed to satisfy the Δ_2-condition and the domain Ω of $\mathbb{R}^n$ is supposed to have the segment property in order to construct a complementary system $(W_0^1L_M(\Omega), W_0^1E_M(\Omega), W^{-1}L_{\overline{M}}(\Omega), W^{-1}E_{\overline{M}}(\Omega))$. It is our purpose in this paper to prove an existence result for the strongly nonlinear elliptic problem (1.1) in the setting of Musielak–Orlicz spaces $W^1L_\varphi(\Omega)$, under the assumption that the conjugate function of φ satisfies the Δ_2-condition.

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For some other existence results for strongly nonlinear elliptic problems see [ABT, AHT].

2. Preliminaries. In this section we briefly list some definitions and facts about Musielak–Orlicz–Sobolev spaces [M].

Let Ω be an open subset of $\mathbb{R}^n$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying the following conditions:

(a) $\varphi(x,\cdot)$ is an N-function, i.e. convex, nondecreasing, continuous, $\varphi(x,0)=0$, $\varphi(x,t)>0$ for all $t>0$, and

$$\lim_{t\to 0}\sup_{x\in\Omega}\frac{\varphi(x,t)}{t}=0, \qquad \lim_{t\to\infty}\inf_{x\in\Omega}\frac{\varphi(x,t)}{t}=\infty,$$

(b) $\varphi(\cdot,t)$ is a measurable function.

Then φ is called a *Musielak–Orlicz function* and we put $\varphi_x(t)=\varphi(x,t)$.

Let $\psi(x,s)=\sup_{t\ge 0}\{st-\varphi(x,t)\}$ be the Musielak–Orlicz function complementary to φ in the sense of Young with respect to the variable s.

The Musielak–Orlicz function φ is said to satisfy the Δ_2-*condition* if there exists $k>0$ independent of $x\in\Omega$ and a nonnegative function h integrable in Ω such that $\varphi(x,2t)\le k\varphi(x,t)+h(x)$ for large values of t.

We define the functional $\varrho_{\varphi,\Omega}(u)=\int_\Omega \varphi(x,|u(x)|)\,dx$ and the Musielak–Orlicz space $L_\varphi(\Omega)=\{u:\Omega\to\mathbb{R} \text{ measurable} : \varrho_{\varphi,\Omega}(|u(x)|/\lambda)<\infty, \lambda>0\}$.

The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. The space $E_\varphi(\Omega)$ is separable and $E_\psi(\Omega)^*=L_\varphi(\Omega)$ (see [M]).

$W^1L_\varphi(\Omega)$ (resp. $W^1E_\varphi(\Omega)$) is the space of all functions u such that u and its distributional derivatives of order 1 lie in $L_\varphi(\Omega)$ (resp. $E_\varphi(\Omega)$). Let $\alpha=(\alpha_1,\dots,\alpha_n)$ with nonnegative integers α_i, $|\alpha|=\alpha_1+\dots+\alpha_n$, and let $D^\alpha u$ denote the distributional derivatives. We set

$$\overline{\varrho}_{\varphi,\Omega}(u)=\sum_{|\alpha|\le 1}\varrho_{\varphi,\Omega}(D^\alpha u), \qquad \|u\|_{1,\varphi,\Omega}=\inf\{\lambda>0:\overline{\varrho}_{\varphi,\Omega}(u/\lambda)\le 1\}.$$

The spaces $W^1L_\varphi(\Omega)$ and $W^1E_\varphi(\Omega)$ can be identified with subspaces of the product of $n+1$ copies of $L_\varphi(\Omega)$. Denoting this product by ΠL_φ, we will use the weak topologies $\sigma(\Pi L_\varphi,\Pi E_\psi)$ and $\sigma(\Pi L_\varphi,\Pi L_\psi)$.

Let $W^{-1}L_\psi(\Omega)$ (resp. $W^{-1}E_\psi(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_\psi(\Omega)$ (resp. $E_\psi(\Omega)$).

If ψ satisfies the Δ_2-condition, then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ for the topology $\sigma(\Pi L_\varphi,\Pi L_\psi)$ (see [BS, Corollary 1]).

LEMMA 2.1. *Let Ω be an open subset of $\mathbb{R}^N$ of finite measure. Let φ, ψ and ϕ be Musielak functions such that $\phi\ll\psi$, and let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a*

Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$,

$$|f(x,s)| \le c(x) + k_1 \psi_x^{-1} \varphi(x, k_2|s|), \tag{2.1}$$

where k_1, k_2 are positive real constants and $c \in E_\phi(\Omega)$. Then the Nemytskiĭ operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is strongly continuous from

$$P(E_\varphi(\Omega), 1/k_2) = \{u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < 1/k_2\}$$

into $E_\phi(\Omega)$.

3. Main results. Let Ω be a bounded open subset of $\mathbb{R}^n$. Let φ be a Musielak–Orlicz function, and ψ the Musielak–Orlicz function complementary (or conjugate) to φ. We assume here that ψ satisfies the Δ_2-condition near infinity, and let γ be a Musielak–Orlicz function such that $\gamma \ll \varphi$.

Let $A : D(A) \subset W_0^1 L_\varphi(\Omega) \to W^{-1} L_\psi(\Omega)$ be a mapping (not defined everywhere) given by $A(u) = -\operatorname{div} a(x, u, \nabla u)$ where:

(A_1) $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function,

(A_2) for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$

$$|a(x,s,\xi)| \le c(x) + k_1 \psi_x^{-1}(\gamma(x, k_2|s|)) + k_3 \psi_x^{-1}(\varphi(x, k_4|\xi|)),$$

for some $c \in E_\psi(\Omega)$, and $k_1, k_2, k_3, k_4 \ge 0$,

(A_3) for each $x \in \Omega$, and all $s \in \mathbb{R}$, $\xi, \xi^* \in \mathbb{R}^n$ with $\xi \ne \xi^*$,

$$[a(x,s,\xi) - a(x,s,\xi^*)][\xi - \xi^*] > 0,$$

(A_4) $a(x,s,\xi) \cdot \xi \ge \alpha \cdot \varphi(x, |\xi|/\lambda)$ for some $\alpha, \lambda > 0$.

Furthermore, let $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$,

(G_1) $g(x,s,\xi) \cdot s \ge 0$,

(G_2) $|g(x,s,\xi)| \le b(|s|)(c'(x) + \varphi(x, |\xi|/\lambda'))$,

where $b : \mathbb{R} \to \mathbb{R}$ is a continuous and non-decreasing function and $c'(x)$ is a given non-negative function in $L^1(\Omega)$ and $\lambda' > 0$. Finally, we assume that

$$f \in WE_\psi^{-1}(\Omega). \tag{3.1}$$

Consider the following elliptic problem with Dirichlet boundary condition:

$$\begin{cases} u \in W_0^1 L_\varphi(\Omega),\ g(x,u,\nabla u) \in L^1(\Omega),\ g(x,u,\nabla u)u \in L^1(\Omega), \\ \int_\Omega a(x,u,\nabla u)\nabla v\,dx + \int_\Omega g(x,u,\nabla u)v\,dx = \langle f, v\rangle, \\ \text{for all } v \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega) \text{ and for } v = u. \end{cases} \tag{3.2}$$

We shall prove the following existence theorem:

MAIN THEOREM 3.1. *Assume that conditions* (A_1)–(A_4), (G_1), (G_2) *and* (3.1) *hold true. Then there exists a solution u of problem* (3.2).

Proof. Step 1. Consider the sequence of approximate equations

$$u_n \in W_0^1 L_\varphi(\Omega), \quad A(u_n) + g_n(x, u_n, \nabla u_n) = f \qquad \text{in } \mathcal{D}'(\Omega), \tag{3.3}$$

where $n \in \mathbb{N}^*$ and

$$g_n(x, s, \xi) = \frac{g(x, s, \xi)}{1 + (1/n)|g(x, s, \xi)|}.$$

Note that $g_n(x, s, \xi) \cdot s \geq 0$, $|g_n(x, s, \xi)| \leq |g(x, s, \xi)|$ and $|g_n(x, s, \xi)| \leq n$.

Since $g_n(x, s, \xi)$ is bounded for any fixed $n > 0$, there exists a solution u_n of (3.3) (see [BS, Theorem 1, Theorem 5 and Remark 1]).

Using in (3.3) the test function u_n we get

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla u_n \, dx \leq \langle f, u_n \rangle. \tag{3.4}$$

By Theorems 1 and 5 of [BS],

$$(u_n) \text{ is bounded in } W_0^1 L_\varphi(\Omega) \quad \text{and} \quad \int_\Omega a(x, u_n, \nabla u_n)\, dx \leq C_1, \tag{3.5}$$

$$a(x, u_n, \nabla u_n) \text{ is bounded in } (L_\psi(\Omega))^n, \tag{3.6}$$

$$\int_\Omega g_n(x, u_n, \nabla u_n) \cdot u_n \, dx \leq C_2. \tag{3.7}$$

Passing to a subsequence if necessary, we can assume that

$$u_n \rightharpoonup u \qquad \text{weakly in } W_0^1 L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) = \sigma(\Pi L_\varphi, \Pi L_\psi).$$

Then

$$u_n \to u \quad \text{strongly in } E_\varphi \quad \text{and} \quad u_n \to u \quad \text{a.e. in } \Omega. \tag{3.8}$$

Step 2. Let $\phi(t) = t \exp(\gamma t^2)$, $\gamma > 0$. It is easy to see that when $\gamma \geq (b(k)K/2\alpha)^2$ one has

$$\phi'(t) - (b(k)K/\alpha)|\phi(t)| \geq 1/2, \qquad \forall t \in \mathbb{R},$$

where $K > 0$ is a constant which will be specified later.

Take $z_n = T_k(u_n) - T_k(u)$ and use $v_n = \phi(z_n) \in W_0^1 L_\varphi(\Omega)$ as a test function in (3.3) to get

$$\langle A(u_n), v_n \rangle + \int_\Omega g_n(x, u_n, \nabla u_n) v_n \, dx \to 0 \qquad \text{as } n \to \infty$$

since $v_n \rightharpoonup 0$ weakly in $W_0^1 L_\varphi(\Omega)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi) = \sigma(\Pi L_\varphi, \Pi L_\psi)$, as is easily seen.

Below we denote by $\varepsilon_i(n)$ $(i = 1, 2, \ldots)$ various sequences of real numbers which tend to 0 as $n \to \infty$.

Since $g_n(x,u_n(x),\nabla u_n(x))v_n(x)\ge 0$ on the subset $\{x\in\Omega:|u_n(x)|>k\}$, we have

$$\langle A(u_n),v_n\rangle+\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)v_n\,dx\le\varepsilon_1(n). \tag{3.9}$$

Fix a real number $r>0$, define $\Omega_r=\{x\in\Omega:|\nabla T_k(u(x))|\le r\}$ and denote by χ_r the characteristic function of Ω_r.

Taking $s\ge r$ we have

(3.10)
$$\begin{aligned}0&\le\int_{\Omega_r}[a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]\,dx\\&\le\int_{\Omega_s}[a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]\,dx\\&\le\int_{\Omega}[a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle A(u_n),v_n\rangle&=\int_\Omega a(x,u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(u)]\phi'(z_n)\,dx\\&=\int_\Omega a(x,u_n,\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\phi'(z_n)\,dx\\&\quad-\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u)\phi'(z_n)\,dx\\&\quad+\int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u)\chi_s\phi'(z_n)\,dx.\end{aligned}$$

Then

$$\begin{aligned}\langle A(u_n),v_n\rangle&=\int_\Omega[a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u)\chi_s)]\\&\qquad\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\phi'(z_n)\,dx\\&\quad-\int_\Omega a(x,u_n,\nabla T_k(u_n))\,\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\,\phi'(z_n)\,dx\\&\quad+\int_\Omega a(x,u_n,\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,\phi'(z_n)\,dx.\end{aligned}\tag{3.11}$$

Denoting by χ_{G_n} the characteristic function of $G_n=\{|u_n(x)|>k\}$, the second term on the right-hand side of (3.11) reads

$$-\int_\Omega[a(x,u_n,\nabla u_n)-a(x,u_n,0)]\chi_{G_n}\nabla T_k(u)\phi'(z_n)\,dx;$$

this tends to 0 since $\chi_{G_n}\nabla T_k(u)\phi'(z_n)\to 0$ strongly in $(E_\varphi(\Omega))^n$ by Lebesgue's theorem while $a(x,u_n,\nabla u_n)-a(x,u_n,0)$ is bounded in $(L_\psi(\Omega))^n$ by (3.6) and (A_1).

Since $|a(x,u_n,\nabla T_k(u_n))|\le |a(x,u_n,\nabla u_n)|+|a(x,u_n,0)|$ it follows that $a(x,u_n,\nabla T_k(u_n))$ is bounded in $(L_\psi(\Omega))^n$ for $\sigma(\Pi L_\psi,\Pi E_\varphi)$, for some $h\in(L_\psi(\Omega))^n$.

We deduce that the third term on the right-hand side of (3.11) tends to

$$-\int_{\Omega\setminus\Omega_s} a(x,u,0)\nabla T_k(u)\,dx$$

since $a(x,u_n,\nabla T_k(u)\chi_s)$ tends strongly to $a(x,u,\nabla T_k(u)\chi_s)$ in $(E_\psi(\Omega))^n$ by Lemma 2.1 while $\nabla T_k(u_n)$ tends weakly to $\nabla T_k(u)$ by (3.8).

This implies that

$$\begin{aligned}\langle A(u_n),v_n\rangle = \int_\Omega &[a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u)\chi_s)]\\ &\times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\phi'(z_n)\,dx\\ &+\int_{\Omega\setminus\Omega_s}(a(x,u,0)-h)\nabla T_k(u)\,dx+\varepsilon_2(n).\end{aligned}\tag{3.12}$$

We now turn to the second term of the left-hand side of (3.9):

$$\begin{aligned}\Big|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)v_n\,dx\Big| &\le \int_{\{|u_n|\le k\}} b(k)\bigg(c'(x)+\varphi\Big(x,\frac{|\nabla u_n|}{\lambda'}\Big)\bigg)|v_n|\,dx\\ &\le \varepsilon_3(n)+b(k)\int_\Omega \varphi\Big(x,\frac{|\nabla T_k(u_n)|}{\lambda'}\Big)|v_n|\,dx\end{aligned}$$

since (v_n) is bounded in $L^\infty(\Omega)$ and $v_n\to 0$ a.e in Ω.

Using (A_4) we can write

$$\begin{aligned}\Big|\int_{\{|u_n|\le k\}} &g_n(x,u_n,\nabla u_n)v_n\,dx\Big|\\ &\le \varepsilon_3(n)+\frac{b(k)}{\alpha}\int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u_n)|v_n|\,dx\\ &= \varepsilon_3(n)+\frac{b(k)}{\alpha}\int_\Omega [a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u)\chi_s)]\\ &\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]|v_n|\,dx\\ &\quad+\frac{b(k)}{\alpha}\int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u)\chi_s|v_n|\,dx\\ &\quad+\frac{b(k)}{\alpha}\int_\Omega a(x,u_n,\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]|v_n|\,dx.\end{aligned}\tag{3.13}$$

The second term on the right-hand side of (3.13) tends to 0 since $a(x,u_n,\nabla T_k(u_n))$ is bounded in $(L_\psi(\Omega))^n$ while $\nabla T_k(u)\chi_s|v_n|$ tends strongly to 0 in $(E_{\varphi(\Omega)})^n$ by Lebesgue's theorem.

The third term on the right-hand side of (3.13) tends to 0 since $a(x,u_n,\nabla T_k(u)\chi_s)|v_n|$ tends strongly to 0 in $(E_\psi(\Omega))^n$ by condition (A_2) while $\nabla T_k(u_n)-\nabla T_k(u)\chi_s$ is bounded in $(L_\varphi(\Omega))^n$.

We deduce that

$$\Big|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)v_n\,dx\Big| \le \varepsilon_4(n)+\frac{b(k)}{\alpha}\int_\Omega [a(x,u_n,\nabla u_n)-a(x,u_n,\nabla T_k(u)\chi_s)] \times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]|v_n|\,dx. \tag{3.14}$$

Combining (3.9), (3.12) and (3.14) we obtain

$$\int_\Omega [a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u)\chi_s)] \times[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\Big(\phi'(z_n)-\frac{b(k)}{\alpha}|\phi(z_n)|\Big)\,dx \le \varepsilon_5(n)-\int_{\Omega\setminus\Omega_s}(a(x,u,0)-h)\nabla T_k(u)\,dx,$$

which gives, by using the inequality $\phi'(t)-(b(k)K/\alpha)|\phi(t)|\ge 1/2$,

$$\int_\Omega [a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx \le 2\varepsilon_5(n)-2\int_{\Omega\setminus\Omega_s}(a(x,u,0)-h)\nabla T_k(u)\,dx.$$

Using (3.10) yields

$$\int_{\Omega_r} [a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]\,dx \le 2\varepsilon_5(n)-2\int_{\Omega\setminus\Omega_s}(a(x,u,0)-h)\nabla T_k(u)\,dx.$$

This implies that

$$0\le \limsup_{n\to\infty}\int_{\Omega_r} [a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]\,dx \le 2\int_{\Omega\setminus\Omega_s}(a(x,u,0)-h)\nabla T_k(u)\,dx.$$

Using the fact that $(a(x,u,0)-h)\nabla T_k(u)\in L^1(\Omega)$ and letting $s\to\infty$ we get

$$\int_{\Omega_r} [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\,dx \to 0.$$

Passing to a subsequence if necessary, we can assume that

$$[a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)] \to 0 \quad \text{a.e. in } \Omega_r.$$

As in [BE2], we deduce that there exists a subsequence, still denoted by u_n, such that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega.$$

Step 3. We shall prove that $g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u)$ strongly in $L^1(\Omega)$ by using Vitali's theorem.

To prove that $g_n(x,u_n,\nabla u_n)$ are uniformly equi-integrable in Ω, let $E \subset \Omega$ be a measurable subset of Ω. We have, for any $m > 0$,

$$\begin{aligned}\int_E |g_n(x,u_n,\nabla u_n)|\,dx \le &\int_{E\cap\{|u_n|\le m\}} |g_n(x,u_n,\nabla u_n)|\,dx \\ &+ \int_{E\cap\{|u_n|> m\}} |g_n(x,u_n,\nabla u_n)|\,dx\end{aligned}$$

Moreover,

$$\begin{aligned}\int_{E\cap\{|u_n|\le m\}} |g_n(x,u_n,\nabla u_n)|\,dx &\le \int_{E\cap\{|u_n|\le m\}} |b(m)|\Big[c'(x) + \varphi\Big(x, \frac{|\nabla u_n|}{\lambda'}\Big)\Big]\,dx \\ &\le b(m)\int_E c'(x)\,dx + \frac{b(m)}{\alpha}\int_E a(x,u_n,\nabla T_m(u_n))\nabla T_m(u_n)\,dx \\ &\le b(m)\int_E c'(x)\,dx + \frac{b(m)}{\alpha}\Big[2\varepsilon_5(n) + 2\int_{\Omega\setminus\Omega_s}(a(x,u,0)-h)\nabla T_m(u)\,dx\Big] \\ &\quad + \frac{b(m)}{\alpha}\int_E a(x,u_n,\nabla T_m(u_n))\nabla T_m(u)\chi_s\,dx \\ &\quad + \frac{b(m)}{\alpha}\int_E a(x,u_n,\nabla T_m(u)\chi_s)[\nabla T_m(u_n) - \nabla T_m(u)\chi_s]\,dx.\end{aligned}$$

We claim that $a(x,u_n,\nabla T_m(u_n))\nabla T_m(u)\chi_s \to a(x,u,\nabla T_m(u))\nabla T_m(u)\chi_s$ and $a(x,u_n,\nabla T_m(u)\chi_s)[\nabla T_m(u_n)) - \nabla T_m(u)\chi_s] \to a(x,u,0)\nabla T_m(u)\chi_{\Omega\setminus\Omega_s}$ strongly in $L^1(\Omega)$. To prove this claim we can use Lemma 2.4 of [BE1].

Let $\varepsilon > 0$. We have

$$\int_{E\cap\{|u_n|>m\}} |g_n(x,u_n,\nabla u_n)|\,dx \le \frac{1}{m}\int_\Omega g_n(x,u_n,\nabla u_n)u_n\,dx \le \frac{C_2}{m}.$$

Thus for m sufficiently large, we can write

$$\int_{E\cap\{|u_n|>m\}} |g_n(x,u_n,\nabla u_n)|\,dx \le \frac{\varepsilon}{2}, \qquad \forall n.$$

Furthermore, there exists $n_0>0$ such that $2(b(m)/\alpha)\varepsilon_5(n)\le \varepsilon/10$ for all $n\ge n_0$, and there exists s large such that

$$2\frac{b(m)}{\alpha}\int_{\Omega\setminus\Omega_s} (a(x,u,0)-h)\nabla T_m(u)\,dx \le \frac{\varepsilon}{10}.$$

There exists $\delta_1>0$ such that $|E|<\delta_1$ implies

$$\frac{b(m)}{\alpha}\int_E a(x,u,\nabla T_m(u))\nabla T_m(u)\chi_s\,dx \le \frac{\varepsilon}{10}, \qquad \forall n,$$

$$\frac{b(m)}{\alpha}\int_E a(x,u_n,\nabla T_m(u)\chi_s)[\nabla T_m(u_n)-\nabla T_m(u)\chi_s]\,dx \le \frac{\varepsilon}{10}, \quad \forall n,$$

$$b(m)\int_E c'(x)\,dx \le \frac{\varepsilon}{10}.$$

Thus when $|E|<\delta_1$ one has $\int_{E\cap\{|u_n|\le m\}}|g_n(x,u_n,\nabla u_n)|\,dx\le\varepsilon/2$ for all $n\ge n_0$. Consequently, $|E|<\delta_1$ implies $\int_E|g_n(x,u_n,\nabla u_n)|\,dx\le\varepsilon$ for all $n\ge n_0$. But $\int_E|g_n(x,u_n,\nabla u_n)|\,dx\le n_0|E|$ for all $n<n_0$. Thus $|E|<\delta=\inf(\delta_1,\varepsilon/n_0)$ implies $\int_E|g_n(x,u_n,\nabla u_n)|\,dx\le\varepsilon$ for all n. This shows that $g_n(x,u_n,\nabla u_n)$ are uniformly equi-integrable in Ω. Applying Vitali's theorem yields $g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u)$ strongly in $L^1(\Omega)$.

Going back to the approximate equation (3.3), one has

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla v\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)v\,dx=\langle f,v\rangle \qquad \forall v\in W_0^1L_\varphi(\Omega)\cap L^\infty(\Omega). \tag{3.15}$$

Note that $a(x,u_n,\nabla u_n)\rightharpoonup a(x,u,\nabla u)$ weakly in $(L_\psi(\Omega))^n$ for $\sigma(\Pi L_\psi,\Pi E_\varphi)$ by Lemma 2 of [BS].

Letting $n\to\infty$ in (3.15), we get

$$\int_\Omega a(x,u,\nabla u)\nabla v\,dx+\int_\Omega g_n(x,u,\nabla u)v\,dx=\langle f,v\rangle. \tag{3.16}$$

This equality also holds for $v=u$.

Indeed, taking $v=T_k(u)\in W_0^1L_\varphi(\Omega)\cap L^\infty(\Omega)$ in (3.16), one has

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u)\,dx+\int_\Omega g(x,u,\nabla u)T_k(u)\,dx=\langle f,T_k(u)\rangle.$$

From (3.7) we deduce by Fatou's Lemma that $g(x,u,\nabla u)u\in L^1(\Omega)$.

Observe that $T_k(u)\to u$ in $W_0^1L_\varphi(\Omega)$ for modular convergence and a.e. in Ω when $k\to\infty$.

Note also that $|g(x,u,\nabla u)T_k(u)| \le g(x,u,\nabla u)u \in L^1(\Omega)$.

Hence, by Lebesgue's theorem, letting $k \to \infty$ we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla u\,dx + \int_\Omega g(x,u,\nabla u)u\,dx = \langle f,u\rangle.$$

This completes the proof of Theorem 3.1.

EXAMPLE 3.2. As an application of this result, we can treat the following model problem:

$$\begin{cases} -\Delta_\varphi u + u\varphi(x,|\nabla u|) = f & \text{on } \Omega,\\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

where Δ_φ is the φ-Laplacian operator $\Delta_\varphi u = \operatorname{div}\big(\frac{a(x,|\nabla u|)}{|\nabla u|}\nabla u\big)$ and where a is the derivative of φ with respect to t. The second member f is supposed to lie in the dual space $W^{-1}E_\psi(\Omega)$ where ψ is the Musielak–Orlicz conjugate to φ.

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Abdelmoujib Benkirane, Fatimazahra Blali
Laboratory LAMA Department of Mathematics
Faculty of Sciences Dhar El Mahraz
University Sidi Mohamed Ben Abdellah
B.P. 1796 Atlas
Fez, Morocco
E-mail: abd.benkirane@gmail.com
fatimazahra.blali@gmail.com

Mohamed Sidi El Vally
Department of Mathematics
Faculty of Sciences
King Khalid University
Abha 61413, Kingdom of Saudi Arabia
E-mail: med.medvall@gmail.com

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