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EXISTENCE RESULT FOR A CLASS OF DOUBLY NONLINEAR PARABOLIC SYSTEMS

Abstract. We prove the existence of a renormalized solution to a class of doubly nonlinear parabolic systems.

1. Introduction. We consider the following nonlinear parabolic system:

$$(1.1)\qquad \begin{cases} \dfrac{\partial b_i(x,u_i)}{\partial t} - \operatorname{div}(a(x,t,u_i,\nabla u_i)) + \operatorname{div}(\phi_i(x,t,u_i)) \\ \qquad\qquad = f_i(x,u_1,u_2) - \operatorname{div}(F_i) \quad \text{in } Q_T, \\ u_i(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ b_i(x,u_i(x,0)) = b_i(x,u_{0,i}(x)) \quad \text{in } \Omega, \end{cases}$$

where $i = 1, 2$.

In (1.1), Ω is a bounded domain of $\mathbb{R}^N$ $(N \ge 2)$; T is a positive real number; $Q_T = \Omega \times (0,T)$; $-\operatorname{div}(a(x,t,u_i,\nabla u_i))$ is a Leray–Lions operator defined on $L^p(0,T;W_0^{1,p}(\Omega))$; $\phi_i(x,t,u_i)$ is a Carathéodory function (see assumptions (2.5)–(2.6)); $b_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for every $x \in \Omega$, $b_i(x,\cdot)$ is a strictly increasing C^1-function; $u_{0,i}$ is in $L^1(\Omega)$ with $b_i(\cdot,u_{0,i})$ in $L^1(\Omega)$; $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (see Assumptions H4); and $F_i \in (L^{p'}(Q))^N$.

Under our assumptions, problem (1.1) does not admit, in general, a weak solution since the terms $\phi_i(x,t,u_i)$ and $f_i(x,u_1,u_2)$ may not belong to $(L^1_{\rm loc}(Q))^N$. In order to overcome this difficulty, we work in the framework of renormalized solutions (see Definition 3.1). This notion was introduced by R.-J. DiPerna and P.-L. Lions [7] for the study of the Boltzmann equation. It was adapted to the study of some nonlinear elliptic or parabolic problems in

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fluid mechanics in [5]. In the case where $b(x,u)=u$, the existence of renormalized solutions for (1.1) has been established by R. Di Nardo et al. [6].

In the case where $\phi(x,t,u)=0$ and $f\in L^1(Q_T)$, the existence of renormalized solutions has been established by H. Redwane [12] in the classical Sobolev space; existence results have also been proved in [1], [9] in the case where $f_i(x,u_1,u_2)$ is replaced by $f-\operatorname{div}(g)$ where $f\in L^1(Q)$ and $g\in(L^{p'}(Q))^N$.

It is our purpose in this paper to generalize the result of [6] and prove the existence of a renormalized solution of system (1.1).

The plan of the paper is as follows: In Section 2 we give the basic assumptions. In Section 3 we give the definition of a renormalized solution of (1.1), and we establish (Theorem 3.1) the existence of such a solution.

2. Assumptions on data. Let Ω be a bounded open set in $\mathbb{R}^N$ $(N\geq 2)$, T a positive real number, and $Q_T=\Omega\times(0,T)$.

2.1. Assumptions. Throughout this paper, we assume that the following assumptions hold true:

Assumption (H1). $b_i:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that for every $x\in\Omega$, $b_i(x,\cdot)$ is a strictly increasing $C^1(\mathbb{R})$-function with $b_i(x,0)=0$ for any $k>0$, and there exists a constant $\lambda_i>0$ and functions $A_k^i\in L^\infty(\Omega)$ and $B_k^i\in L^p(\Omega)$ such that for almost every x in Ω,

$$\lambda_i\leq\frac{\partial b_i(x,s)}{\partial s}\leq A_k^i(x),\qquad\left|\nabla_x\left(\frac{\partial b_i(x,s)}{\partial s}\right)\right|\leq B_k^i(x)\qquad\forall|s|\leq k. \tag{2.1}$$

Assumption (H2). $a:Q_T\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function such that, for any $k>0$, there exist ν_k and a function $h_k\in L^{p'}(Q_T)$ with

$$|a(x,t,s,\xi)|\leq\nu_k(h_k(x,t)+|\xi|^{p-1})\qquad\forall|s|\leq k, \tag{2.2}$$

$$a(x,t,s,\xi)\xi\geq\alpha|\xi|^p\qquad\text{with some }\alpha>0, \tag{2.3}$$

$$(a(x,t,s,\xi)-a(x,t,s,\eta))(\xi-\eta)>0\qquad\text{when }\xi\neq\eta. \tag{2.4}$$

Assumption (H3). $\phi_i:Q_T\times\mathbb{R}\to\mathbb{R}^N$ is a Carathéodory function such that for almost every $(x,t)\in Q_T$ and every $s\in\mathbb{R}$,

$$|\phi_i(x,t,s)|\leq c_i(x,t)|s|^\gamma, \tag{2.5}$$

$$c_i\in L^\tau(Q_T)\quad\text{with}\quad\tau=\frac{N+p}{p-1},\qquad\gamma=\frac{N+2}{N+p}(p-1). \tag{2.6}$$

Assumption (H4). For $i=1,2$, $f_i:\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function with $f_1(x,0,s)=f_2(x,s,0)=0$ for a.e. $x\in\Omega$, and all $s\in\mathbb{R}$; and for almost every $x\in\Omega$, and every $s_1,s_2\in\mathbb{R}$,

$$\operatorname{sign}(s_i)f_i(x,s_1,s_2)\geq 0.$$

The growth assumptions on f_i are as follows: for each $k > 0$ there exist $\sigma_k > 0$ and $F_k \in L^1(\Omega)$ such that

$$|f_1(x, s_1, s_2)| \le F_k + \sigma_k |b_2(x, s_2)| \quad \text{a.e. } x \in \Omega,\ \forall |s_1| \le k,\ \forall s_2 \in \mathbb{R}; \tag{2.7}$$

and for each $k > 0$ there exist $\mu_k > 0$ and $G_k \in L^1(\Omega)$ such that

$$|f_2(x, s_1, s_2)| \le G_k(x) + \mu_k |b_1(x, s_1)| \quad \text{a.e. } x \in \Omega,\ \forall |s_2| \le k,\ \forall s_1 \in \mathbb{R}. \tag{2.8}$$

Finally, $u_{0,i}$ is a measurable function such that $b_i(\cdot, u_{0,i}) \in L^1(\Omega)$ for $i = 1, 2$.

3. Main results. In this section, we study the existence of renormalized solutions to systems (1.1).

DEFINITION 3.1. A couple of measurable functions (u_1, u_2) defined on Q_T is called a *renormalized* solution of (1.1) if for $i = 1, 2$ the function u_i satisfies

$$b_i(x, u_i) \in L^\infty(0, T; L^1(\Omega)), \tag{3.1}$$

$$T_k(u_i) \in L^p(0, T; W_0^{1,p}(\Omega)) \quad \text{for any } k > 0, \tag{3.2}$$

$$\lim_{n\to\infty} \frac{1}{n} \int_{\{(x,t)\in Q_T:\, |u_i(x,t)|\le n\}} a(x, t, u_i, \nabla u_i)\nabla u_i \, dx\, dt = 0, \tag{3.3}$$

and for every function S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has compact support,

$$\begin{aligned}\frac{\partial B_{i,S}(x, u_i)}{\partial t} &- \operatorname{div}(a(x, t, u_i, \nabla u_i) S'(u_i)) + S''(u_i) a(x, t, u_i, \nabla u_i)\nabla u_i \\ &+ \operatorname{div}(\phi_i(x, t, u_i) S'(u_i)) - S''(u_i)\phi_i(x, t, u_i)\nabla u_i \\ &= f_i(x, u_1, u_2) S'(u_i) - \operatorname{div}(S'(u_i) F_i) + S''(u_i) F_i \nabla u_i \quad \text{in } \mathcal{D}'(Q_T),\end{aligned} \tag{3.4}$$

and

$$B_{i,S}(x, u_i)(t = 0) = B_{i,S}(x, u_{i,0}) \quad \text{in } \Omega, \tag{3.5}$$

where $B_{i,S}(x, z) = \int_0^z \frac{\partial b_i(x,s)}{\partial s} S'(s)\, ds$.

Equation (3.4) is formally obtained through pointwise multiplication of (1.1) by $S'(u)$. However $a(x, t, u_i, \nabla u_i)$ and $\phi_i(x, t, u_i)$ do not in general make sense in (1.1). Recall that for a renormalized solution, due to (3.2), each term in (3.4) has a meaning in $L^1(Q) + L^{p'}(0, T; W^{-1,p'}(\Omega))$ (see e.g. [5]). We have

$$\frac{\partial B_{i,S}(x, u_i)}{\partial t} \in L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q), \tag{3.6}$$

$$B_{i,S}(x, u_i) \in L^p(0, T; W_0^{1,p}(\Omega)). \tag{3.7}$$

Then (3.6) and (3.7) imply that $B_{i,S}(x, u_i)$ belongs to $C^0([0, T]; L^1(\Omega))$ (for a proof of this trace result see [11]), so that the initial condition (3.5) makes sense.

Main Theorem 3.2. *Let* $b(x,u_0) \in L^1(\Omega)$ *and assume that* (H1)–(H4) *hold true. Then there exists a renormalized solution* (u_1,u_2) *of problem* (1.1) *in the sense of Definition* 3.1.

Proof. Step 1. Let us introduce the following regularization of the data: for $i=1,2$ and $\epsilon>0$,

$$b_{i,\epsilon}(x,r) = b(x,T_{1/\epsilon}(r)) + \epsilon r \quad \forall r \in \mathbb{R}, \tag{3.8}$$

$$a_\epsilon(x,t,s,\xi) = a(x,t,T_{1/\epsilon}(s),\xi) \quad \text{a.e. } (x,t)\in Q_T,\ \forall s\in\mathbb{R},\ \forall \xi \in \mathbb{R}^N, \tag{3.9}$$

$$\phi_{i,\epsilon}(x,t,r) = \phi_i(x,t,T_{1/\epsilon}(r)) \quad \text{a.e. } (x,t)\in Q_T,\ \forall r\in\mathbb{R}, \tag{3.10}$$

$$\begin{aligned} f_{1,\epsilon}(x,s_1,s_2) &= f_1(x,T_{1/\epsilon}(s_1),T_{1/\epsilon}(s_2)) \quad \text{a.e. } x\in\Omega,\ \forall s_1,s_2\in\mathbb{R},\\ f_{2,\epsilon}(x,s_1,s_2) &= f_2(x,T_{1/\epsilon}(s_1),T_{1/\epsilon}(s_2)) \quad \text{a.e. } x\in\Omega,\ \forall s_1,s_2\in\mathbb{R}.\end{aligned} \tag{3.11}$$

Let $u_{i,0\epsilon} \in C_0^\infty(\Omega)$ be such that

$$b_{i,\epsilon}(x,u_{i,0\epsilon}) \to b_i(x,u_{i,0}) \quad \text{strongly in } L^1(\Omega). \tag{3.12}$$

In view of (3.8), for $i=1,2$, $b_{i,\epsilon}$ is a Carathéodory function and satisfies (2.1), so there exists $\lambda_i>0$ such that

$$\lambda_i + \epsilon \le \frac{\partial b_{i,\epsilon}(x,s)}{\partial s}, \quad |b_{i,\epsilon}(x,s)| \le \max_{|s|\le 1/\epsilon} |b_i(x,s)| \quad \text{a.e. } x\in\Omega,\ \forall s\in\mathbb{R}.$$

Let us now consider the regularized problem

$$\begin{cases} \dfrac{\partial b_{i,\epsilon}(x,u_{i,\epsilon})}{\partial t} - \operatorname{div}(a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})) + \operatorname{div}(\phi_{i,\epsilon}(x,t,u_{i,\epsilon}))\\ \qquad\qquad = f_{i,\epsilon}(x,u_1,u_2) - \operatorname{div}(F_i) \quad \text{in } Q_T,\\ u_{i,\epsilon}(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T),\\ b_{i,\epsilon}(x,u_{i,\epsilon})(t=0) = b_{i,\epsilon}(x,u_{i,0\epsilon}) \quad \text{in } \Omega. \end{cases} \tag{3.13}$$

In view of (2.7)–(2.8), there exist $F_{1,\epsilon}, F_{2,\epsilon} \in L^1(\Omega)$ and $\sigma_\epsilon,\mu_\epsilon>0$ such that

$$|f_{1,\epsilon}(x,s_1,s_2)| \le F_{1,\epsilon}(x) + \sigma_\epsilon \max_{|s|\le 1/\epsilon} |b_i(x,s)| \quad \text{a.e. } x\in\Omega,\ \forall s_1,s_2\in\mathbb{R},$$

$$|f_{2,\epsilon}(x,s_1,s_2)| \le F_{2,\epsilon}(x) + \mu_\epsilon \max_{|s|\le 1/\epsilon} |b_i(x,s)| \quad \text{a.e. } x\in\Omega,\ \forall s_1,s_2\in\mathbb{R}.$$

Hence, proving the existence of a weak solution $u_{i,\epsilon} \in L^p(0,T;W_0^{1,p}(\Omega))$ of (3.13) is an easy task (see e.g. [10], [8]).

Step 2. The estimates derived in this step rely on standard techniques for problems of type (3.13), and we just sketch their proof (referring the reader to [4]) for the elliptic version. Let $\tau_1 \in (0,T)$ and t fixed in $(0,\tau_1)$. For $i=1,2$, using $T_k(u_{i,\epsilon})\chi_{(0,t)}$ as a test function in (3.13), we integrate over $(0,\tau_1)$, and by the condition (2.5) we have

$$(3.14)\qquad \int_\Omega B^\epsilon_{i,k}(x,u_{i,\epsilon}(t))\,dx + \int_{Q_t} a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})\nabla T_k(u_{i,\epsilon})\,dx\,ds$$
$$\le \int_{Q_t} c(x,t)|u_{i,\epsilon}|^\gamma|\nabla T_k(u_{i,\epsilon})|\,dx\,ds + \int_{Q_t} f_{i,\epsilon}(x,u_1^\epsilon,u_2^\epsilon)T_k(u_{i,\epsilon})\,dx\,ds$$
$$+ \int_\Omega B_k^{i,\epsilon}(x,u_{i,0}^\epsilon)\,dx + \int_{Q_t} F_i\nabla T_k(u_i^\epsilon)\,dx\,ds$$

where $B^\epsilon_{i,k}(x,r) = \int_0^r T_k(s)\frac{\partial b_{i,\epsilon}(x,s)}{\partial s}\,ds$. Due to the definition of $B^\epsilon_{i,k}$ we have

$$(3.15)\qquad 0 \le \int_\Omega B^\epsilon_{i,k}(x,u_{i,0\epsilon})\,dx \le k\int_\Omega |b_{i,\epsilon}(x,u_{i,0\epsilon})|\,dx = k\|b_i(x,u_{i,0\epsilon})\|_{L^1(\Omega)} \quad \forall k>0.$$

Using (3.14) and (2.3) and (3.11) we obtain

$$\int_\Omega B^\epsilon_{i,k}(x,u_{i,\epsilon}(t))\,dx + \alpha\int_{Q_t} |\nabla T_k(u_{i,\epsilon})|^p\,dx\,ds$$
$$\le \int_{Q_t} c(x,t)|u_{i,\epsilon}|^\gamma|\nabla T_k(u_{i,\epsilon})|\,ds\,dx$$
$$+ k(\|b_i(x,u_{i,0\epsilon})\|_{L^1(\Omega)} + \|f_{i,\epsilon}\|_{L^1(Q_T)}) + \int_{Q_t} F_i\nabla T_k(u_{i,\epsilon})\,dx\,ds.$$

Let $M_i = \sup_\epsilon \|f_{i,\epsilon}\|_{L^1(Q_T)} + \|b_i(x,u_{i,0\epsilon})\|_{L^1(\Omega)}$. Note that

$$B^\epsilon_{i,k}(x,s) = \int_0^s T_k(\sigma)\frac{\partial b_{i,\epsilon}(x,\sigma)}{\partial\sigma}\,d\sigma \ge \frac{\lambda_i+\epsilon}{2}|T_k(s)|^2 > \frac{\lambda_i}{2}|T_k(s)|^2.$$

We deduce from (3.14) and (3.15) that

$$(3.16)\qquad \frac{\lambda_i}{2}\int_\Omega |T_k(u_{i,\epsilon})|^2\,dx + \alpha\int_{Q_t} |\nabla T_k(u_{i,\epsilon})|^p\,dx\,ds$$
$$\le M_ik + \int_{Q_t} c_i(x,t)|u_{i,\epsilon}|^\gamma|\nabla T_k(u_{i,\epsilon})|\,dx\,ds + \int_{Q_t} F_i\nabla T_k(u_{i,\epsilon})\,dx\,ds.$$

By the Gagliardo–Nirenberg and Young inequalities we have

$$(3.17)\qquad \int_{Q_t} c_i(x,t)|u_{i,\epsilon}|^\gamma|\nabla T_k(u_{i,\epsilon})|\,dx\,ds$$
$$\le C_i\frac{\gamma}{N+2}\|c_i(x,t)\|_{L^\tau(Q_{\tau_1})}\sup_{t\in(0,\tau_1)}\int_\Omega |T_k(u_{i,\epsilon})|^2\,dx$$
$$+ C_i\frac{N+2-\gamma}{N+2}\|c_i(x,t)\|_{L^\tau(Q_{\tau_1})}\Big(\int_{Q_{\tau_1}} |\nabla T_k(u_{i,\epsilon})|^p\,dx\,ds\Big)^{(\frac{1}{p}+\frac{N\gamma}{(N+2)p})\frac{N+2}{N+2-\gamma}}.$$

Since $\gamma = \frac{(N+2)}{N+p}(p-1)$, by using (3.16) and (3.17) we obtain

$$\begin{aligned}\frac{\lambda_i}{2}\int_{\Omega}|T_k(u_{i,\epsilon})|^2\,dx &+ \alpha\int_{Q_t}|\nabla T_k(u_{i,\epsilon})|^p\,dx\,ds\\ &\le M_i k + C_i\frac{\gamma}{N+2}\|c_i(x,t)\|_{L^\tau(Q_{\tau_1})}\sup_{t\in(0,\tau_1)}\int_{\Omega}|T_k(u_{i,\epsilon})|^2\,dx\\ &\quad + \left(\frac{\alpha}{p}\right)^{-(p-1)}\|F_i\|_{(L^{p'}(Q))^N}\\ &\quad + C_i\frac{N+2-\gamma}{N+2}\|c_i(x,t)\|_{L^\tau(Q_{\tau_1})}\int_{Q_{\tau_1}}|\nabla T_k(u_{i,\epsilon})|^p\,dx\,ds\\ &\quad + \frac{\alpha}{p}\int_{Q_t}|\nabla T_k(u_{i,\epsilon})|^p\,dx\,ds,\end{aligned}$$

which is equivalent to

$$\begin{aligned}\left(\frac{\lambda_i}{2}-C_i\frac{\gamma}{N+2}\|c_i(x,t)\|_{L^\tau(Q_{\tau_1})}\right)\sup_{t\in(0,\tau_1)}\int_{\Omega}|T_k(u_{i,\epsilon})|^2\,dx+\alpha\int_{Q_{\tau_1}}|\nabla T_k(u_{i,\epsilon})|^p\,dx\,ds\\ -\left(C_i\frac{N+2-\gamma}{N+2}\|c_i(x,t)\|_{L^\tau(Q_{\tau_1})}+\frac{\alpha}{p}\right)\int_{Q_{\tau_1}}|\nabla T_k(u_{i,\epsilon})|^p\,dx\,ds \le M_i k.\end{aligned}$$

If we choose τ_1 such that

$$\begin{aligned}\frac{\lambda_i}{2}-C_i\frac{\gamma}{N+2}\|c_i(x,t)\|_{L^\tau(Q_{\tau_1})} &\ge 0,\\ \frac{\alpha}{p'}-C_i\frac{N+2-\gamma}{N+2}\|c_i(x,t)\|_{L^\tau(Q_{\tau_1})} &\ge 0,\end{aligned}$$

then, denoting by C_i the minimum of

$$\frac{\lambda_i(N+2)}{2\gamma\|c_i(x,t)\|_{L^\tau(Q_{\tau_1})}} \quad\text{and}\quad \frac{\alpha(N+2)}{p'(N+2-\gamma)\|c_i(x,t)\|_{L^\tau(Q_{\tau_1})}},$$

we obtain

$$\sup_{t\in(0,\tau_1)}\int_{\Omega}|T_k(u_{i,\epsilon})|^2\,dx+\int_{Q_{\tau_1}}|\nabla T_k(u_{i,\epsilon})|^p\,dx\,dt \le C_i M_i k. \tag{3.18}$$

Then, by (3.18) and Lemma 3.1 ([1], [6]), we conclude that $T_k(u_{i,\epsilon})$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ independently of ϵ for any $k \ge 0$, so there exists a subsequence still denoted by $u_{i,\epsilon}$ such that

$$T_k(u_{i,\epsilon}) \rightharpoonup H_{i,k} \quad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)). \tag{3.19}$$

LEMMA 3.3 (see [1]). *We have*

$$u_{i,\epsilon}\to u_i \quad a.e.\ Q_T, \qquad b_i(x,u_i)\in L^\infty(0,T;L^1(\Omega)), \tag{3.20}$$

where u_i is a measurable function defined on Q_T for $i = 1, 2$. Moreover,

$$\lim_{n\to\infty} \limsup_{\epsilon\to 0} \frac{1}{n} \int_{\{|u_{i,\epsilon}|\le n\}} a(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})\nabla u_{i,\epsilon}\,dx\,dt = 0. \tag{3.21}$$

STEP 4. In this step we prove that the weak limit $X_{i,k}$ of $a(x,t,T_k(u_{i,\epsilon}), \nabla T_k(u_{i,\epsilon}))$ can be identified with $a(x,t,T_k(u_i),\nabla T_k(u_i))$, for $i = 1, 2$. To prove this we recall the following lemma (see [1]):

LEMMA 3.4. *For $i = 1, 2$, a subsequence of $u_{i,\epsilon}$ satisfies, for any $k \ge 0$,*

$$\limsup_{\epsilon\to 0} \int_{Q_T}\int_0^t a(x,s,u_{i,\epsilon},\nabla T_k(u_{i,\epsilon}))\nabla T_k(u_{i,\epsilon})\,ds\,dx\,dt \le \int_{Q_T}\int_0^t X_{i,k}\nabla T_k(u_i)\,dx\,ds\,dt,$$

$$\lim_{\epsilon\to 0} \int_{Q_T}\int_0^t \big(a(x,t,T_k(u_{i,\epsilon}),\nabla T_k(u_{i,\epsilon})) - a(x,t,T_k(u_{i,\epsilon}),\nabla T_k(u_i))\big) \times (\nabla T_k(u_{i,\epsilon}) - \nabla T_k(u_i)) = 0,$$

$$X_{i,k} = a(x,t,T_k(u_i),\nabla T_k(u_i))) \quad \textit{a.e. in } Q_T, \tag{3.22}$$

and as ϵ tends to 0,

$$a(x,t,T_k(u_{i,\epsilon}),\nabla T_k(u_{i,\epsilon}))\nabla T_k(u_{i,\epsilon}) \rightharpoonup a(x,t,T_k(u_i),\nabla T_k(u_i))\nabla T_k(u_i) \tag{3.23}$$

weakly in $L^1(Q_T)$.

Proof. For $i = 1, 2$, we introduce a time regularization of $T_k(u_i)$ for $k > 0$ in order to apply the monotonicity method. This regularization was introduced for the first time by R. Landes [9]. Let v_0^μ be a sequence of functions in $L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$ such that $\|v_0^\mu\|_{L^\infty(\Omega)} \le k$ for all $\mu > 0$ and v_0^μ converges to $T_k(u_0)$ a.e. in Ω and $\frac{1}{\mu}\|v_0^\mu\|_{L^p(\Omega)}$ converges to 0. For $k \ge 0$ and $\mu > 0$, we use the sequence $(T_k(u))_\mu$ as approximation of $T_k(u)$. We define the regularization in time of the function $T_k(u)$ by

$$(T_k(u))_\mu(x,t) = \mu \int_{-\infty}^t e^{\mu(s-t)}T_k(u(x,s))\,ds,$$

extending $T_k(u)$ by 0 for $s < 0$. It is differentiable for a.e. $t \in (0,T)$ with

$$|(T_k(u))_\mu(x,t)| \le k(1-e^{-\mu t}) < k \quad \text{a.e. in } Q,$$

$$\frac{\partial (T_k(u))_\mu}{\partial t} + \mu((T_k(u))_\mu - T_k(u)) = 0 \quad \text{in } \mathcal{D}'(\Omega),$$

Note that $(T_k(u))_\mu \to T_k(u)$ a.e. in Q_T, weakly-$*$ in $L^\infty(Q)$ and strongly in $L^p(0,T;W_0^p(\Omega))$ as $\mu \to \infty$ and

$$\|(T_k(u))_\mu\|_{L^\infty(Q)} \le \max(\|(T_k(u))\|_{L^\infty(Q)}, \|\nu_0^\mu\|_{L^\infty(\Omega)}) \le k, \quad \forall \mu > 0, \forall k > 0.$$

Lemma 3.5 (see H. Redwane [12]). *Let $k \ge 0$ be fixed. Let S be an increasing $C^\infty(\mathbb{R})$-function such that $S(r) = r$ for $|r| \le k$, and $\operatorname{supp} S'$ is compact. Then*

$$\liminf_{\mu\to\infty} \lim_{\epsilon\to 0} \int_0^T\int_0^t \left\langle \frac{\partial b_{i,\epsilon}(x,u_{i,\epsilon})}{\partial t}, S'(u_{i,\epsilon})\big(T_k(u_{i,\epsilon}) - (T_k(u_i))_\mu\big) \right\rangle \ge 0,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $L^1(\Omega) + W^{-1,p'}(\Omega)$ and $L^\infty(\Omega)\cap W^{1,p}(\Omega)$.

Let S_n be a sequence of increasing C^∞-functions such that

$$S_n(r) = r \quad \text{for } |r| \le n, \qquad \operatorname{supp} S_n' \subset [-(n+1), n+1],$$
$$\|S_n''\|_{L^\infty(\mathbb{R})} \le 1 \quad \text{for any } n \ge 1.$$

For $i = 1, 2$, we use the sequence $(T_k(u_i))_\mu$ of approximations of $T_k(u_i)$, and plug the test function $S_n'(u_{i,\epsilon})(T_k(u_{i,\epsilon}) - (T_k(u_i))_\mu)$ in (3.4) for $n, \mu > 0$. For fixed $k \ge 0$, let $W_\mu^\epsilon = T_k(u_{i,\epsilon}) - (T_k(u_i))_\mu$. Upon integration over $(0,t)$ and then over $(0,T)$ we obtain

$$\begin{aligned}
(3.24)\qquad & \int_0^T\int_0^t \left\langle \frac{\partial b_{i,\epsilon}(x,u_{i,\epsilon})}{\partial t}, S_n'(u_{i,\epsilon})W_\mu^\epsilon \right\rangle ds\,dt \\
&+ \int_{Q_T}\int_0^t a_\epsilon(x,s,u_{i,\epsilon},\nabla u_{i,\epsilon}) S_n'(u_{i,\epsilon})\nabla W_\mu^\epsilon \,ds\,dt\,dx \\
&+ \int_{Q_T}\int_0^t a_\epsilon(x,s,u_{i,\epsilon},\nabla u_{i,\epsilon}) S_n''(u_{i,\epsilon})\nabla u_{i,\epsilon}\nabla W_\mu^\epsilon \,ds\,dt\,dx \\
&- \int_{Q_T}\int_0^t \phi_{i,\epsilon}(x,s,u_{i,\epsilon}) S_n'(u_{i,\epsilon})\nabla W_\mu^\epsilon \,ds\,dt\,dx \\
&- \int_{Q_T}\int_0^t S_n''(u_{i,\epsilon})\phi_{i,\epsilon}(x,s,u_{i,\epsilon})\nabla u_{i,\epsilon}\nabla W_\mu^\epsilon \,ds\,dt\,dx \\
= & \int_{Q_T}\int_0^t f_{i,\epsilon} S_n'(u_{i,\epsilon})W_\mu^\epsilon \,dx\,ds\,dt + \int_{Q_T}\int_0^t F_i S_n'(u_{i,\epsilon})\nabla W_\mu^\epsilon \,ds\,dt\,dx \\
&+ \int_{Q_T}\int_0^t F_i S_n''(u_{i,\epsilon})\nabla u_{i,\epsilon}\nabla W_\mu^\epsilon \,ds\,dt\,dx.
\end{aligned}$$

We pass to the limit in (3.24) as $\epsilon \to 0$, $\mu \to \infty$ and then $n \to \infty$ for k fixed. We use Lemma 3.5 and proceeding as in [4], [12], we conclude that

$$\liminf_{\mu\to\infty} \lim_{\epsilon\to 0} \int_0^T \int_0^t \left\langle \frac{\partial b_{i,\epsilon}(x,u_{i,\epsilon})}{\partial t}, W_\mu^\epsilon \right\rangle ds\,dt \geq 0 \quad \text{for any } n \geq k,$$

$$\lim_{n\to\infty} \limsup_{\mu\to\infty} \limsup_{\epsilon\to 0} \int_{Q_T} \int_0^t a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon}) S_n''(u_{i,\epsilon}) \nabla u_{i,\epsilon} \nabla W_\mu^\epsilon\,ds\,dt\,dx = 0,$$

$$\lim_{\mu\to\infty} \lim_{\epsilon\to 0} \int_{Q_T} \int_0^t f_{i,\epsilon} S_n'(u_{i,\epsilon}) W_\mu^\epsilon\,ds\,dt\,dx = 0,$$

$$\lim_{\mu\to\infty} \int_{Q_T} \int_0^t F_i S_n'(u_{i,\epsilon}) \nabla W_\mu^\epsilon\,ds\,dt\,dx = 0,$$

$$\lim_{\mu\to\infty} \int_{Q_T} \int_0^t F_i S_n''(u_{i,\epsilon}) \nabla u_{i,\epsilon} W_\mu^\epsilon\,ds\,dt\,dx = 0,$$

and finally,

$$\lim_{\mu\to\infty} \lim_{\epsilon\to 0} \int_{Q_T} \int_0^t \phi_{i,\epsilon}(x,t,u_{i,\epsilon}) S_n'(u_{i,\epsilon}) \nabla W_\mu^\epsilon\,ds\,dt\,dx = 0, \tag{3.25}$$

$$\lim_{\mu\to\infty} \lim_{\epsilon\to 0} \int_{Q_T} \int_0^t S_n''(u_\epsilon) \phi_{i,\epsilon}(x,t,u_{i,\epsilon}) \nabla u_{i,\epsilon} \nabla W_\mu^\epsilon\,ds\,dt\,dx = 0. \tag{3.26}$$

For the proof of (3.25) and (3.26) the reader is referred to [1]; here (3.22) and (3.23) are used. Note that, letting $\epsilon \to 0$ in (3.21) and using (3.23) shows that u satisfies (3.3).

Now we want to prove that u satisfies (3.4). Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that $\operatorname{supp} S' \subset [-k,k]$ where k is a positive real number. Pointwise multiplication of (3.13) by $S'(u_\epsilon)$ leads to

$$\begin{aligned} &\frac{\partial B_{i,S}^\epsilon(x,u_{i,\epsilon})}{\partial t} - \operatorname{div}\big(a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon}) S'(u_{i,\epsilon})\big) \\ &\quad + S''(u_{i,\epsilon}) a(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon}) \nabla u_{i,\epsilon} + \operatorname{div}\big(\phi_{i,\epsilon}(x,t,u_{i,\epsilon}) S'(u_{i,\epsilon})\big) \\ &\quad - S''(u_{i,\epsilon}) \phi_{i,\epsilon}(x,t,u_{i,\epsilon}) \nabla u_{i,\epsilon} \\ &= f_{i,\epsilon} S'(u_{i,\epsilon}) - \operatorname{div}(F_i S'(u_{i,\epsilon})) + S''(u_{i,\epsilon}) F_i \nabla u_{i,\epsilon} \quad \text{in } \mathcal{D}'(Q_T), \end{aligned} \tag{3.27}$$

where $B_{i,S}^\epsilon(x,r) = \int_0^r \frac{\partial b_{i,\epsilon}(x,s)}{\partial s} S'(s)\,ds$.

In what follows we let $\epsilon \to 0$ in each term of (3.27). Since $u_{i,\epsilon}$ converging to u_i a.e. in Q_T implies that $B_{i,S}^\epsilon(x,u_{i,\epsilon})$ converges to $B_{i,S}(x,u_i)$ a.e. in Q_T and weakly-$*$ in $L^\infty(Q_T)$, it follows that $\partial B_{i,S}^\epsilon(x,u_{i,\epsilon})/\partial t$ converges to $\partial B_{i,S}(x,u_i)/\partial t$ in $\mathcal{D}'(Q_T)$. We observe that $a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})S'(u_{i,\epsilon})$ can be identified with $a(x,t,T_k(u_{i,\epsilon}),\nabla T_k(u_{i,\epsilon}))S'(u_{i,\epsilon})$ for $\epsilon \leq 1/k$, so using the pointwise convergence of $u_{i,\epsilon}$ to u_i in Q_T, and the weak convergence of $T_k(u_{i,\epsilon})$ to $T_k(u_i)$in $L^p(0,T;W_0^{1,p}(\Omega))$, we get

$$a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})S'(u_{i,\epsilon}) \rightharpoonup a(x,t,T_k(u_i),\nabla T_k(u_i))S'(u_i) \quad \text{in } L^{p'}(Q_T),$$

and

$$S''(u_{i,\epsilon})a_\epsilon(x,t,u_{i,\epsilon},\nabla u_{i,\epsilon})\nabla u_{i,\epsilon} \rightharpoonup S''(u_i)a(x,t,T_k(u_i),\nabla T_k(u_i))\nabla T_k(u_i)$$

in $L^1(Q_T)$. Furthermore, since

$$\phi_{i,\epsilon}(x,t,u_{i,\epsilon})S'(u_{i,\epsilon}) = \phi_{i,\epsilon}(x,t,T_k(u_{i,\epsilon}))S'(u_{i,\epsilon})$$

a.e. in Q_T, by (3.10) we obtain

$$|\phi_{i,\epsilon}(x,t,T_k(u_{i,\epsilon}))S'(u_{i,\epsilon})| \leq |c_i(x,t)|k^\gamma.$$

It follows that

$$\phi_{i,\epsilon}(x,t,T_k(u_{i,\epsilon}))S'(u_{i,\epsilon}) \to \phi_i(x,t,T_k(u_i))S'(u_i) \quad \text{strongly in } L^{p'}(Q_T).$$

In a similar way

$$S''(u_{i,\epsilon})\phi_{i,\epsilon}(x,t,u_{i,\epsilon})\nabla u_{i,\epsilon} = S''(T_k(u_{i,\epsilon}))\phi_{i,\epsilon}(x,t,T_k(u_{i,\epsilon}))\nabla T_k(u_{i,\epsilon})$$

a.e. in Q_T. Using the weak convergence of $T_k(u_{i,\epsilon})$ in $L^p(0,T;W_0^{1,p}(\Omega))$ it is possible to prove that

$$S''(u_{i,\epsilon})\phi_\epsilon(x,t,u_{i,\epsilon})\nabla u_{i,\epsilon} \to S''(u_i)\phi_i(x,t,u_i)\nabla u_i \quad \text{in } L^1(Q_T),$$

and $S''(u_{i,\epsilon})F_i\nabla u_{i,\epsilon} \to S''(u_i)F_i\nabla u_i$ in $L^1(Q_T)$. Since $|S'(u_{i,\epsilon})| \leq C$, it follows that $F_iS''(u_{i,\epsilon}) \to F_iS''(u_i)$ strongly in $L^{p'}(Q_T)$. Finally by (3.11) we deduce that $f_\epsilon S'(u_{i,\epsilon}) \to f_iS'(u_i)$ in $L^1(Q_T)$.

Now, it remains to prove that $B_{i,S}(x,u_i)$ satisfies the initial condition $B_{i,S}(x,u_i)(t=0) = B_{i,S}(x,u_{i,0})$ in Ω. To this end, first note that $B_S^\epsilon(x,u_\epsilon)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$. Secondly the above consideration of the behavior of the terms of this equation shows that $\partial B_{i,S}^\epsilon(x,u_{i,\epsilon})/\partial t$ is bounded in $L^1(Q_T) + L^{p'}(0,T;W^{-1,p'}(\Omega))$. As a consequence, $B_{i,S}^\epsilon(u_{i,\epsilon})(t=0) = B_{i,S}^\epsilon(x,u_{i,0\epsilon})$ converges to $B_{i,S}(x,u_i)(t=0)$ strongly in $L^1(\Omega)$ (for a proof of this trace result see [11]). Finally, the smoothness of S implies that $B_{i,S}(x,u_i)(t=0) = B_{i,S}(x,u_{i,0})$ in Ω. The proof of Theorem 3.1 is complete. ■

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