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ON THE SPECTRUM OF THE p-BIHARMONIC OPERATOR INVOLVING p-HARDY'S INEQUALITY

Abstract. In this paper, we study the spectrum for the following eigenvalue problem with the p-biharmonic operator involving the Hardy term:

$$\Delta(|\Delta u|^{p-2}\Delta u)=\lambda\frac{|u|^{p-2}u}{\delta(x)^{2p}}\quad \text{in } \Omega,\quad u\in W_0^{2,p}(\Omega).$$

By using the variational technique and the Hardy–Rellich inequality, we prove that the above problem has at least one increasing sequence of positive eigenvalues.

1. Introduction. Let Ω be a regular bounded domain in $\mathbb{R}^N$. Consider the fourth order nonlinear eigenvalue problem

$$\Delta(|\Delta u|^{p-2}\Delta u)=\lambda\frac{|u|^{p-2}u}{\delta(x)^{2p}}\quad \text{in } \Omega,\quad u\in W_0^{2,p}(\Omega), \tag{1.1}$$

where λ is a real parameter which plays the role of eigenvalue, $1<p<N/2$ and $\delta(x)=d(x,\partial\Omega)$.

$\Delta_p^2 u:=\Delta(|\Delta u|^{p-2}\Delta u)$ is the operator of fourth order called the *p-biharmonic operator*. For $p=2$, the linear operator $\Delta_2^2=\Delta^2=\Delta\Delta$ is the iterated Laplacian that up to a multiplicative positive constant often appears in the equations of Navier–Stokes as the viscosity term, and its inverse operator denoted by $(\Delta^2)^{-1}$ is the celebrated Green operator [L].

Note that the biharmonic equation $\Delta^2 u=0$ is a linear partial differential equation of fourth order which appears in quantum mechanics and in the theory of linear elasticity modeling Stokes flows.

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This paper is motivated by recent advances in mathematical modeling of non-Newtonian fluids and elastic mechanics, in particular, electrorheological fluids (smart fluids). This important class of fluids is characterized by the change of viscosity which depends on the electric field. These fluids, also known as ER fluids, have many applications in elastic mechanics, fluid dynamics etc. For more information, the reader can refer to [H, R].

Recently, El Khalil [E] proved the existence of nontrivial solutions of the nonlinear eigenvalue problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda\rho|u|^{q-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where p, q are real numbers and ρ is an indefinite weight function, by applying the Mountain Pass Theorem and local minimization. In [EKT], the authors studied the spectrum of the p-biharmonic operator in the homogeneous case $p = q$. In the present paper, we use a variational technique to prove the existence of a sequence of positive eigenvalues of problem (1.1).

Notice that we also use some technical setting based on the Hardy–Rellich inequality.

2. Preliminaries.

First, we introduce some preliminary results.

DEFINITION 2.1. We say that $u \in W_0^{2,p}(\Omega)$ is a *weak solution* of (1.1) if

$$\int_\Omega |\Delta u|^{p-2}\Delta u \Delta v\,dx = \lambda \int_\Omega \frac{|u|^{p-2}u}{\delta(x)^{2p}} v\,dx \quad \text{for all } v \in W_0^{2,p}(\Omega).$$

If u is not identically zero, then we say that λ is an *eigenvalue* of (1.1) corresponding to the eigenfunction u.

The main objective of this work is to show that problem (1.1) has at least one increasing sequence $(\lambda_k)_{k\geq 1}$ of positive eigenvalues, by using a variational technique based on Ljusternik–Schnirelmann theory on C^1-manifolds [S]. In fact, we give a direct characterization of λ_k involving a mini-max argument over sets of genus greater than k. We set

$$\lambda_1 = \inf\left\{ \|\Delta v\|_p^p : v \in W_0^{2,p}(\Omega),\ \int_\Omega \frac{|v|^p}{\delta(x)^{2p}}\,dx = 1 \right\}, \tag{2.1}$$

where $\|\Delta v\|_p = (\int_\Omega |\Delta v|^p)^{1/p}$ denotes the norm of $v \in W_0^{2,p}(\Omega)$. Let us notice that $W_0^{2,p}(\Omega)$ equipped with this norm is a uniformly convex Banach space for $1 < p < \infty$. The norm $\|\Delta \cdot\|_p$ is uniformly equivalent on $W_0^{2,p}(\Omega)$ to the usual norm of $W_0^{2,p}(\Omega)$ [GT].

We see that the value defined in (2.1) can be written as

$$\lambda_1 = \inf_{u\in W_0^{2,p}(\Omega),\, u\not\equiv 0} \frac{\int_\Omega |\Delta u|^p\,dx}{\int_\Omega \frac{|u|^p}{\delta(x)^{2p}}\,dx}. \tag{2.2}$$

Finally, by the Hardy–Rellich inequality (see [DH, Mi]), we know that

$$\int_\Omega \frac{|u|^p}{\delta(x)^{2p}}\,dx \le \frac{1}{\overline{\lambda}} \int_\Omega |\Delta u|^p\,dx, \qquad \forall u \in C_c^\infty(\Omega), \tag{2.3}$$

where

$$\overline{\lambda} = [N(p-1)(N-2p)/p^2]^p,$$

hence the problem (1.1) is naturally well defined.

Definition 2.2. Let X be a real reflexive Banach space and let X^* stand for its dual with respect to the pairing $\langle\cdot,\cdot\rangle$. We shall deal with mappings T acting from X into X^*. Strong convergence in X (and in X^*) is denoted by $\to$, and weak convergence by $\rightharpoonup$. A mapping T is said to belong to the class (S^+) if for any sequence u_n in X converging weakly to $u \in X$ with $\limsup_{n\to\infty}\langle \Delta_p^2 u_n, u_n - u\rangle \le 0$, the sequence u_n converges strongly to u in X. We then write $T \in (S^+)$.

Consider now the following two functionals defined on $W_0^{2,p}(\Omega)$:

$$\Phi(u) = \frac{1}{p}\int_\Omega |\Delta u|^p\,dx \quad \text{and} \quad \varphi(u) = \frac{1}{p}\int_\Omega \frac{|u|^p}{\delta(x)^{2p}}\,dx,$$

and set

$$\mathcal{M} = \{u \in W_0^{2,p}(\Omega) : p\varphi(u) = 1\}.$$

Lemma 2.3.

(i) *Φ and φ are even, and of class C^1 on $W_0^{2,p}(\Omega)$.*
(ii) *$\mathcal{M}$ is a closed C^1-manifold.*

Proof. See [EMT, Lemma 2.3]. ■

Remark 2.4 ([EKT, Remark 3.2]). The functional

$$J : W_0^{2,p}(\Omega) \to W_0^{-2,p'}(\Omega)$$

defined by

$$J(u) = \begin{cases} \|\Delta u\|_p^{2-p}\Delta_p^2 u & \text{if } u \neq 0,\\ 0 & \text{if } u = 0,\end{cases}$$

is the duality mapping of $(W_0^{2,p}(\Omega), \|\Delta\cdot\|_p)$ associated with the gauge function $t \mapsto |t|^{p-2}t$.

The following lemma is the key to showing existence.

LEMMA 2.5.

(i) *φ' is completely continuous.*

(ii) *The functional Φ satisfies the Palais–Smale condition on $\mathcal{M}$, i.e., for $\{u_k\} \subset \mathcal{M}$, if $\{\Phi(u_k)\}_k$ is bounded and*

$$\alpha_k := \Phi'(u_k) - \beta_k \varphi'(u_k) \to 0 \quad \text{as } k \to \infty, \tag{2.4}$$

where $\beta_k = \langle \Phi'(u_k), u_k\rangle / \langle \varphi'(u_k), u_k\rangle$, then $\{u_k\}_{k\geq 1}$ has a subsequence convergent in $W_0^{2,p}(\Omega)$.

Proof. (i) First let us prove that φ' is well defined. Let $u, v \in W_0^{2,p}(\Omega)$. We have

$$\langle \varphi'(u), v\rangle = \int_\Omega \frac{|u|^{p-1}uv}{\delta(x)^{2p}}\,dx.$$

Thus

$$|\langle \varphi'(u), v\rangle| \leq \int_{\{x\in\Omega:\,\delta(x)>1\}} \frac{|u|^{p-1}}{\delta(x)^{2p}}|v|\,dx + \int_{\{x\in\Omega:\,\delta(x)\leq 1\}} \frac{|u|^{p-1}}{\delta(x)^{2p}}|v|\,dx.$$

Hence

$$|\langle \varphi'(u), v\rangle| \leq \int_{\{x\in\Omega:\,\delta(x)>1\}} |u|^{p-1}|v|\,dx + \int_{\{x\in\Omega:\,\delta(x)\leq 1\}} \frac{1}{\delta(x)^2}\,\frac{|u|^{p-1}}{\delta(x)^{2(p-1)}}|v|\,dx.$$

By applying Hölder's inequality, we obtain

$$\begin{aligned}|\langle \varphi'(u), v\rangle| &\leq \Big(\int_{\{x\in\Omega:\,\delta(x)>1\}} |u|^{(p-1)p'}\,dx\Big)^{1/p'}\Big(\int_{\{x\in\Omega:\,\delta(x)>1\}} |v|^p\,dx\Big)^{1/p}\\ &\quad + \Big(\int_{\{x\in\Omega:\,\delta(x)\leq 1\}} \frac{|u|^{(p-1)p'}}{\delta(x)^{2(p-1)p'}}\,dx\Big)^{1/p'}\Big(\int_{\{x\in\Omega:\,\delta(x)\leq 1\}} \frac{|v|^p}{\delta(x)^{2p}}\,dx\Big)^{1/p},\end{aligned}$$

and by the Hardy–Rellich inequality (2.3), we have

$$|\langle \varphi'(u), v\rangle| \leq \|u\|_{p'}^{p-1}\|v\|_{L^p(\Omega)} + \frac{1}{(\overline{\lambda})^2}\Big(\int_\Omega |\Delta u|^{(p-1)p'}\,dx\Big)^{1/p'}\Big(\int_\Omega |\Delta v|^p\,dx\Big)^{1/p}.$$

Thus

$$|\langle \varphi'(u), v\rangle| \leq \|u\|_{p'}^{p-1}\|v\|_{L^p(\Omega)} + \frac{1}{(\overline{\lambda})^2}\|\Delta u\|_{p'}^{p-1}\|\Delta v\|_p,$$

where p and p' are conjugate by the equality $pp' = p + p'$. Therefore

$$|\langle \varphi'(u), v\rangle| \leq K\|\Delta u\|_{p'}^{p-1}\|\Delta v\|_p + \frac{1}{(\overline{\lambda})^2}\|\Delta u\|_{p'}^{p-1}\|\Delta v\|_p.$$

Hence

$$\|\varphi'(u)\|_* \leq \Big(K + \frac{1}{(\overline{\lambda})^2}\Big)\|\Delta u\|_{p'}^{p-1},$$

where K is the constant given by the embedding of $W_0^{2,p}(\Omega)$ in $L^p(\Omega)$. Here $\|\cdot\|_*$ is the dual norm associated with $\|\Delta\cdot\|_p$.

The complete continuity of φ' is proved exactly as in [EMT]. This proves (i).

(ii) By the definition of Φ, $\|\Delta(u_k)\|_p$ is bounded in $\mathbb{R}$. Thus, without loss of generality, we can assume that u_k converges weakly in $W_0^{2,p}(\Omega)$ to some function $u \in W_0^{2,p}(\Omega)$ and $\|\Delta u_k\|_p \to c$. We distinguish two cases:

If $c = 0$, then u_k converges strongly to 0 in $W_0^{2,p}(\Omega)$.

If $c \neq 0$, then let us prove that

$$\limsup_{k\to\infty} \langle \Delta_p^2 u_k, u_k - u\rangle \le 0.$$

Indeed, notice that

$$\langle \Delta_p^2 u_k, u_k - u\rangle = \|\Delta u_k\|_p^p - \langle \Delta_p^2 u_k, u\rangle.$$

Applying α_k of (2.4) to u, we have

$$\theta_k := \langle \Delta_p^2 u_k, u\rangle - \beta_k \langle \varphi'(u_k), u\rangle \to 0 \qquad \text{as } k \to \infty.$$

Therefore

$$\langle \Delta_p^2 u_k, u_k - u\rangle = \|\Delta u_k\|_p^p - \theta_k - (\langle \Phi'(u_k), u_k\rangle / \langle \varphi'(u_k), u_k\rangle) \cdot \langle \varphi'(u_k), u\rangle.$$

That is,

$$\langle \Delta_p^2 u_k, u_k - u\rangle = \frac{\|\Delta u_k\|_p^p}{\langle \varphi'(u_k), u_k\rangle} (\langle \varphi'(u_k), u_k\rangle - \langle \varphi'(u_k), u\rangle) - \theta_k.$$

On the other hand, from (i), φ' is completely continuous. Thus

$$\varphi'(u_k) \to \varphi'(u) \quad \text{and} \quad \langle \varphi'(u_k), u_k\rangle \to \langle \varphi'(u), u\rangle.$$

Then

$$|\varphi'(u_k), u_k\rangle - \langle \varphi'(u_k), u\rangle| \le |\varphi'(u_k), u_k\rangle - \langle \varphi'(u), u\rangle| + |\langle \varphi'(u_k), u\rangle - \langle \varphi'(u), u\rangle|.$$

It follows that

$$\begin{aligned} &|\varphi'(u_k), u_k\rangle - \langle \varphi'(u_k), u\rangle| \\ &\qquad \le |\langle \varphi'(u_k), u_k\rangle - \langle \varphi'(u), u\rangle| + \|\varphi'(u_k) - \varphi'(u)\|_* \|\Delta u\|_p. \end{aligned}$$

This implies that

$$\langle \varphi'(u_k), u_k\rangle - \langle \varphi'(u_k), u\rangle \to 0 \qquad \text{as } k \to \infty.$$

We deduce that

$$\limsup_{k\to\infty} \langle \Delta_p^2 u_k, u_k - u\rangle \le \frac{c^p}{\langle \varphi'(u), u\rangle} \limsup_{k\to\infty} (\langle \varphi'(u_k), u_k\rangle - \langle \varphi'(u_k), u\rangle).$$

Thus

$$\limsup_{k\to\infty} \langle \Delta_p^2 u_k, u_k - u\rangle \le 0. \tag{2.5}$$

We can write $\Delta_p^2 u_k = \|\Delta u_k\|_p^{p-2} J(u_k)$, since $\|\Delta u_k\|_p \neq 0$ for k large enough. Therefore

$$\limsup_{k\to\infty} \langle \Delta_p^2 u_k, u_k - u\rangle = c^{p-2} \limsup_{k\to\infty} \langle J u_k, u_k - u\rangle.$$

According to (2.5), we conclude that

$$\limsup_{k\to\infty} \langle J u_k, u_k - u\rangle \leq 0.$$

In view of Remark 2.4, J is the duality mapping. Thus $J \in (S^+)$. Therefore, $u_n \to u$ in $W_0^{2,p}(\Omega)$. This completes the proof of the lemma. ∎

3. Main results. Set

$$\Gamma_j = \{K \subset \mathcal{M} : K \text{ symmetric, compact and } \gamma(K) \geq j\},$$

where $\gamma(K) = j$ is the genus of K, i.e., the smallest positive integer j such that there exists an odd continuous map from K to $\mathbb{R}^j \setminus \{0\}$.

Let us now state our first main result using Ljusternik–Schnirelmann theory.

Main Theorem 3.1. *For any integer $j \geq 1$,*

$$\lambda_j := \inf_{K\in\Gamma_j} \max_{u\in K} p\Phi(u)$$

is a critical value of Φ restricted to $\mathcal{M}$. More precisely, there exists $u_j \in K$ such that

$$\lambda_j = p\Phi(u_j) = \sup_{u\in K} p\Phi(u)$$

and u_j is a solution of (1.1) *associated to the positive eigenvalue λ_j. Moreover,*

$$\lambda_j \to \infty \quad \text{as } j \to \infty.$$

Proof. We only need to prove that $\Gamma_j \neq \emptyset$ for any integer $j \geq 1$, and the last assertion. Indeed, since $W_0^{2,p}(\Omega)$ is separable, there exists $(e_i)_{i\geq 1}$ linearly dense in $W_0^{2,p}(\Omega)$ such that $\operatorname{supp} e_i \cap \operatorname{supp} e_n = \emptyset$ if $i \neq n$. We may assume that $e_i \in \mathcal{M}$ (if not, we take $e_i' = e_i/[p\varphi(e_i)]^{1/p}$). Let now $j \geq 1$ and denote

$$F_j = \operatorname{span}\{e_1, \ldots, e_j\}.$$

Clearly, F_j is a vector subspace with $\dim F_j = j$. If $v \in F_j$, then there exist $\alpha_1, \ldots, \alpha_j$ in $\mathbb{R}$ such that $v = \sum_{i=1}^j \alpha_i e_i$. Thus

$$\varphi(v) = \sum_{i=1}^{j} |\alpha_i|^p \varphi(e_i) = \frac{1}{p} \sum_{i=1}^{j} |\alpha_i|^p.$$

It follows that the map

$$v \mapsto (p\varphi(v))^{1/p} =: \|v\|$$

defines a norm on F_j. Consequently, there is a constant $c > 0$ such that

$$c\,\|\Delta v\|_p \le \|v\| \le \frac{1}{c}\|\Delta v\|_p.$$

This implies that the set

$$V = F_k \cap \{v \in W_0^{2,p}(\Omega) : p\varphi(v) \le 1\}$$

is bounded, since $V \subset B(0, 1/p)$ where

$$B(0, 1/c) = \{u \in W_0^{2,p}(\Omega) : \|\Delta u\| \le 1/c\}.$$

Thus, V is a symmetric bounded neighborhood of $0 \in F_j$. Moreover, $F_j \cap \mathcal{M}$ is a compact set. By [S, Proposition 2.3(f)], we conclude that $\gamma(F_j \cap \mathcal{M}) = j$ and then we finally obtain $\Gamma_j \ne \emptyset$. This completes the proof of the first assertion of the theorem.

Now, we claim that

$$\lambda_j \to \infty \quad \text{as } j \to \infty.$$

Let $(e_k, e_n^*)_{k,n}$ be a bi-orthogonal system such that $e_k \in W_0^{2,p}(\Omega)$ and $e_n^* \in W_0^{-2,p'}(\Omega)$, the $(e_k)_k$ are linearly dense in $W_0^{2,p}(\Omega)$ and the $(e_n^*)_n$ are total for the dual $W_0^{-2,p'}(\Omega)$. For $k \ge 1$, set

$$F_k = \operatorname{span}\{e_1, \dots, e_k\} \quad \text{and} \quad F_k^\perp = \operatorname{span}\{e_{k+1}, e_{k+2}, \dots\}.$$

By [S, Proposition 2.3(g)], we have $K \cap F_k^\perp \ne \emptyset$ for any $K \in \Gamma_k$. Thus

$$t_k := \inf_{K \in \Gamma_k} \sup_{u \in K \cap F_{k-1}^\perp} p\Phi(u) \to \infty \quad \text{as } k \to \infty.$$

Indeed, if not, then for k large, there exists $u_k \in F_{k-1}^\perp$ with $\|u_k\|_p = 1$ such that

$$t_k \le p\Phi(u_k) \le M,$$

for some $M > 0$ independent of k. Thus $\|\Delta u_k\|_p \le M$. This implies that $(u_k)_k$ is bounded in $W_0^{2,p}(\Omega)$. Taking a subsequence of u_k if necessary, we can assume that (u_k) converges weakly in $W_0^{2,p}(\Omega)$ and strongly in $L^p(\Omega)$. By our choice of $F_{k-1}^\perp$, we have $u_k \rightharpoonup 0$ in $W_0^{2,p}(\Omega)$, because $\langle e_n^*, e_k\rangle = 0$ for any $k \ge n$. This contradicts the fact that $\|u_k\|_p = 1$ for all k. Since $\lambda_k \ge t_k$, the claim is proved. This completes the proof of the theorem. ■

Remark 3.2. From the theorem above we have the following statements:

(i) $\lambda_1 = \inf\bigl\{\|\Delta v\|_p^p : v \in W_0^{2,p}(\Omega),\ \int_\Omega \frac{|v|^p}{\delta(x)^{2p}}\,dx = 1\bigr\}$;
(ii) $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \to +\infty$.
(iii) (1.1) has spectrum

$$\Lambda = \{\lambda \in \mathbb{R} : \lambda \text{ is an eigenvalue of (1.1)}\},$$

λ_1 is the smallest eigenvalue in the spectrum of (1.1).

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