APPLICATIONES MATHEMATICAE
41,2-3 (2014), pp. 247–255

Khalil Ben Haddouch, Najib Tsouli and Zakaria El Allali (Oujda)

THE THIRD ORDER SPECTRUM OF THE p-BIHARMONIC OPERATOR WITH WEIGHT

Abstract. We show that the spectrum of $\Delta_p^2 u + 2\beta \cdot \nabla(|\Delta u|^{p-2}\Delta u) + |\beta|^2|\Delta u|^{p-2}\Delta u = \alpha m|u|^{p-2}u$, where $\beta \in \mathbb{R}^N$, under Navier boundary conditions, contains at least one sequence of eigensurfaces.

1. Introduction. We are concerned here with the eigenvalue problem

$$(1.1)\quad \begin{cases} \text{Find } (\beta,\alpha,u) \in \mathbb{R}^N \times \mathbb{R}_+^* \times (X \setminus \{0\}) \text{ such that} \\ \Delta_p^2 u + 2\beta \cdot \nabla(|\Delta u|^{p-2}\Delta u) + |\beta|^2|\Delta u|^{p-2}\Delta u = \alpha m|u|^{p-2}u \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$), $\beta \in \mathbb{R}^N$, Δ_p^2 denotes the p-biharmonic operator defined by $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$, $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, and $m \in M = \{m \in L^\infty(\Omega) : \text{meas}\{x \in \Omega : m(x) > 0\} \neq 0\}$.

Set $\Omega^+ = \{x \in \Omega : m(x) > 0\}$; we suppose that $|\Omega^+| \neq 0$.

A. Anane, O. Chakrone and J.-P. Gossez [A] have studied the eigenvalue problem

$$\begin{cases} \text{Find } (\beta,\alpha,u) \in \mathbb{R}^N \times \mathbb{R} \times (W_0^{1,p}(\Omega) \setminus \{0\}) \text{ such that} \\ -\Delta_p u = \alpha m|u|^{p-2}u + \beta \cdot |\nabla u|^{p-2}\nabla u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

They showed that the spectrum of this problem, denoted by $\sigma_1(-\Delta_p, m)$, contains at least one sequence of eigensurfaces in $\mathbb{R}^N \times \mathbb{R}$.

Motivated by this work, we define the third-order spectrum for the p-biharmonic operator, denoted by $\sigma_3(\Delta_p^2, m)$, to be the set of couples $(\beta, \alpha) \in \mathbb{R}^N \times \mathbb{R}$ such that the problem (1.1) has a non-trivial solution $u \in X$. We will show that this spectrum contains a sequence of eigensurfaces in $\mathbb{R}^N \times \mathbb{R}$.

2010 *Mathematics Subject Classification*: 35P30, 35J40.
Key words and phrases: p-biharmonic operator, nonlinear spectral theory.

DOI: 10.4064/am41-2-12

In the case where $\beta = 0$, the zero order spectrum, denoted by $\sigma_0(\Delta_p^2, m)$, is defined to be the set of eigenvalues $\alpha \in \mathbb{R}$ such that the problem

$$\begin{cases} \text{Find } (\alpha, u) \in \mathbb{R}_+^* \times (X \setminus \{0\}) \text{ such that} \\ \Delta_p^2 u = \alpha m |u|^{p-2} u \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

has a non-trivial solution $u \in X$.

Problem (1.2) was considered by P. Drábek and M. Ôtani [DR] for $m = 1$. They showed that it has a principal positive eigenvalue which is simple and isolated.

A. El Khalil, S. Kellati and A. Touzani [E] have studied the spectrum of the p-biharmonic operator with weight and with Dirichlet boundary conditions. They showed that this spectrum contains at least one non-decreasing sequence of positive eigenvalues.

In 2007, M. Talbi and N. Tsouli [T] considered the spectrum of the weighted p-biharmonic operator with weight and showed that the eigenvalue problem

$$\begin{cases} \Delta(\rho |\Delta u|^{p-2} \Delta u) = \alpha m |u|^{p-2} u \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\rho \in C(\overline{\Omega})$ and $\rho > 0$, has a non-decreasing sequence of eigenvalues, and studied the one-dimensional case.

J. Benedikt [B] found the spectrum of the p-biharmonic operator with Dirichlet and Neumann boundary conditions in the case $N = 1$, $m = 1$, and $\rho = 1$.

In this article we consider the transformation of the Poisson problem used by P. Drábek and M. Ôtani. We use Ljusternik–Schnirelmann theory to prove that the spectrum of (1.1) contains a sequence of eigensurfaces $(G(\Gamma_n^p(\cdot, m)))_{n \geq 1}$ such that for all $\beta \in \mathbb{R}^N$, $\Gamma_n^p(\beta, m) \to \infty$.

2. Preliminaries. Let $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. We denote by:

- $\|u\|_p = (\int_\Omega |u|^p \, dx)^{1/p}$ the norm in $L^p(\Omega)$,
- $\|u\|_{2,p} = (\|\Delta u\|_p^p + \|u\|_p^p)^{1/p}$ the norm in X,
- $\|u\|_\infty$ the norm in $L^\infty(\Omega)$,
- $\langle \cdot, \cdot \rangle$ the duality bracket between $L^p(\Omega)$ and $L^{p'}(\Omega)$, where $1/p + 1/p' = 1$.

For all $f \in L^p(\Omega)$ the Dirichlet problem for the Poisson equation,

$$\begin{cases} -\Delta u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

is uniquely solvable in X (cf. [G]). We denote by $\varLambda$ the inverse operator of $-\varDelta : X \to L^p(\Omega)$. In the following lemma we give some properties of the operator $\varLambda$ (cf. [AG]).

LEMMA 2.1.

(i) (Continuity) *There exists a constant* $C_p > 0$ *such that* $\|\varLambda f\|_{2,p} \le C_p\|f\|_p$ *for all* $p \in]1,\infty[$ *and* $f \in L^p(\Omega)$.
(ii) (Continuity) *Given* $k \in \mathbb{N}^*$, *there exists a constant* $C_{p,k} > 0$ *such that* $\|\varLambda f\|_{W^{k+2,p}} \le C_{p,k}\|f\|_{W^{k,p}}$.
(iii) (Symmetry) *The identity* $\int_\Omega \varLambda u \cdot v\,dx = \int_\Omega u \cdot \varLambda v\,dx$ *holds for all* $u \in L^p(\Omega)$ *and* $v \in L^{p'}(\Omega)$ *with* $p \in]1,\infty[$.
(iv) (Regularity) *Given* $f \in L^\infty(\Omega)$, *we have* $\varLambda f \in C^{1,\alpha}(\overline{\Omega})$ *for all* $\alpha \in]0,1[$; *moreover, there exists* $C_\alpha > 0$ *such that* $\|\varLambda f\|_{C^{1,\alpha}} \le C_\alpha\|f\|_\infty$.
(v) (Regularity and Hopf-type maximum principle) *Let* $f \in C(\overline{\Omega})$ *and* $f \ge 0$. *Then* $w = \varLambda f \in C^{1,\alpha}(\overline{\Omega})$ *for all* $\alpha \in]0,1[$, *and* $w > 0$ *in* Ω, $\partial w/\partial n < 0$ *on* $\partial\Omega$.
(vi) (Order preserving property) *Given* $f, g \in L^p(\Omega)$, *if* $f \le g$ *in* Ω *then* $\varLambda f < \varLambda g$ *in* Ω.

REMARK 2.2.

$$\forall u \in X\ \forall v \in L^p(\Omega) \qquad v = -\varDelta u \ \Leftrightarrow\ u = \varLambda v.$$

Let N_p be the Nemytskiĭ operator defined by

$$N_p(v)(x) = \begin{cases} |v(x)|^{p-2}v(x) & \text{if } v(x) \ne 0, \\ 0 & \text{if } v(x) = 0. \end{cases} \tag{2.2}$$

We have

$$\forall v \in L^p(\Omega)\ \forall w \in L^{p'}(\Omega) \qquad N_p(v) = w \Leftrightarrow v = N_{p'}(w).$$

PROPOSITION 2.3 (cf. [D], [K]). *Let* $q, r \in [1,\infty[$. *If there exist* $c > 0$ *and* $b \in L^r(\Omega)$ *such that*

$$|f(x,\xi)| \le c|\xi|^{q/r} + b(x) \qquad \text{a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^m,$$

then N_f *is well defined from* $(L^q(\Omega))^m$ *to* $L^r(\Omega)$, *continuous and bounded. Moreover, if* $m = 1$ *and* $r = q' \ne 1$, *then the functional* $\Psi : L^q(\Omega) \to \mathbb{R}$, $\Psi(u) = \int_\Omega F(x,u)\,dx$, *where* $F(x,s) = \int_0^s f(x,t)\,dt$, *is well defined, of class* C^1 *on* $L^q(\Omega)$, *and* $\Psi'(u) = f(x,u)$ *for all* $u \in L^q(\Omega)$.

DEFINITION 2.4. Let E be a real Banach space and A be a closed, symmetric subset of $E \setminus \{0\}$. We define the *genus* of A to be the number

$$\gamma(A) = \inf\{m : \exists f \in C^0(A, \mathbb{R}^m \setminus \{0\})\ \forall u \in A,\ f(-u) = f(u)\},$$

and $\gamma(A) = \infty$ if no such f exists; $\gamma(\emptyset) = 0$ by definition.

LEMMA 2.5 (cf. [C], [R]). *Let* E *be a real Banach space and* A, B *be symmetric subsets of* $E \setminus \{0\}$ *which are closed in* E. *Then:*

(a) *If there exists an odd continuous mapping $f : A \to B$, then $\gamma(A) \leq \gamma(B)$.*
(b) *If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
(c) *$\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.*
(d) *If $\gamma(B) < \infty$, then $\gamma(\overline{A - B}) \geq \gamma(A) - \gamma(B)$.*
(e) *If A is compact, then $\gamma(A) < \infty$ and there exists a neighborhood N of A which is a symmetric subset of $E \setminus \{0\}$, closed in E, and such that $\gamma(N) = \gamma(A)$.*
(f) *If N is a symmetric and bounded neighborhood of the origin in $\mathbb{R}^k$ and if A is homeomorphic to the boundary of N by an odd homeomorphism, then $\gamma(A) = k$.*
(g) *If E_0 is a subspace of E of codimension k and if $\gamma(A) > k$, then $A \cap E_0 = \emptyset$.*

LEMMA 2.6 ([L, Corollary 4.1]). *Suppose that M is a closed symmetric C^1-submanifold of a real Banach space E and $0 \notin M$. Suppose also that $f \in C^1(M, \mathbb{R})$ is even and bounded below. Define*

$$c_j = \inf_{K \in \Gamma_j} \sup_{x \in K} f(x),$$

where

$$\Gamma_j = \{K \subset M : K \text{ is symmetric, compact and } \gamma(K) \geq j\}.$$

If $\Gamma_k \neq \emptyset$ for some $k \geq 1$ and if f satisfies $(\mathrm{PS})_c$ for all $c = c_j$, $j = 1, \ldots, k$, then f has at least k distinct pairs of critical points.

LEMMA 2.7 (cf. [AD]). *Let Ω be a domain of class C^1 in $\mathbb{R}^N$.*

(i) *If $p < N/2$, then $W^{2,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [1, p_2^*[$.*
(ii) *If $p = N/2$, then $W^{2,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [1, \infty[$.*
(iii) *If $p > N/2$, then $W^{2,p}(\Omega) \hookrightarrow C(\overline{\Omega})$.*

The above injections are compact, and

$$p_2^* = \begin{cases} \dfrac{Np}{N - 2p} & \text{if } p < N/2, \\ \infty & \text{if } p \geq N/2. \end{cases}$$

3. Third order spectrum of the p-biharmonic operator

DEFINITION 3.1. The set of couples $(\beta, \alpha) \in \mathbb{R}^N \times \mathbb{R}$ such that there exists a solution (β, α, u) of (1.1) is called the *third order spectrum* of the p-biharmonic operator.

The couple (β, α) is then called a *third-order eigenvalue* and u is said to be the associated eigenfunction.

A set of third-order eigenvalues of the form $(\beta, f(\beta))$, for $\beta \in \mathbb{R}^N$ and some function $f : \mathbb{R}^N \to \mathbb{R}$, is called an *eigensurface* of (1.1).

Lemma 3.2. *The problem* (1.1) *is equivalent to the problem*

$$
\begin{cases} \textit{Find } (\alpha, u) \in \mathbb{R}_+^* \times (X \setminus \{0\}) \textit{ such that} \\ \Delta_p^{2,\beta} u = \alpha m e^{\beta\cdot x}|u|^{p-2}u \quad \textit{in } \Omega, \\ u = \Delta u = 0 \quad \textit{on } \partial\Omega, \end{cases} \tag{3.1}
$$

where $\Delta_p^{2,\beta} u = \Delta(e^{\beta\cdot x}|\Delta u|^{p-2}\Delta u)$.

Proof. For all $\beta \in \mathbb{R}^N$,

$$
\begin{aligned}
\Delta(e^{\beta\cdot x}|\Delta u|^{p-2}\Delta u) &= \nabla[\nabla(e^{\beta\cdot x}|\Delta u|^{p-2}\Delta u)] \\
&= \nabla[\nabla(e^{\beta\cdot x})|\Delta u|^{p-2}\Delta u + e^{\beta\cdot x}\nabla(|\Delta u|^{p-2}\Delta u)] \\
&= e^{\beta\cdot x}[\Delta_p^2 u + 2\beta\cdot\nabla(|\Delta u|^{p-2}\Delta u) + |\beta|^2|\Delta u|^{p-2}\Delta u],
\end{aligned}
$$

hence (1.1) is equivalent to (3.1).

The operator Λ enables us to transform problem (3.1) to another problem which we shall study in the space $L^p(\Omega)$.

Lemma 3.3. *The problem* (3.1) *is equivalent to the problem*

$$
\begin{cases} \textit{Find } (\alpha, v) \in \mathbb{R}_+^* \times (L^p(\Omega) \setminus \{0\}) \textit{ such that} \\ e^{\beta\cdot x}N_p(v) = \alpha\Lambda(e^{\beta\cdot x} m N_p(\Lambda v)) \quad \textit{in } L^{p'}(\Omega). \end{cases} \tag{3.2}
$$

A pair $(\alpha, u) \in \mathbb{R}_+^* \times X \setminus \{0\}$ is a solution of problem (3.1) if and only if (α, v), where $v = -\Delta u$, is a solution of problem (3.2).

By using Ljusternik–Schnirelmann theory (cf. [S]), we will give a sequence of eigensurfaces of problem (1.1).

We consider the functionals $F_\beta, G_\beta : L^p(\Omega) \to \mathbb{R}$ defined

$$
F_\beta(v) = \frac{1}{p}\int_\Omega e^{\beta\cdot x}|v|^p\,dx, \qquad G_\beta(v) = \frac{1}{p}\int_\Omega e^{\beta\cdot x} m|\Lambda v|^p\,dx.
$$

F_β and G_β are of class C^1 in $L^p(\Omega)$ and for all $v \in L^p(\Omega)$,

$$
F'_\beta(v) = e^{\beta\cdot x}N_p(v), \qquad G'_\beta(v) = \Lambda(me^{\beta\cdot x}N_p(\Lambda v)) \quad \text{in } L^{p'}(\Omega).
$$

Set

$$
\begin{aligned}
\mathcal{M}_\beta &= \{v \in L^p(\Omega) : pG_\beta(v) = 1\}, \\
\Gamma_n &= \{K \subset \mathcal{M}_\beta : K \text{ is symmetric, compact and } \gamma(K) \ge n\},
\end{aligned}
$$

where $\gamma(K)$ indicates the genus of K.

Since $|\Omega^+| \neq 0$, there exists $v \in L^p(\Omega)$ such that $\int_\Omega me^{\beta\cdot x}|\Lambda v|^p\,dx = 1$, so $\mathcal{M}_\beta \neq \emptyset$. Furthermore $\mathcal{M}_\beta$ is a C^1-manifold.

For all $\beta \in \mathbb{R}^N$, define

$$
\Gamma_n^p(\beta, m) = \inf_{K\in\Gamma_n}\sup_{v\in K} pF_\beta(v).
$$

Lemma 3.4.

(i) *For all* $\beta \in \mathbb{R}^N$, F'_β *satisfies condition* (S_+), *i.e.*

$$v_n \rightharpoonup v \text{ in } L^p(\Omega) \quad \text{and} \quad \limsup_{n\to\infty} \int_\Omega F'_\beta(v_n)(v_n - v)\,dx \le 0$$

implies $v_n \to v$ *strongly in* $L^p(\Omega)$.

(ii) *For all* $\beta \in \mathbb{R}^N$, G'_β *is completely continuous in* $L^p(\Omega)$.

Proof. (i) Let (v_n) be a sequence in $L^p(\Omega)$ such that

$$v_n \rightharpoonup v \text{ in } L^p(\Omega) \quad \text{and} \quad \limsup_{n\to\infty} \langle e^{\beta\cdot x} N_p(v_n), v_n - v\rangle \le 0. \tag{3.3}$$

We have

$$\limsup_{n\to\infty} \int_\Omega e^{\beta\cdot x} N_p(v_n)(v_n - v)\,dx = \limsup_{n\to\infty} \int_\Omega e^{\beta\cdot x}(N_p(v_n) - N_p(v))(v_n - v)\,dx.$$

By the monotonicity of N_p and by Hölder's inequality we obtain

$$\int_\Omega e^{\beta\cdot x}(N_p(v_n) - N_p(v))(v_n - v)\,dx \ge C_0(\|v_n\|_p^{p-1} - \|v\|_p^{p-1})(\|v_n\|_p - \|v\|_p) \ge 0,$$

where $C_0 = \min\{e^{\beta\cdot x} : x \in \overline{\Omega}\}$. Hence (3.3) implies that $\|v_n\|_p \to \|v\|_p$. Since $v_n \rightharpoonup v$ in $L^p(\Omega)$ and $L^p(\Omega)$ is uniformly convex, $v_n \to v$ in $L^p(\Omega)$.

(ii) Let (v_n) be a sequence in $L^p(\Omega)$ such that $v_n \rightharpoonup v$ in $L^p(\Omega)$. By Lemma 2.1(i) we obtain $\Lambda v_n \rightharpoonup \Lambda v$ in X. Sobolev's embedding theorem (cf. Lemma 2.7) and the properties of the Nemytskiĭ operator N_p (cf. Proposition 2.3) imply that $\Lambda v_n \to \Lambda v$ in $L^p(\Omega)$ and $G'_\beta(v_n) \to G'_\beta(v)$ in $L^{p'}(\Omega)$.

Lemma 3.5. *For all* $\beta \in \mathbb{R}^N$:

(i) F_β *is* C^1 *in* $\mathcal{M}_\beta$, *even and bounded below.*

(ii) *For all* $n \in \mathbb{N}^*$, $\Gamma_n \neq \emptyset$.

(iii) *The functional* F_β *satisfies* $(\mathrm{PS})_c$ *on* $\mathcal{M}_\beta$ *for every* $c \neq 0$.

Proof. (i) is evident. (ii) Since $|\Omega^+| \neq 0$, for all $n \in \mathbb{N}^*$ there exist $u_1, \ldots, u_n \in X$ which satisfy

$$\begin{cases} \operatorname{supp} u_i \cap \operatorname{supp} u_j = \emptyset & \text{if } i \neq j, \\ \int_\Omega m e^{\beta\cdot x}|u_i|^p\,dx = 1, & \end{cases} \qquad i, j \in \{1, \ldots, n\}.$$

For all $i \in \{1, \ldots, n\}$, there exists $v_i \in L^p(\Omega)$ such that $u_i = \Lambda v_i$.

Let $F_n = \operatorname{span}\{v_1, \ldots, v_n\} \subset L^p(\Omega)$. Then

$$\forall v \in F_n\ \exists(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \quad v = \sum_{i=1}^n \alpha_i v_i$$

and

$$\int_\Omega me^{\beta\cdot x}|\Lambda v|^p = \int_\Omega me^{\beta\cdot x}\Big|\sum_{i=1}^n \alpha_i \Lambda v_i\Big|^p = \sum_{i=1}^n |\alpha_i|^p \int_\Omega me^{\beta\cdot x}|\Lambda v_i|^p$$
$$= \sum_{i=1}^n |\alpha_i|^p > 0.$$

It follows that the map $v \mapsto (pG_\beta(v))^{1/p}$ defines a norm on F_n. Hence $S_\beta = F_n \cap \mathcal{M}_\beta$ is the unit sphere of F_n which is homeomorphic to the unit sphere of $\mathbb{R}^N$ and this homeomorphism is odd. Then Lemma 2.5(f) yields $\gamma(S_\beta) = n$. Therefore $S_\beta \in \Gamma_n$.

(iii) Let (v_n) be a sequence in $\mathcal{M}_\beta$ and (t_n) be a sequence in $\mathbb{R}$ such that

$$F_\beta(v_n) \to c \quad \text{and} \quad F'_\beta(v_n) - t_n G'_\beta(v_n) \to 0. \tag{3.4}$$

We will show that (v_n) has a subsequence which converges strongly in $L^p(\Omega)$. We have $(C_0/p)\|v_n\|_p^p \le F_\beta(v_n)$, so (v_n) is bounded in $L^p(\Omega)$. Hence for a subsequence still denoted by (v_n) we have $v_n \rightharpoonup v$ in $L^p(\Omega)$. As G'_β is completely continuous, we have $G'_\beta(v_n) \to G'_\beta(v)$ in $L^{p'}(\Omega)$.

It follows from (3.4) that

$$\langle F'_\beta(v_n) - t_n G'_\beta(v_n), v_n\rangle = pF_\beta(v_n) - t_n \to 0$$

and $t_n \to pc$. So the sequence $(F'_\beta(v_n))$ is strongly convergent in $L^{p'}(\Omega)$. Hence

$$\lim_{n\to\infty} \int_\Omega F'_\beta(v_n)(v_n - v)\,dx = 0.$$

Since F'_β is of type (S_+), we have $v_n \to v$ in $L^p(\Omega)$.

Our main result is the following theorem:

THEOREM 3.6. *Problem* (1.1) *has a sequence of positive eigensurfaces* $(G(\Gamma_n^p(\cdot, m)))_{n\ge1}$, *where* $G(\Gamma_n^p(\cdot, m))$ *is the graph of the function* $\Gamma_n^p(\cdot, m)$. *Moreover, we have*

$$\forall \beta \in \mathbb{R}^N \qquad \Gamma_n^p(\beta, m) = \inf_{K\in\mathcal{B}_n} \sup_{u\in K} \int_\Omega e^{\beta\cdot x}|\Delta u|^p\,dx \tag{3.5}$$

where

$$\mathcal{B}_n = \{K \subset \mathcal{N}_\beta : K \text{ is compact, symmetric and } \gamma(K) \ge n\},$$
$$\mathcal{N}_\beta = \Big\{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : \int_\Omega me^{\beta\cdot x}|u|^p\,dx = 1\Big\}$$

and

$$\Gamma_n(\beta, m) \to \infty \qquad \text{as } n \to \infty. \tag{3.6}$$

Proof. Lemmas 2.6 and 3.5 enable us to claim that $(G(\Gamma_n^p(\cdot, m)))_{n\ge1}$ is an infinite sequence of positive eigensurfaces of problem (3.2).

We now prove (3.6). Since $L^p(\Omega)$ is separable, there exists a biorthogonal system $(e_i, e_j^*)_{i,j\in\mathbb{N}}$ such that $e_i \in L^p(\Omega)$, $e_j^* \in L^{p'}(\Omega)$. The e_i's are linearly dense in $L^p(\Omega)$ and the e_j^*'s are total in $L^{p}(\Omega)$ (cf. [L]).

For $n\in\mathbb{N}^*$, set $F_n = \operatorname{span}\{e_1, \ldots, e_n\}$ and $F_n^\perp = \operatorname{cl}\operatorname{span}\{e_{n+1}, e_{n+2}, \ldots\}$, where cl denotes closure. From Lemma 2.5(g) we deduce that $K \cap F_{n-1}^\perp \neq \emptyset$ for any $K \in \Gamma_n$.

We claim that

$$d_n = \inf_{K\in\Gamma_n} \sup_{v\in K\cap F_{n-1}^\perp} pF_\beta(v) \to \infty \quad \text{ as } n \to \infty.$$

Indeed, if not, there exists $M > 0$ such that for every $n \in \mathbb{N}^*$ there exists $v_n \in F_{n-1}^\perp$ with $pG_\beta(v_n) = 1$ and $d_n \le pF_\beta(v_n) \le M$. We deduce that (v_n) is bounded in $L^p(\Omega)$. Thus for a subsequence still denoted by (v_n), $v_n \rightharpoonup v$ in $L^p(\Omega)$. Since G'_β is completely continuous we get

$$G'_\beta(v_n) \to G'_\beta(v) \text{ in } L^{p'}(\Omega) \quad \text{and} \quad \lim_{n\to\infty} \langle G'_\beta(v_n), v_n\rangle = \langle G'_\beta(v), v\rangle.$$

Then $pG_\beta(v_n) \to pG_\beta(v)$. The fact that $pG_\beta(v_n) = 1$ implies $pG_\beta(v) = 1$.

On the other hand, for every $n \ge j$, $\langle e_j^*, e_n\rangle = 0$. Hence $v_n \rightharpoonup 0$, therefore $v = 0$ and $G_\beta(v) = 0$, which leads to a contradiction. Since $\Gamma_n^p(\beta, m) \ge d_n$, we get (3.6).

Finally we verify (3.5). We know that $-\Delta : X \to L^p(\Omega)$ and $\Lambda : L^p(\Omega) \to X$ are odd homeomorphisms. Consequently, by the properties of genus we get: $K \in \Gamma_n \Leftrightarrow \Lambda K \in \mathcal{B}_n$. Hence for every $n \in \mathbb{N}^*$, $\Gamma_n^p(\beta, m) = \inf_{K\in\mathcal{B}_n} \sup_{u\in K} \int_\Omega e^{\beta\cdot x} |\Delta u|^p \, dx$.

Remark 3.7. We can prove that the spectrum of (1.1) is closed and the set of eigenfunctions associated with the same eigensurface of problem (1.1) is compact.

References

[AD] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[AG] S. Agmon, A. Douglis and L. Nirenberg , *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I*, Comm. Pure Appl. Math. 12 (1959), 623–727.

[A] A. Anane, O. Chakrone et J.-P. Gossez, *Spectre d'ordre supérieur et problèmes aux limites quasi-linéaires*, Boll. Un. Mat. Ital. B (8) 4 (2001), 483–519.

[B] J. Benedikt, *On the discreteness of the spectra of the Dirichlet and Neumann p-biharmonic problems*, Abstr. Appl. Anal. 2004, 777–792.

[C] S. N. Chow and J. K. Hale, *Methods of Bifurcation Theory*, Springer, New York, 1982.

[D] D. de Figueiredo, *Lectures on the Ekeland Variational Principle with Applications and Detours*, Tata Inst. Fundam. Res., Bombay, and Springer, Berlin, 1989.

[DR] P. Drábek and M. Ôtani, *Global bifurcation result for the p-biharmonic operator*, Electron. J. Differential Equations 2001, no. 48, 19 pp.

[E] A. El Khalil, S. Kellati and A. Touzani, *On the spectrum of the p-biharmonic operator*, Electron. J. Differential Equations Conf. 9 (2002), 161–170.

[G] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer, New York, 1983.

[I] T. Idogawa and M. Ôtani, *The first eigenvalues of some abstract elliptic operator*, Funkcial. Ekvac. 38 (1995), 1–9.

[K] M. A. Krasnosel'skiĭ, *Topological Methods in the Theory of Nonlinear Integral Equations*, Macmillan, New York, 1964.

[L] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer, Berlin, 1977.

[R] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conf. Ser. Math. 65, Amer. Math. Soc., Providence, RI, 1986.

[S] A. Szulkin, *Ljusternik–Schnirelmann theory on $\mathcal{C}^1$ manifolds*, Ann. Inst. H. Poincaré Anal. Non Linéaire 5 (1988), 119–139.

[T] M. Talbi and N. Tsouli, *On the spectrum of the weighted p-biharmonic operator with weight*, Mediterr. J. Math. 4 (2007), 73–86.

Khalil Ben Haddouch, Najib Tsouli
Department of Mathematics
University Mohamed I
Oujda, Morocco
E-mail: ayasch1@hotmail.com
tsouli@hotmail.com

Zakaria El Allali
Department of Mathematics
University Mohamed I
Oujda, Multidisciplinary Faculty of Nador, Morocco
E-mail: elallali@hotmail.com

Received on 25.2.2014;
revised version on 18.7.2014 and 23.9.2014 (2231)