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THREE SOLUTIONS FOR A NONLINEAR NEUMANN BOUNDARY VALUE PROBLEM

Abstract. The aim of this paper is to establish the existence of at least three solutions for the nonlinear Neumann boundary-value problem involving the $p(x)$-Laplacian of the form

$$-\Delta_{p(x)}u + a(x)|u|^{p(x)-2}u = \mu g(x,u) \quad \text{in } \Omega,$$
$$|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \lambda f(x,u) \quad \text{on } \partial\Omega.$$

Our technical approach is based on the three critical points theorem due to Ricceri.

1. Introduction. In this paper we are interested in the multiplicity of weak solutions of the following nonlinear Neumann boundary-value problem involving the $p(x)$-Laplacian:

$$\text{(P)} \qquad \begin{cases} -\Delta_{p(x)}u + a(x)|u|^{p(x)-2}u = \mu g(x,u) & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \nu} = \lambda f(x,u) & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a smooth boundary $\partial\Omega$, $\Delta_{p(x)} = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator with $p \in C(\overline{\Omega})$ and $p(x) > 1$ for every $x \in \overline{\Omega}$, $\partial u/\partial\nu$ is the outer unit normal derivative on $\partial\Omega$, $\lambda > 0$ and $\mu \geq 0$ are real numbers, and $a \in L^\infty(\Omega)$ with $a^- := \operatorname{ess\,inf}_{x\in\Omega} a(x) > 0$. We denote $p^- := \inf_{x\in\overline{\Omega}} p(x) > 1$ and $p^+ := \inf_{x\in\overline{\Omega}} p(x)$. Throughout this paper we assume the following assumptions:

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(F1) $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies

$$|f(x,s)| \le h_1(x) + b_1|s|^{\alpha(x)-1}, \quad \forall (x,s) \in \partial\Omega \times \mathbb{R},$$

where $h_1(\cdot)$ is in $L^{\frac{\alpha(\cdot)}{\alpha(\cdot)-1}}(\partial\Omega)$, $b_1 \ge 0$ is a constant, $\alpha(\cdot) \in C(\overline{\Omega})$, and $1 < \alpha^- := \inf_{x\in\overline{\Omega}} \alpha(x) \le \alpha^+ := \sup_{x\in\overline{\Omega}} \alpha(x) < p^-$.

(F2) There exists a constant $t_0 > 1$ such that $f(x,t) < 0$ when $|t| \in (0,1)$, and $f(x,t) \ge m > 0$ when $|t| \in (t_0,\infty)$, where m is a positive constant.

(G) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies

$$|g(x,s)| \le h_2(x) + b_2|s|^{\beta(x)-1}, \quad \forall (x,s) \in \Omega \times \mathbb{R},$$

where $h_2(\cdot)$ is in $L^{\frac{\beta(\cdot)}{\beta(\cdot)-1}}(\Omega)$, $b_2 \ge 0$ is a constant, $\beta(\cdot) \in C(\overline{\Omega})$, and $1 < \beta^- := \inf_{x\in\overline{\Omega}} \beta(x) \le \beta^+ := \sup_{x\in\overline{\Omega}} \beta(x) < p^-$.

The main result of this paper is the following

THEOREM 1.1. *Assume that $p^- > N$ and f satisfy* (F1)–(F2). *Then there exist an open interval $\Lambda \subset (0,\infty)$ and a positive real number ρ such that for each $\lambda \in \Lambda$ and every function g satisfying* (G), *the problem* (P) *has at least three solutions whose norms are less than ρ.*

The study of differential equations involving the $p(x)$-Laplacian has received considerable attention in recent years. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics and calculus of variations; for information on modeling physical phenomena by equations involving the $p(x)$-growth condition we refer the reader to [D, H, My, PMBD, R, W, Z]. Recently elliptic problems with nonlinear boundary conditions have attracted much interest: for example, see [AEO, DS, DW, Y, TD].

In [DS], the authors obtained the existence of an unbounded sequence of weak solutions for problem (P). In [DW], the authors considered problem (P) in the case $\lambda = \mu = 1$ and $a(x) =$ constant and obtained nonexistence, existence and multiplicity results. In [Y], the author obtained a number of interesting results on existence and multiplicity of solutions of problem (P) when $a(x) = 1$, using the variational method. In [AEO], the authors obtained the existence of three solutions for problem (P) when $g(x,u) = 0$ and $a(x) = 1$ using the three critical points theorem established by Ricceri.

Following the same lines as in [AEO, M, SD], we will prove that there exist three weak solutions of problem (P) using a version of Ricceri's three critical points theorem.

This article is organized as follows. In Section 2, we recall some basic results on the theory of Lebesgue–Sobolev spaces with variable exponent.

We also recall a version of the three critical points theorem due to Ricceri. In Section 3, we give the proof of our main result.

2. Preliminaries. For completeness, we first recall some facts on the variable exponent spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$. For more details, see [ER1, ELN, ER2, KR, FZS, FZ1, FZ2]. Suppose that Ω is a bounded open domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $p \in C_+(\overline{\Omega})$ where

$$C_+(\overline{\Omega}) = \Big\{p \in C(\overline{\Omega}) : \inf_{x\in\overline{\Omega}} p(x) > 1\Big\}.$$

Denote $p^- := \inf_{x\in\overline{\Omega}} p(x)$ and $p^+ := \sup_{x\in\overline{\Omega}} p(x)$. Define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ by

$$L^{p(\cdot)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ measurable: } \int_\Omega |u|^{p(x)}\,dx < \infty\Big\},$$

with the norm

$$|u|_{p(x)} = \inf\Big\{\tau > 0 : \int_\Omega \Big|\frac{u}{\tau}\Big|^{p(x)} dx \le 1\Big\}.$$

Define the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

with the norm

$$\|u\| = \inf\Big\{\tau > 0 : \int_\Omega \Big(\Big|\frac{\nabla u}{\tau}\Big|^{p(x)} + \Big|\frac{u}{\tau}\Big|^{p(x)}\Big)\,dx \le 1\Big\} = |\nabla u|_{p(\cdot)} + |u|_{p(\cdot)}.$$

Lemma 2.1 (see [FZ1]). *Both $(L^{p(\cdot)}(\Omega), |\cdot|_{p(x)})$ and $(W^{1,p(\cdot)}(\Omega), \|\cdot\|)$ are separable, reflexive and uniformly convex Banach spaces.*

Lemma 2.2 (see [FZ1]). *The Hölder inequality holds, namely*

$$\int_\Omega |uv|\,dx \le \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(\cdot)}|v|_{q(\cdot)} \quad \forall u \in L^{p(\cdot)}(\Omega), v \in L^{q(\cdot)}(\Omega),$$

where $1/p(x) + 1/q(x) = 1$.

Hereafter, let

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N-p(x)} & \text{if } p(x) < N,\\ +\infty & \text{if } p(x) \ge N, \end{cases} \qquad p^\partial(x) = \begin{cases} \dfrac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N,\\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

Lemma 2.3 (see [Y, FZ1, FZ2]). *Assume that the boundary of Ω has the cone property.*

(1) *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the imbedding of $W^{1,p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$ is compact and continuous.*

(2) *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) < p^{\partial}(x)$ *for any* $x \in \partial\Omega$, *then the trace imbedding of* $W^{1,p(\cdot)}(\Omega)$ *into* $L^{q(\cdot)}(\partial\Omega)$ *is compact and continuous.*

An important role in manipulating generalized Lebesgue–Sobolev spaces is played by the mapping defined in the following

LEMMA 2.4 (see [KR, De, FZ1]). *Let*

$$I(u) = \int_{\Omega} (|\nabla u|^{p(x)} + a(x)|u|^{p(x)})\,dx.$$

For $u \in W^{1,p(\cdot)}(\Omega)$ *we have*

- $\|u\| < 1\ (=1, >1) \Leftrightarrow I(u) < 1\ (=1, >1)$.
- $\|u\| \le 1 \Rightarrow \|u\|^{p^+} \le I(u) \le \|u\|^{p^-}$.
- $\|u\| \ge 1 \Rightarrow \|u\|^{p^-} \le I(u) \le \|u\|^{p^+}$.

REMARK 2.5 (see [M]). If $N < p^- < p(x)$ for any $x \in \overline{\Omega}$ then by Theorem 2.2 in [FZ2], we deduce that $W^{1,p(\cdot)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$. Since $N < p^-$ it follows that $W^{1,p^-}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Thus, $W^{1,p(\cdot)}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Defining $\|u\|_\infty = \sup_{x\in\overline{\Omega}} u(x)$, we find that there exists a positive constant k such that

$$\|u\|_\infty \le k\|u\|_a, \quad \forall u \in W^{1,p(\cdot)}(\Omega).$$

PROPOSITION 2.6 (see [AEO]). *Suppose* $f : \partial\Omega\times\mathbb{R}\to\mathbb{R}$ *is a Carathéodory function and*

$$|f(x,s)| \le h(x) + b|s|^{\alpha(x)-1} \quad \textit{for all } (x,s) \in \partial\Omega \times \mathbb{R},$$

where $h(\cdot) \in L^{\frac{\alpha(\cdot)}{\alpha(\cdot)-1}}(\partial\Omega)$ *and* $b \ge 0$ *is a constant,* $\alpha(\cdot) \in C_+(\partial\Omega)$ *such that* $\alpha(x) < p^{\partial}(x)$ *for all* $x \in \partial\Omega$. *Set*

$$X = W^{1,p(\cdot)}(\Omega), \quad F(x,u) = \int_0^u f(x,t)\,dt, \quad \psi(u) = -\int_{\partial\Omega} F(x,u(x))\,d\sigma.$$

Then $\psi(\cdot) \in C^1(X,\mathbb{R})$ *and*

$$D\psi(u,\varphi) = \langle \psi'(u), \varphi\rangle = -\int_{\partial\Omega} f(x,u(x))\varphi\,d\sigma,$$

moreover, the operator $\psi' : X \to X^*$ *is compact.*

Finally, to prove our result in the next section, we use the following theorem. It is equivalent to the three critical points theorem of Ricceri [Ri]. (See also [B, Theorem 2.3].)

THEOREM 2.7. *Let* X *be a separable and reflexive real Banach space;* $\phi : X \to \mathbb{R}$ *a continuously Gâteaux differentiable and sequentially weakly* *lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on* X^*; *and* $\psi : X \to \mathbb{R}$ *a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that:*

(i) $\lim_{\|u\|_X \to \infty} (\phi(u) + \lambda\psi(u)) = \infty$ *for all* $\lambda > 0$,

(ii) *there are* $r \in \mathbb{R}$ *and* $u_0, u_1 \in X$ *such that* $\phi(u_0) < r < \phi(u_1)$,

(iii) $$\inf_{u \in \phi^{-1}((-\infty, r])} \psi(u) > \frac{(\phi(u_1) - r)\psi(u_0) + (r - \phi(u_0))\psi(u_1)}{\phi(u_1) - \phi(u_0)}.$$

Then there exist an open interval $\Lambda \subset (0,\infty)$ *and a real number* $\rho > 0$ *with the following property: for every* $\lambda \in \Lambda$ *and every* C^1 *functional* $J : X \to \mathbb{R}$ *with compact derivative, there exists* $\delta > 0$ *such that for each* $\mu \in [0, \delta]$ *the equation* $\phi'(u) + \lambda\psi'(u) + \mu J'(u) = 0$ *has at least three solutions in* X *whose norms are less than* ρ.

3. Proof of the main result. In this part, we will prove that for problem (P), there exist at least three weak solutions, by using Theorem 2.7.

Definition 3.1. $u \in W^{1,p(\cdot)}(\Omega)$ is called a *weak solution* of problem (P) if for all $v \in W^{1,p(\cdot)}(\Omega)$,

$$\int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla v + a(x)|u|^{p(x)-2} uv\right) dx = \lambda \int_{\partial\Omega} f(x,u)v \, d\sigma_x + \mu \int_\Omega g(x,u)v \, dx.$$

Let X denote the generalized Sobolev space $W^{1,p(\cdot)}(\Omega)$. Define

$$F(x,t) := \int_0^t f(x,s)\, ds \quad \text{and} \quad G(x,t) := \int_0^t g(x,s)\, ds,$$

and the functionals

$$\phi(u) = \int_\Omega \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + a(x)|u|^{p(x)}\right) dx,$$

$$\psi(u) = -\int_{\partial\Omega} F(x,u)\, d\sigma_x, \qquad J(u) = -\int_\Omega G(x,u)\, dx, \qquad \forall u \in X.$$

Arguments similar to those used in the proof of [MR, Proposition 3.1], and Proposition 2.6, imply that ϕ, ψ and J are C^1-functionals on $W^{1,p(\cdot)}(\Omega)$ with the derivatives given by

$$(\phi'(u), v) = \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla v + a(x)|u|^{p(x)-2} uv\right) dx,$$

$$(\psi'(u), v) = -\int_{\partial\Omega} f(x,u)v\, d\sigma_x, \qquad (J'(u), v) = -\int_\Omega g(x,u)v\, dx,$$

for any $u, v \in X$. (See also [FD].) Thus, there exist $\lambda, \mu > 0$ such that u is a critical point of the operator $\phi(u) + \lambda\psi(u) + \mu J(u)$, that is, $\phi'(u) + \lambda\psi'(u) + \mu J'(u) = 0$. To prove our result, it is enough to verify that ϕ, ψ and J satisfy the hypotheses of Theorem 2.7.

It is obvious that $(\phi')^{-1} : X^* \to X$ exists and is continuous, because $\phi' : X \to X^*$ is a homeomorphism by [CCD, Lemma 3.1]. Moreover $J' : X \to X^*$ is completely continuous because of the assumption (G) and [KR], which implies J' is compact. ψ' is also compact according to (F1) and Proposition 2.6.

Next, we will verify that condition (i) of Theorem 2.7 is fulfilled. In fact, by Lemma 2.4, we have

$$\begin{aligned}\phi(u) &\geq \frac{1}{p^+}\int_{\Omega}(|\nabla u|^{p(x)} + a(x)|u|^{p(x)})\,dx \\ &= \frac{1}{p^+}I(u) \geq \frac{1}{p^+}\|u\|_a^{p^-}, \qquad u \in X,\ \|u\|_a^{p^-} > 1.\end{aligned}$$

On the other hand, for $u \in X$ such that $\|u\|_a \geq 1$, we have

$$\begin{aligned}\psi(u) &= -\int_{\partial\Omega} F(x,u)\,d\sigma = -\int_{\partial\Omega}\Big(\int_0^{u(x)} f(x,t)\,dt\Big)\,d\sigma_x \\ &\leq \int_{\partial\Omega}\Big(h_1(x)|u(x)| + \frac{b_1}{\alpha(x)}|u|^{\alpha(x)}\Big)\,d\sigma_x.\end{aligned}$$

Using the Hölder inequality and the Sobolev embedding theorem, we have for some positive constants C and C',

$$\int_{\partial\Omega} h_1(x)|u(x)|\,d\sigma \leq 2\|h_1\|_{L^{\frac{\alpha(\cdot)}{\alpha(\cdot)-1}}(\partial\Omega)}\|u\|_{L^{\alpha(\cdot)}(\partial\Omega)} \leq 2C\|h_1\|_{L^{\frac{\alpha(\cdot)}{\alpha(\cdot)-1}}(\partial\Omega)}\|u\|_a,$$

and

$$\int_{\partial\Omega}|u|^{\alpha(x)}\,d\sigma_x \leq \max\{\|u\|^{\alpha^+}_{L^{\alpha(\cdot)}(\partial\Omega)}, \|u\|^{\alpha^-}_{L^{\alpha(\cdot)}(\partial\Omega)}\} \leq C'\|u\|_a^{\alpha^+}.$$

Altogether we obtain

$$|\psi(u)| \leq 2C\|h_1\|_{L^{\frac{\alpha(\cdot)}{\alpha(\cdot)-1}}(\partial\Omega)}\|u\|_a + \frac{b_1}{\alpha^-}C'\|u\|_a^{\alpha^+}.$$

Consequently, for any $\lambda > 0$ we have

$$\phi(u) + \lambda\psi(u) \geq \frac{1}{p^+}\|u\|_a^{p^-} - 2\lambda C\|h_1\|_{L^{\frac{\alpha(\cdot)}{\alpha(\cdot)-1}}(\partial\Omega)}\|u\|_a - \frac{\lambda b_1 C'}{\alpha^-}\|u\|_a^{\alpha^+}.$$

For $p^- > \alpha^+$ we have

$$\lim_{\|u\|_a\to\infty}(\phi(u) + \lambda\psi(u)) = \infty,$$

and (i) of Theorem 2.7 is verified.

It remains to verify conditions (ii) and (iii) in Theorem 2.7. By (F2), it is clear that $F(x,t)$ is increasing for $t \in (t_0,\infty)$ and decreasing for $t \in (0,1)$ uniformly with respect to $x \in \partial\Omega$, and $F(x,0) = 0$ is obvious. Moreover $F(x,t) \to \infty$ when $t \to \infty$ because $F(x,t) \geq mt$ uniformly for $x \in \partial\Omega$. Then there exists a real number $\delta > t_0$ such that

$$F(x,t) \geq 0 = F(x,0) \geq F(x,\tau) \quad \forall u \in X,\ t > \delta, \tau \in (0,1).$$

Let c, b be real numbers such that $0 < c < \min\{1,k\}$ where k is given in Remark 2.5, and $b > \delta$ satisfies

$$b^{p^-} \int_\Omega a(x)\,dx > 1.$$

When $t \in [0,c]$ we have $F(x,t) \leq F(x,0) = 0$. Then

$$\int_{\partial\Omega} \sup_{0<t<c} F(x,t)\,d\sigma \leq \int_{\partial\Omega} F(x,0)\,d\sigma_x = 0.$$

Furthermore, since $b > \delta$ we have

$$\int_{\partial\Omega} F(x,b)\,d\sigma_x > 0.$$

Moreover,

$$\frac{1}{k^{p^+}} \frac{c^{p^+}}{b^{p^-}} \int_{\partial\Omega} F(x,b)\,d\sigma_x > 0.$$

This implies

$$\int_{\partial\Omega} \sup_{0<t<c} F(x,t)\,d\sigma_x \leq 0 < \frac{1}{k^{p^+}} \frac{c^{p^+}}{b^{p^-}} \int_{\partial\Omega} F(x,b)\,d\sigma_x.$$

Let $u_0, u_1 \in X$, where $u_0(x) = 0$ and $u_1(x) = b$ for any $x \in \overline{\Omega}$. We define

$$r = \frac{1}{p^+}\left(\frac{c}{k}\right)^{p^+}.$$

Clearly $r \in (0,1)$, $\phi(u_0) = \psi(u_0) = 0$,

$$\phi(u_1) = \int_\Omega \frac{a(x)}{p(x)} b^{p(x)}\,dx \geq \frac{b^{p^-}}{p^+} \int_\Omega a(x)\,dx > \frac{1}{p^+}\cdot 1 > \frac{1}{p^+}\left(\frac{c}{k}\right)^{p^+} = r,$$

and

$$\psi(u_1) = -\int_{\partial\Omega} F(x,u_1(x))\,d\sigma_x = -\int_{\partial\Omega} F(x,b)\,d\sigma_x < 0.$$

So we have $\phi(u_0) < r < \phi(u_1)$. Thus (ii) of Theorem 2.7 is verified.

On the other hand, we have

$$-\frac{(\phi(u_1)-r)\psi(u_0)+(r-\phi(u_0))\psi(u_1)}{\phi(u_1)-\phi(u_0)} = -r\frac{\psi(u_1)}{\phi(u_1)} = r\frac{\int_{\partial\Omega} F(x,b)\,d\sigma_x}{\int_\Omega \frac{a(x)}{p(x)} b^{p(x)}\,dx} > 0.$$

Let $u \in X$ be such that $\phi(u) \le r < 1$. Set $I(u) = \int_\Omega (|\nabla u|^{p(x)} + a(x)|u|^{p(x)})\,dx$. Since $\frac{1}{p^+} I(u) \le \phi(u) \le r$, for $u \in W^{1,p(\cdot)}(\Omega)$, we obtain

$$I(u) \le p^+ r = \left(\frac{c}{k}\right)^{p^+} < 1.$$

It follows that $\|u\|_a < 1$ by Lemma 2.4. Furthermore, we have

$$\frac{1}{p^+}\|u\|_a^{p^+} \le \frac{1}{p^+} I(u) \le \phi(u) \le r.$$

Thus, using Remark 2.5, we obtain

$$|u(x)| \le k\|u\|_a \le k(p^+ r)^{1/p^+} = c \quad \forall u \in X,\ x \in \overline{\Omega},\ \phi(u) \le r.$$

The above inequality shows that

$$-\inf_{u\in\phi^{-1}((-\infty,r])} \psi(u) = \sup_{u\in\phi^{-1}((-\infty,r])} -\psi(u) \le \int_{\partial\Omega} \sup_{0<t<c} F(x,t)\,d\sigma_x \le 0.$$

Then

$$\inf_{u\in\phi^{-1}((-\infty,r])} \psi(u) > \frac{(\phi(u_1)-r)\psi(u_0)+(r-\phi(u_0))\psi(u_1)}{\phi(u_1)-\phi(u_0)}.$$

This means that condition (iii) in Theorem 2.7 is satisfied. Thus the proof of Theorem 1.1 is complete.

Remark 3.2. Applying [B, Theorem 2.1] in the proof of Theorem 1.1, an upper bound of the interval of parameters λ for which (P) has at least three weak solutions is obtained when $\mu = 0$. To be precise, in the conclusion of Theorem 1.1 one has

$$\Lambda \subseteq \left]0, h\frac{\int_\Omega \frac{a(x)}{p(x)} b^{p(x)}\,dx}{\int_{\partial\Omega} F(x,b)\,d\sigma_x}\right[$$

for each $h > 1$ and b as in the proof of Theorem 1.1.

Remark 3.3. We observe that the roles of the functions f and g can be reversed. For instance, we can study the problem

$$(\mathrm{P}')\qquad \begin{cases} -\Delta_{p(x)} u + a(x)|u|^{p(x)-2}u = \lambda f(x,u) & \text{in } \Omega,\\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \nu} = \mu g(x,u) & \text{on } \partial\Omega, \end{cases}$$

and consider the assumptions:

(F′1) $|f(x,s)| \le h_1(x) + b_1|s|^{\alpha(x)-1}$ for all $(x,s) \in \Omega \times \mathbb{R}$, where $h_1(\cdot)$ is in $L^{\frac{\alpha(\cdot)}{\alpha(\cdot)-1}}(\Omega)$, $b_1 \ge 0$ is a constant, $\alpha(\cdot) \in C(\overline{\Omega})$, $1 < \alpha^- := \inf_{x\in\overline{\Omega}} \alpha(x) \le \alpha^+ := \sup_{x\in\overline{\Omega}} \alpha(x) < p^-$.

(F′2) There exists a constant $t_0 > 1$ such that $f(x,t) < 0$ when $|t| \in (0,1)$, and $f(x,t) \ge m > 0$ when $|t| \in (t_0,\infty)$, where m is a positive constant.

(G′) $|g(x,s)| \le h_2(x) + b_2|s|^{\beta(x)-1}$ for all $(x,s) \in \partial\Omega \times \mathbb{R}$, where $h_2(\cdot)$ is in $L^{\frac{\beta(\cdot)}{\beta(\cdot)-1}}(\partial\Omega)$, $b_2 \ge 0$ is a constant, $\beta(\cdot) \in C(\overline{\Omega})$, $1 < \beta^- := \inf_{x\in\overline{\Omega}} \beta(x) \le \beta^+ := \sup_{x\in\overline{\Omega}} \beta(x) < p^-$.

Then one can easily obtain a similar result, namely

THEOREM 3.4. *Assume that $p^- > N$ and let f satisfy* (F′1)–(F′2). *Then there exist an open interval $\Lambda \subset (0,\infty)$ and a positive real number ρ such that for each $\lambda \in \Lambda$ and every function g satisfying* (G′), *problem* (P′) *has at least three solutions whose norms are less than ρ.*

References

[AEO] M. Allaoui, A. El Amrouss and A. Ourraoui, *Existence and multiplicity for a Steklov problem involving the $p(x)$-Laplace operator*, Electron. J. Differential Equations 2012, no. 132, 12 pp.

[B] G. Bonanno, *Some remarks on a three critical points theorem*, Nonlinear Anal. 54 (2003), 651–665.

[CCD] F. Cammaroto, A. Chinni and B. Di Bella, *Multiple solutions for a Neumann problem involving the $p(x)$-Laplacian*, Nonlinear Anal. 71 (2009), 4486–4492.

[DS] G. D'Aguì and A. Sciammetta, *Infinitely many solutions to elliptic problems with variable exponent and nonhomogeneous Neumann conditions*, Nonlinear Anal. 75 (2012), 5612–5619.

[De] S. G. Deng, *A local mountain pass theorem and applications to a double perturbed $p(x)$-Laplacian equations*, Appl. Math. Comput. 211 (2009), 234–241.

[DW] S. G. Deng and Q. Wang, *Nonexistence, existence and multiplicity of positive solutions to the $p(x)$-Laplacian nonlinear Neumann boundary value problem*, Nonlinear Anal. 73 (2010), 2170–2183.

[D] L. Diening, *Theoretical and numerical results for electrorheological fluids*, Ph.D. thesis, Univ. of Freiburg, 2002.

[ELN] D. E. Edmunds, J. Lang and A. Nekvinda, *On $L^{p(x)}$ norms*, Proc. Roy. Soc. London Ser. A 455 (1999), 219–225.

[ER1] D. E. Edmunds and J. Rákosník, *Density of smooth functions in $W^{1,p(x)}$*, Proc. Roy. Soc. London Ser. A 437 (1992), 229–236.

[ER2] D. E. Edmunds and J. Rákosník, *Sobolev embedding with variable exponent*, Studia Math. 143 (2000), 267–293.

[FD] X. L. Fan and S. G. Deng, *Remarks on Ricceri's variational principle and applications to the $p(x)$-Laplacian equations*, Nonlinear Anal. 67 (2007), 3064–3075.

[FZS] X. Fan, J. Shen and D. Zhao, *Sobolev embedding theorems for spaces $W^{1,p(x)}$*, J. Math. Anal. Appl. 262 (2001), 749–760.

[FZ1] X. L. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. 263 (2001), 424–446.

[FZ2] X. L. Fan and D. Zhao, *On the generalized Orlicz–Sobolev space $W^{k,p(x)}(\Omega)$*, J. Gansu Educ. College 12 (1998), 1–6.

[H] T. C. Halsey, *Electrorheological fluids*, Science 258 (1992), 761–766.

[KR] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math. J. 41 (1991), 592–618.

[M] M. Mihăilescu, *Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator*, Nonlinear Anal. 67 (2007), 1419–1425.

[MR] M. Mihăilescu and V. Rădulescu, *A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fuids*, Proc. Roy. Soc. London Ser. A Math. Phys. Engrg. Sci. 462 (2006), 2625–2641.

[My] T. G. Myers, *Thin films with high surface tension*, SIAM Rev. 40 (1998), 441–462.

[PMBD] C. Pfeiffer, C. Mavroidis, Y. Bar-Cohen and B. Dolgin, *Electrorheological fluid based force feedback device*, in: Proceedings of the 1999 SPIE Telemanipulator and Telepresence Technologies VI (Boston, MA, 1999), 88–99.

[Ri] B. Ricceri, *A three critical points theorem revisited*, Nonlinear Anal. 70 (2009), 3084–3089.

[R] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer, Berlin, 2000.

[SD] X. Shi and X. Ding, *Existence and multiplicity of solutions for a general $p(x)$-Laplacian Neumann problem*, Nonlinear Anal. 70 (2009), 3715–3720.

[TD] N. Tsouli and O. Darhouche, *Existence and multiplicity results for nonlinear problems involving the $p(x)$-Laplace operator*, Opuscula Math. 34 (2014), 621–638.

[W] W. M. Winslow, *Induced fibration of suspensions*, J. Appl. Phys. 20 (1949), 1137–1140.

[Y] J. H. Yao, *Solution for Neumann boundary problems involving $p(x)$-Laplace operators*, Nonlinear Anal. 68 (2008), 1271–1283.

[Z] V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR-Izv. 29 (1987), 33–66.

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