APPLICATIONES MATHEMATICAE
41,2-3 (2014), pp. 267–275

Omar Chakrone, Najib Tsouli, Mostafa Rahmani,
Omar Darhouche (Oujda)

BOUNDARY EIGENCURVE PROBLEMS INVOLVING THE BIHARMONIC OPERATOR

Abstract. The aim of this paper is to study the spectrum of the fourth order eigenvalue boundary value problem

$$\begin{cases} \Delta^2 u = \alpha u + \beta \Delta u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $(\alpha, \beta) \in \mathbb{R}^2$. We prove the existence of a first nontrivial curve of this spectrum and we give its variational characterization. Moreover we prove some properties of this curve, e.g., continuity, convexity, and asymptotic behavior. As an application, we study the non-resonance of solutions below the first principal eigencurve of the biharmonic problem

$$\begin{cases} \Delta^2 u = f(u, x) + \beta \Delta u + h & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and h is a given function in $L^2(\Omega)$.

1. Introduction. In this paper, we are concerned with the fourth order eigenvalue problem

$$(1.1)\qquad \begin{cases} \text{find } \ (u, \alpha, \beta) \in (X \setminus \{0\}) \times \mathbb{R}^2 \text{ such that} \\ \Delta^2 u = \alpha u + \beta \Delta u \quad \text{on } \Omega, \\ u = \Delta u = 0 \quad \text{in } \partial\Omega. \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $X := H_0^1(\Omega) \cap H^2(\Omega)$ and Δ^2 denotes the biharmonic operator defined by $\Delta^2 u = \Delta(\Delta u)$. We define the *second-order spectrum* of the biharmonic operator to be the set of $(\alpha, \beta) \in \mathbb{R}^2$

2010 *Mathematics Subject Classification*: Primary 35J35; Secondary 35J40.
Key words and phrases: biharmonic operator, eigencurve, resonance.

such that problem (1.1) has a nontrivial solution. Many authors have studied the fourth order elliptic problems

$$\begin{cases} \Delta^2 u + c\Delta u = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In [LL], the authors studied problem (1.2) in the case where $f(x,u)$ is asymptotically linear with respect to u at infinity. Using an equivalent version of Cerami's condition and the symmetric mountain pass lemma, they obtained the existence of multiple solutions. Problem (1.2) is usually used to describe some phenomena appearing in physical, engineering and other sciences [LM1, LM2, M]. In recent years, there are many results for fourth order elliptic equations: we refer the reader to [MP1, MP2, T, XZ, TCDR, ZW, LL].

To establish the existence of the first principal curve of problem (1.1), we consider the auxiliary problem

$$\begin{cases} \Delta^2 u = \alpha(r)(u + r\Delta u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

with $\alpha(r)$ is a real function defined on $\mathbb{R}$. We set $\Sigma^+ = \{(\alpha,\beta) \in \Sigma : \alpha \geq 0\}$ and

$$C(r) = \sup_{\substack{u\in X \\ \|u\|=1}} \int_\Omega (u^2 + r|\nabla u|^2). \tag{1.4}$$

Note that $u \in X$ is a weak solution of (1.1) if

$$\int_\Omega \Delta u \Delta v\,dx = \alpha \int_\Omega uv\,dx + \beta \int_\Omega \nabla u \nabla v\,dx, \qquad \forall v \in X.$$

1.1. The first principal curve of second order. In this section, we will prove the existence of the principal curve Γ_1 and give its variational characterization. Moreover, we prove the continuity and a certain asymptotic property of Γ_1.

LEMMA 1.1. *If $\mu_1(-\Delta)$ is the first eigenvalue of the Laplacian, then*

(1) $C(r) > 0$ *if and only if* $r > -1/\mu_1(-\Delta)$.
(2) $C(r) = 0$ *if and only if* $r = -1/\mu_1(-\Delta)$.
(3) $C(r) < 0$ *if and only if* $r < -1/\mu_1(-\Delta)$.

Proof. (1) Suppose that $C(r) > 0$. By (1.4) there exists u in X such that $\|u\| = 1$ and

$$\int_\Omega (|u|^2 + r|\nabla u|^2) > 0.$$

For all v in X we have

$$\mu_1(-\Delta) \int_\Omega v^2 \leq \int_\Omega |\nabla v|^2.$$

Then

$$\frac{\int_\Omega v^2}{\int_\Omega |\nabla v|^2} \le \frac{1}{\mu_1(-\Delta)},$$

and if we put $u = v$, we obtain

$$r > -\frac{\int_\Omega u^2}{\int_\Omega |\nabla u|^2} \ge -\frac{1}{\mu_1(-\Delta)}.$$

Conversely, suppose that $r > -1/\mu_1(-\Delta)$. Let φ_1 be the eigenfunction associated with the first eigenvalue $\mu_1(-\Delta)$ satisfying $\|\varphi_1\| = 1$. Then we have

$$C(r) \ge \int_\Omega (\varphi_1^2 + r|\nabla\varphi_1|^2),$$

and therefore

$$C(r) > \int_\Omega \left(\varphi_1^2 - \frac{1}{\mu_1(-\Delta)}|\nabla\varphi_1|^2\right) = 0.$$

(2) It is easy to see that $C(-1/\mu_1(-\Delta)) = 0$.

Conversely, suppose that $C(r) = 0$. By (1.4), there exists $(u_n)_n \subset X$ such that $\|u_n\| = 1$ and

$$\int_\Omega (u_n^2 + r|\nabla u_n|^2) \to 0.$$

Since $(u_n)_n$ is bounded in X, for a further subsequence we have

$$\begin{cases} u_n \rightharpoonup u & \text{in } X, \\ u_n \to u & \text{in } W_0^{1,2}(\Omega), \\ u_n \to u & \text{in } L^2(\Omega), \end{cases} \tag{1.5}$$

so

$$\begin{aligned} \nabla u_n \to \nabla u &\quad \text{in } (L^2(\Omega))^N, \\ \Delta u_n \to \Delta u &\quad \text{in } W_0^{-1,2}(\Omega). \end{aligned}$$

Since u and Δu_n are in X, we deduce that

$$\Delta u_n \to \Delta u \quad \text{in } L^2(\Omega).$$

On the other hand,

$$1 = \|u_n\|^2 = \int_\Omega (u_n^2 + |\Delta u_n|^2) \to \int_\Omega (u^2 + |\Delta u|^2),$$

which implies that $u \not\equiv 0$. As $\int_\Omega (u^2 + r|\nabla u|^2) = 0$, we get

$$-r\int_\Omega |\nabla u|^2 = \int_\Omega u^2 \le \frac{1}{\mu_1(-\Delta)} \int_\Omega |\nabla u|^2,$$

so $-r \le 1/\mu_1(-\Delta)$. Since $C(r) = 0$ we get

$$\int_\Omega u^2 \le -r \int_\Omega |\nabla u|^2$$

for u in X, which implies that $-r \ge 1/\mu_1(-\Delta)$ and so $-r = 1/\mu_1(-\Delta)$.

(3) From (1) and (2) we deduce (3).

REMARK. By Lemma 1.1 the function α is well defined and we have

$$\alpha(r) = \frac{1}{C(r)}, \qquad \forall r \ne -\frac{1}{\mu_1(-\Delta)}.$$

THEOREM 1.2. *$\Gamma_1 := \{(\alpha(r), r\alpha(r)) : r > -\mu_1^{-1}(-\Delta)\}$ is the first eigencurve of second order in the sense that if $(\alpha', \beta') \in \Sigma^+$, then $\alpha(\beta'/\alpha') \le \alpha'$.*

Let $(u_n)_n \subset X$ be such that

$$\|u_n\| = 1, \qquad \int_\Omega (u_n^2 + r|\nabla u_n|) \to \frac{1}{\alpha(r)}. \tag{1.6}$$

So there exists a subsequence also denoted (u_n) such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } X, \\ u_n \to u & \text{in } W_0^{1,2}(\Omega), \\ u_n \to u & \text{in } L^2(\Omega), \end{cases} \tag{1.7}$$

so $\nabla u_n \to \nabla u$ in $(L^2(\Omega))^N$. Furthermore

$$\frac{1}{\alpha(r)} = \int_\Omega (u^2 + r|\nabla u|^2). \tag{1.8}$$

Now using Lemma 1.1, we have $u \not\equiv 0$, and

$$\frac{1}{\alpha(r)} \ge \frac{1}{\|u\|^2} \int_\Omega (u^2 + r|\nabla u|^2) = \frac{1}{\|u\|^2 \alpha(r)},$$

which implies that $\|u\|^2 \ge 1$. On the other hand, $\|u\| \le \liminf \|u_n\| = 1$, so that $\|u\| = 1$.

Let $v \in X$. For t small enough we have

$$\frac{1}{\alpha(r)} \ge \int_\Omega \bigg(\frac{|u + tv|^2}{\|\Delta u + t\Delta v\|^2} + r \frac{|\nabla u + t\nabla v|^2}{\|\Delta u + t\Delta v\|^2} \bigg),$$

hence

$$\|\Delta u + t\Delta v\|^2 \ge \alpha(r) \int_\Omega \big[u^2 + 2tuv + t^2 v^2 + r(|\nabla u|^2 + 2t\nabla u \nabla v + t^2 |\nabla v|^2)\big].$$

Hence

$$\|\Delta u\|^2-\alpha(r)\int_\Omega(u^2+r|\nabla u|^2)+2t\int_\Omega\Delta u\Delta v+t^2\int_\Omega\|\Delta v\|^2$$
$$\geq 2t\alpha(r)\int_\Omega(uv+r\nabla u\nabla v)+t^2\alpha(r)\int_\Omega(v^2+r|\nabla v|^2).$$

Dividing by t, letting $t\to 0$ and using (1.8), we get

$$\int_\Omega\Delta u\Delta v\geq\alpha(r)\Big[\int_\Omega uv+r\int_\Omega\nabla u\nabla v\Big]$$

for all v in X. Similarly for $-v$ in X we have

$$\int_\Omega\Delta u\Delta v\leq\alpha(r)\Big[\int_\Omega uv+r\int_\Omega\nabla u\nabla v\Big].$$

Hence we deduce that

$$\int_\Omega\Delta u\Delta v=\alpha(r)\Big[\int_\Omega uv+r\int_\Omega\nabla u\nabla v\Big],$$

which means that u is a weak solution of problem (1.3).

On the other hand, if $(\alpha',\beta')\in\Sigma^+$, then there exists $u\in X\setminus\{0\}$ such that

$$\Delta^2u=\alpha'u+\beta'\Delta u\quad\text{ in } X'.$$

It is clear that by Lemma 1.1, $\alpha'>0$ and therefore

$$\frac{1}{\alpha'}=\int_\Omega\bigg(\frac{u^2}{\|u\|^2}+\frac{\beta'}{\alpha'}\,\frac{|\nabla u|^2}{\|u\|^2}\bigg)\leq C\Big(\frac{\beta'}{\alpha'}\Big).$$

Finally by definition of $\alpha(r)$, we have $\alpha'\geq\alpha(\beta'/\alpha')$.

Proposition 1.3. *The function $r\mapsto C(r)$ is convex on $I=[-\mu_1^{-1}(-\Delta),\infty[$, concave on $]-\infty,-\mu_1^{-1}(-\Delta)[$, continuous on $\mathbb{R}$ and differentiable; moreover*

$$\alpha'(r)=-\alpha^2(r)\int_\Omega|\nabla u|^2.$$

Proof. For r_1 , r_2 in I and t in $[0,1]$, we have

$$\begin{aligned}C(tr_1+(1-t)r_2)&=\sup\int_\Omega\big(u^2+[tr_1+(1-t)r_2]|\nabla u|^2\big)\,dx\\&\leq t\sup\int_\Omega(u^2+r_1|\nabla u|^2)\,dx+(1-t)\sup\int_\Omega(u^2+r_2|\nabla u|^2)\,dx\\&=tC(r_1)+(1-t)C(r_2).\end{aligned}$$

Thus $r\mapsto C(r)$ is convex. The proof of concavity is similar.

It is easy to show that the function is differentiable, it suffices to use the characterization of the upper bound.

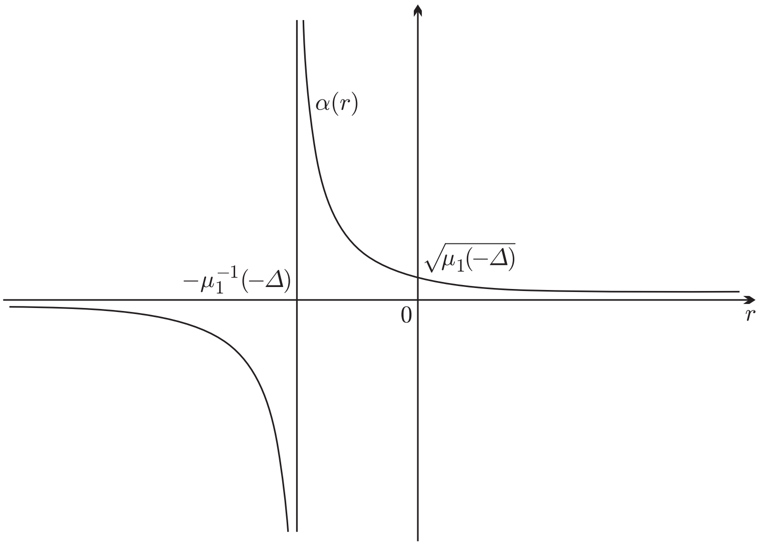

Fig. 1. First eigencurve

PROPOSITION 1.4. *We have*

$$\lim_{r\to\infty}\alpha(r)=0 \quad \textit{and} \quad \lim_{\substack{r\to-\mu_1^{-1}(-\Delta)\\ r>-\mu_1^{-1}(-\Delta)}}\alpha(r)=\infty.$$

Proof. For u fixed in X such that $\|u\|=1$, we have

$$r\int_\Omega|\nabla u|^2\le\int_\Omega(u^2+r|\nabla u|^2)\le\frac{1}{\alpha(r)},$$

so

$$\lim_{r\to\infty}\alpha(r)=0.$$

From Lemma 1.1, we have

$$\lim_{r\to-\mu_1^{-1}(-\Delta)}\alpha(r)=\infty.$$

We will prove that Σ^+ is a sequence of eigencurves of problem (1.3) with $r>-1/\mu_1(-\Delta)$. Indeed, let $T:X\to X$ be the operator defined by

$$Tu=(\Delta^2)^{-1}(u+r\Delta u).$$

Then problem (1.3) is equivalent to the problem

$$\begin{cases}\text{find } (\lambda,u)\in\mathbb{R}\times(X\setminus\{0\}) \text{ such that}\\ Tu=\lambda u.\end{cases} \tag{1.9}$$

PROPOSITION 1.5. *The spectrum* $\sigma(T)$ *of problem* (1.9) *is a sequence* (λ_n) *of eigenvalues such that* $\lambda_n \neq 0$ *and*

$$\lim_{n\to\infty} \lambda_n = 0.$$

Proof. Since T is a symmetric compact linear operator, the conclusion is well known (cf. [CH]).

Moreover, $\alpha_n(r) = 1/\lambda_n$ has the variational characterization

$$\alpha_n(r) = \sup_{F_n} \inf_{u\in F_n} \Big\{ \int_\Omega (u^2 + r|\nabla u|^2) : \int_\Omega |\Delta u|^2 = 1 \Big\}, \tag{1.10}$$

and

$$\Sigma^+ = \{(\alpha_n(r), r\alpha_n(r)) : r > -\mu_1^{-1}(-\Delta)\},$$

where F_n varies over all n-dimensional subspaces of X and $\alpha_n(r)$ is repeated according to its multiplicity.

2. Non-resonance below the first principal eigencurve of the biharmonic problem. In this section, we consider the biharmonic problem

$$\begin{cases} \Delta^2 u = f(u,x) + \beta\Delta u + h & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where h is in $L^2(\Omega)$ and $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function. We consider the associated energy functional $\Phi: X \to \overline{\mathbb{R}}$ defined by

$$\Phi(u) = \frac{1}{2}\int_\Omega |\Delta u|^2 - \int_\Omega F(x,u) - \frac{\beta}{2}\int_\Omega |\nabla u|^2 - \langle h, u\rangle_{X,X'}. \tag{2.2}$$

with $F(x,s) = \int_0^s f(x,t)\,dt$.

We also suppose that there exist $a > 0$ and $b \in L^2(\Omega)$ such that

$$|f(x,s)| \le a|s| + b(x) \quad \text{a.e. } x\in\Omega,\ \forall s\in\mathbb{R}.$$

Note that Φ is in $C^1(X,\mathbb{R})$, and $u \in X$ is a critical point of Φ if and only if u is a solution of problem (2.1).

We are interested in what conditions should be imposed on the nonlinearity f for problem (2.1) to have a solution u in X for any given h in $L^2(\Omega)$. We will suppose that the potential F is below the first principal eigencurve of the biharmonic problem (2.1) in the sense that there exist $\alpha\in\mathbb{R}$ such that for a.e. $x\in\Omega$, we have

$$\limsup_{s\to\pm\infty} \frac{2F(x,s)}{s^2} \le \alpha. \tag{2.3}$$

THEOREM 2.1. *If* $\alpha(\beta/\alpha) > \alpha$ *then problem* (2.1) *has a solution for all* h *in* $L^2(\Omega)$.

Proof. The relation (2.3) signifies that for every $\epsilon > 0$ there exists $b_\epsilon \in L^1(\Omega)$ such that
$$F(x,s) \le \frac{1}{2}(\alpha+\epsilon)s^2 + b_\epsilon(x).$$
We have $\alpha\big(\frac{\beta}{\alpha+\epsilon}\big) > \alpha+\epsilon$, since $r \mapsto \alpha(r)$ is continuous on $\mathbb{R}\setminus\{0\}$ and $\alpha+\epsilon \neq 0$. Thus the functional Φ is well-defined. Moreover we have
$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\int_\Omega |\Delta u|^2 - \int_\Omega F(x,u) - \frac{\beta}{2}\int_\Omega |\nabla u|^2 - \langle h,u\rangle_{X,X'}\\
&\ge \frac{1}{2}\int_\Omega |\Delta u|^2 - \frac{\alpha+\epsilon}{2}\int_\Omega u^2 - \int_\Omega b_\epsilon - \frac{\beta}{2}\int_\Omega |\nabla u|^2 - \|h\|\,\|u\|\\
&\ge \frac{1}{2}\int_\Omega |\Delta u|^2 - \frac{\alpha+\epsilon}{2}\int_\Omega \Big(u^2 + \frac{\beta}{\alpha+\epsilon}|\nabla u|^2\Big) - \int_\Omega b_\epsilon - \|h\|\cdot\|u\|\\
&\ge \frac{1}{2}\|u\|^2\bigg[1-(\alpha+\epsilon)\int_\Omega\Big(\frac{u^2}{\|u\|^2} + \frac{\beta}{\alpha+\epsilon}\,\frac{|\nabla u|^2}{\|u\|^2}\Big)\bigg].
\end{aligned}$$
It is easy to see that Φ is coercive in the following cases:

- $\frac{\beta}{\alpha+\epsilon} > -\frac{1}{\mu_1(-\Delta)}$ and $\alpha+\epsilon<0$,
- $\frac{\beta}{\alpha+\epsilon} < -\frac{1}{\mu_1(-\Delta)}$ and $\alpha+\epsilon>0$,

In the cases

- $\frac{\beta}{\alpha+\epsilon} > -\frac{1}{\mu_1(-\Delta)}$ and $\alpha+\epsilon>0$,
- $\frac{\beta}{\alpha+\epsilon} < -\frac{1}{\mu_1(-\Delta)}$ and $\alpha+\epsilon<0$,

we have
$$\Phi(u) \ge \frac{1}{2}\bigg(1-\frac{\alpha+\epsilon}{\alpha\big(\frac{\beta}{\alpha+\epsilon}\big)}\bigg)\|u\|^2 - \int_\Omega b_\epsilon - \|h\|\,\|u\|,$$
since
$$1-\frac{\alpha+\epsilon}{\alpha\big(\frac{\beta}{\alpha+\epsilon}\big)} > 0,$$
so Φ is also coercive. This completes the proof of Theorem 2.1.

References

[CH] R. Courant and D. Hilbert, *Methods of Mathematical Physics. Vols. 1 and 2*, Interscience, New York, 1962.

[LM1] A. C. Lazer and P. J. McKenna, *Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis*, SIAM Rev. 32 (1990), 537–578.

[LM2] A. C. Lazer and P. J. McKenna, *Global bifurcation and a theorem of Tarantello*, J. Math. Anal. Appl. 18 (1994), 648–655.

[LL] Q. Liu and D. Lü, *Multiplicity of solutions for a class of fourth-order elliptic problems with asymptotically linear term*, J. Appl. Math. 2012, art. ID 749059, 14 pp.

[M] P. J. McKenna and W. Reichel, *Radial solutions of singular nonlinear biharmonic equations and applications to conformal geometry*, Electron. J. Differential Equations 2003, no. 37, 13 pp.

[MP1] A. M. Micheletti and A. Pistoia, *Multiplicity results for a fourth-order semilinear elliptic problem*, Nonlinear Anal. 31 (1998), 895–908.

[MP2] A. M. Micheletti and A. Pistoia, *Nontrivial solutions for some fourth order semilinear elliptic problems*, Nonlinear Anal. 34 (1998), 509–523.

[T] G. Tarantello, *A note on a semilinear elliptic problem*, Differential Integral Equations 5 (1992), 561–565.

[TCDR] N. Tsouli, O. Chakrone, O. Darhouche and M. Rahmani, *Nonresonance between two consecutive eigenvalues of the biharmonic operator*, J. Abstract Differential Equations Appl. 2 (2012), no. 2, 1–6.

[XZ] G. Xu and J. Zhang, *Existence results for some fourth-order nonlinear elliptic problems of local superlinearity and sublinearity*, J. Math. Anal. Appl. 281 (2003), 633–640.

[ZW] J. Zhou and X. Wu, *Sign-changing solutions for some fourth-order nonlinear elliptic problems*, J. Math. Anal. Appl. 342 (2008), 542–558.

Omar Chakrone, Najib Tsouli, Mostafa Rahmani, Omar Darhouche
Department of Mathematics
University Mohamed I
P.O. Box 717
Oujda 60000, Morocco
E-mail: chakrone@yahoo.fr
ntsouli@hotmail.com
rahmani.mostafa.63@hotmail.com
omarda13@hotmail.com

Received on 21.7.2014 (2234)