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ASYMPTOTIC STABILITY OF A LINEAR BOLTZMANN-TYPE EQUATION

Abstract. We present a new necessary and sufficient condition for the asymptotic stability of Markov operators acting on the space of signed measures. The proof is based on some special properties of the total variation norm. Our method allows us to consider the Tjon–Wu equation in a linear form. More precisely a new proof of the asymptotic stability of a stationary solution of the Tjon–Wu equation is given.

1. Introduction. Some problems of mathematical physics can be written as differential equations for functions with values in a space of measures. The vector space of signed measures does not have good analytical properties. For example, this space with the Fortet–Mourier, Kantorovich–Wesserstein or Zolotarev metric is not complete (see [R]). There are two methods to overcome this problem. First, we may replace the original equations by the adjoint ones on the space of bounded continuous functions. Secondly, we may restrict our equations to some complete convex subsets of the vector space of measures. This approach seems to be quite natural and it is related to the classical results concerning differential equations on convex subsets of Banach spaces (see [C]). The convex sets method in studying the Boltzmann equation was used in a series of papers (see for example [G2, GL, L1, LT1, LT2]).

Our goal is to show the utility of some version of the invariance principle established in [G2]. In particular, it can be used to prove a new necessary and sufficient condition for the asymptotic stability of Markov semigroups with respect to the total variation norm.

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Moreover, we discuss the problem of the asymptotic stability of solution of the linear Boltzmann equation.

2. Markov operators. Let (X, ρ) be a Polish space, i.e., a separable, complete metric space. We denote by $\mathcal{B}$ the σ-algebra of Borel subsets of X, and by $\mathcal{M}$ the family of all finite, nonnegative Borel measures on X.

Let $\mathcal{M}_1$ denote the subset of those $\mu \in \mathcal{M}$ such that $\mu(X) = 1$. The elements of $\mathcal{M}_1$ will be called *probability measures*. Further let

$$\mathcal{M}_{\mathrm{sig}} = \{\mu_1 - \mu_2 : \mu_1, \mu_2 \in \mathcal{M}\}$$

be the space of finite signed measures. For arbitrary $\mu \in \mathcal{M}_{\mathrm{sig}}$ we denote by μ_+ and μ_- the positive and negative parts of μ. Then we set

$$\mu_+ - \mu_- = \mu \quad \text{and} \quad \mu_+ + \mu_- = |\mu|, \tag{2.1}$$

where $|\mu|$ is called the *total variation* of the measure μ.

In $\mathcal{M}_{\mathrm{sig}}$ we introduce the *total variation norm* of $\mu \in \mathcal{M}_{\mathrm{sig}}$ by

$$\|\mu\|_T = \sup\Big\{\sum_{i=1}^{n} |\mu(A_i)| : n \in \mathbb{N},\ A_i \in \mathcal{B}\Big\}, \tag{2.2}$$

where the supremum is taken over all finite partitions of X, i.e.

$$X = \bigcup_{i=1}^{n} A_i \quad \text{and} \quad A_i \cap A_j = \emptyset \quad \text{for } i \neq j.$$

Corollary 2.1. *For every $\mu \in \mathcal{M}_{\mathrm{sig}}$ we have*

$$\|\mu\|_T = \mu_+(X) + \mu_-(X) = \mu(X_+) - \mu(X_-),$$

where $X = X_+ \cup X_-$ is the Hahn decomposition of the measure μ.

Definition 2.2. An operator $P : \mathcal{M} \to \mathcal{M}$ is called a *Markov operator* if it satisfies the following conditions:

(i) P is positively linear:

$$P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2\mu_2$$

for $\lambda_1, \lambda_2 \geq 0$ and $\mu_1, \mu_2 \in \mathcal{M}$,

(ii) P preserves the measure of the space:

$$P\mu(X) = \mu(X) \quad \text{for } \mu \in \mathcal{M}. \tag{2.3}$$

Remark 2.3. Every Markov operator P can be uniquely extended as a linear operator onto the space of signed measures. Namely, for $\mu \in \mathcal{M}_{\mathrm{sig}}$ we define

$$P\mu = P\mu_1 - P\mu_2, \quad \text{where} \quad \mu = \mu_1 - \mu_2,\ \mu_1, \mu_2 \in \mathcal{M}.$$

It is easy to verify that this definition does not depend on the choice of μ_1, μ_2.

DEFINITION 2.4. A function $d : \mathcal{M}_{\mathrm{sig}} \times \mathcal{M}_{\mathrm{sig}} \to \mathbb{R}_+$ is called a *distance* if d is continuous and if

$$d(x, y) = 0 \ \Leftrightarrow \ x = y, \qquad \text{for } x, y \in X. \tag{2.4}$$

A Markov operator $P : \mathcal{M}_{\mathrm{sig}} \to \mathcal{M}_{\mathrm{sig}}$ is called *contracting* or *nonexpansive* in a class $\widetilde{\mathcal{M}} \subset \mathcal{M}_{\mathrm{sig}}$ (with respect to d) if

$$d(P\mu_1, P\mu_2) \le d(\mu_1, \mu_2) \quad \text{for } \mu_1, \mu_2 \in \widetilde{\mathcal{M}}. \tag{2.5}$$

A Markov operator $P : \mathcal{M}_{\mathrm{sig}} \to \mathcal{M}_{\mathrm{sig}}$ is called *strongly contracting* in $\widetilde{\mathcal{M}} \subset \mathcal{M}_{\mathrm{sig}}$ (with respect to d) if

$$d(P\mu_1, P\mu_2) < d(\mu_1, \mu_2) \quad \text{for } \mu_1, \mu_2 \in \widetilde{\mathcal{M}}. \tag{2.6}$$

3. A stability criterion. A new proof of the asymptotic stability of Markov semigroup will be given. The proof is based on some special property of the total variation norm which we call the maximum principle. The maximum principle method in studying the asymptotic stability of Markov semigroup with respect to the Kantorovich–Wasserstein metric and Fortet–Mourier metric was used in [LT1] and [G2]. This part of our paper was stimulated by results of Lasota [L1].

We start from the following

DEFINITION 3.1. We say that the measures $\mu_1, \mu_2 \in \mathcal{M}$ are *mutually singular* or *orthogonal* if there are two sets $A, B \in \mathcal{B}$ such that $A \cap B = \emptyset$, $A \cup B = X$ and $\mu_1(B) = \mu_2(A) = 0$.

LEMMA 3.2. *Let $\mu_1, \mu_2 \in \mathcal{M}$. Then*

$$\|\mu_1 - \mu_2\|_T = \|\mu_1\|_T + \|\mu_2\|_T \tag{3.1}$$

if and only if μ_1 and μ_2 are mutually singular.

Proof. Assume that $\|\mu_1 - \mu_2\|_T = \|\mu_1\|_T + \|\mu_2\|_T$. Suppose, contrary to our claim, that there is no $A \in \mathcal{B}$ such that $\mu_1(A) = 0$ and $\mu_2(A^c) = 0$, where $A^c = X \setminus A$. Let H be the set from the corresponding Hahn decomposition of the measure $\mu_1 - \mu_2$. Since there is no $A \in \mathcal{B}$ such that $\mu_1(A) = 0$ and $\mu_2(A^c) = 0$, we have $\mu_1(H^c) > 0$ or $\mu_2(H) > 0$. Thus

$$\begin{aligned} \|\mu_1 - \mu_2\|_T &= (\mu_1 - \mu_2)(H) - (\mu_1 - \mu_2)(H^c) \\ &< \mu_1(H) + \mu_1(H^c) + \mu_2(H^c) + \mu_2(H). \end{aligned}$$

Consequently,

$$\|\mu_1 - \mu_2\|_T < \|\mu_1\|_T + \|\mu_2\|_T,$$

a contradiction.

Conversely, assume that there exists a set $A \in \mathcal{B}$ such that $\mu_1(A) = 0$ and $\mu_2(A^c) = 0$. Let $\mu = \mu_1 - \mu_2$. Then

$$\begin{aligned}\|\mu_1 - \mu_2\|_T &\geq |\mu(A)| + |\mu(A^c)| = |\mu_2(A)| + |\mu_1(A^c)| \\ &= |\mu_1(X)| + |\mu_2(X)| = \|\mu_1\|_T + \|\mu_2\|_T,\end{aligned}$$

and the proof is complete. ∎

Definition 3.3. We say that the measures $\mu, \nu \in \mathcal{M}$ *overlap supports* if there is no $A \in \mathcal{B}$ such that

$$\mu(A) = 0 \quad \text{and} \quad \nu(A^c) = 0.$$

Now using Lemma 3.2 we may easily derive

Theorem 3.4. *Let P be a Markov operator. Assume that $P\mu_+, P\mu_-$ overlap supports for every nontrivial measure $\mu \in \mathcal{M}_{\rm sig}$. Then the Markov operator P is strongly contracting with respect to the total variation norm.*

Proof. Fix $\mu_1, \mu_2 \in \mathcal{M}_{\rm sig}$, $\mu_1 \neq \mu_2$. Define $\mu = \mu_1 - \mu_2$. It follows easily that

$$\|P\mu_+ - P\mu_-\|_T \leq \|P\mu_+\|_T + \|P\mu_-\|_T.$$

By Lemma 3.2, equality holds if and only if there exists $A \in \mathcal{B}$ such that $P\mu_+(A) = 0$ and $P\mu_-(A^c) = 0$. From these properties we have

$$\|P\mu_+ - P\mu_-\|_T < \|P\mu_+\|_T + \|P\mu_-\|_T. \tag{3.2}$$

Since P is a Markov operator, we obtain

$$P\mu_+(X) = \mu_+(X) \quad \text{and} \quad P\mu_-(X) = \mu_-(X).$$

This gives

$$\|P\mu_+\|_T = \|\mu_+\|_T \quad \text{and} \quad \|P\mu_-\|_T = \|\mu_-\|_T.$$

By the above, inequality (3.2) takes the form

$$\|P\mu_+ - P\mu_-\|_T < \|\mu_+ - \mu_-\|_T. \ \blacksquare$$

For the convenience of the reader we recall a few definitions from the theory of dynamical systems. Let T be a *nontrivial semigroup* of nonnegative real numbers. More precisely we assume that $\{0\} \subsetneq T \subset \mathbb{R}_+$ and

$$t_1 + t_2, t_1 - t_2 \in T \quad \text{for } t_1, t_2 \in T,\ t_1 \geq t_2. \tag{3.3}$$

A family $(P^t)_{t \in T}$ of Markov operators is called a *semigroup* if

$$P^{t+s} = P^t P^s \quad \text{for } t, s \in T,$$

and $P^0 = I$ where I is the identity operator.

A semigroup $(P^t)_{t \in T}$ is called a *semidynamical system* if the transformation $\mathcal{M}_{\rm sig} \ni \mu \mapsto P^t\mu \in \mathcal{M}_{\rm sig}$ is continuous for every $t \in T$.

If a semidynamical system $(P^t)_{t \in T}$ is given, then for every fixed $\mu \in \mathcal{M}_{\rm sig}$ the function $T \ni t \mapsto P^t\mu \in \mathcal{M}_{\rm sig}$ will be called the *trajectory* starting from μ, and denoted $(P^t\mu)$. A point $\nu \in \mathcal{M}_{\rm sig}$ is called a *limiting point* of a trajectory $(P^t\mu)$ if there exists a sequence (t_n), $t_n \in T$, such that $t_n \to \infty$ and

$$\lim_{n\to\infty} P^{t_n}\mu = \nu.$$

The set of all limiting points of the trajectory $(P^t\mu)$ will be denoted $\omega(\mu)$.

We say that a trajectory $(P^t\mu)$ is *sequentially compact* if for every sequence (t_n), $t_n \in T$, $t_n \to \infty$, there exists a subsequence (t_{k_n}) such that the sequence $(P^{t_{k_n}}\mu)$ is convergent to a point $\nu \in \mathcal{M}_{\rm sig}$.

REMARK 3.5. If the trajectory $(P^t\mu)$ is sequentially compact, then $\omega(\mu)$ is a nonempty, sequentially compact set.

We say that a semidynamical system $(P^t)_{t\in T}$ is *Lagrange stable* if $(P^t\mu)$ for $t \in T$ is sequentially compact.

A point $\mu_* \in \mathcal{M}_{\rm sig}$ is called *stationary* (or *invariant*) with respect to a semidynamical system $(P^t)_{t\in T}$ if

$$P^t\mu_* = \mu_* \quad \text{for } t \in T. \tag{3.4}$$

A semidynamical system $(P^t)_{t\in T}$ is called *asymptotically stable* if there exists a stationary point $\mu_* \in \mathcal{M}_{\rm sig}$ such that

$$\lim_{t\to\infty} P^t\mu = \mu_* \quad \text{for } \mu \in \mathcal{M}_{\rm sig}. \tag{3.5}$$

REMARK 3.6. Since $(\mathcal{M}_{\rm sig}, \|\cdot\|_T)$ is a Hausdorff space, an asymptotically stable dynamical system has exactly one stationary point.

We say that a Markov semigroup $(P^t)_{t\in T}$ is *contracting* or *nonexpansive* with respect to the distance d in the class $\widetilde{\mathcal{M}} \subset \mathcal{M}_{\rm sig}$ if

$$d(P^t\mu_1, P^t\mu_2) \le d(\mu_1, \mu_2) \quad \text{for } \mu_1, \mu_2 \in \widetilde{\mathcal{M}},\ t \in T. \tag{3.6}$$

A contracting semigroup $(P^t)_{t\in T}$ will be called *strongly contracting* with respect to the distance d in the class $\widetilde{\mathcal{M}} \subset \mathcal{M}_{\rm sig}$ if for any distinct μ_1, μ_2 in $\widetilde{\mathcal{M}}$ there is a $t_0 \in T$ such that

$$d(P^{t_0}\mu_1, P^{t_0}\mu_2) < d(\mu_1, \mu_2).$$

DEFINITION 3.7. We say that a Markov semigroup $(P_t)_{t\in T}$ *overlaps supports* if for any $\mu, \nu \in \mathcal{M}_1$ there is a $t_0 \ge 0$ such that the measures $P^{t_0}(\nu-\mu)_+$ and $P^{t_0}(\nu-\mu)_-$ overlap supports.

Now we consider a semidynamical system $(P^t)_{t\in T}$ which has at least one sequentially compact trajectory. By $\mathcal{Z}$ we denote the set of all $\mu \in \mathcal{M}_{\rm sig}$ such that the trajectory $(P^t\mu)$ is sequentially compact. Since $\mathcal{Z}$ is a nonempty set we have

$$\Omega = \bigcup_{\mu\in\mathcal{Z}} \omega(\mu) \ne \emptyset.$$

In the proof of the main result (Theorem 3.9) we will use the following criterion for the asymptotic stability of trajectories,

THEOREM 3.8. *Let $\mu_* \in \Omega$. Assume that for every $\mu \in \Omega$, $\mu \neq \mu_*$ there is $t(\mu) \in T$ such that*

$$d(P^{t(\mu)}\mu, P^{t(\mu)}\mu_*) < d(\mu, \mu_*). \tag{3.7}$$

Further assume that the semidynamical system $(P^t)_{t\in T}$ is nonexpansive with respect to d, i.e.,

$$d(P^t\mu, P^t\nu) \le d(\mu, \nu) \quad \text{for } \mu, \nu \in \mathcal{M}_{\mathrm{sig}} \text{ and } t \in T. \tag{3.8}$$

Then μ_* is a stationary point of $(P^t)_{t\in T}$ and

$$\lim_{t\to\infty} d(P^t\mu, \mu_*) = 0 \quad \text{for } \mu \in \mathcal{Z}. \tag{3.9}$$

For details see [G2, pp. 28–30].

Now using Theorem 3.4 we can easily derive from the above criterion the following

THEOREM 3.9. *A Markov semigroup $(P_t)_{t\in T}$ is asymptotically stable with respect to the total variation norm if and only if $(P_t)_{t\in T}$ is Lagrange stable and overlaps supports.*

Proof. First assume that a Markov semigroup $(P^t)_{t\in\mathbb{R}_+}$ is asymptotically stable with respect to the total variation norm. Then, evidently, $(P^t)_{t\in\mathbb{R}_+}$ is Lagrange stable. Suppose for contradiction that there exist distinct measures $\mu, \nu \in \mathcal{M}_1$ such that for each $t > 0$ there exists a set $A_t \in \mathcal{B}$ such that

$$P^t(\nu-\mu)_+(A_t) = 0 \quad \text{and} \quad P^t(\nu-\mu)_-(A_t^c) = 0.$$

Thus, by Lemma 3.2 we have

$$\|P^t(\nu-\mu)_+ - P^t(\nu-\mu)_-\|_T = \|P^t(\nu-\mu)_+\|_T + \|P^t(\nu-\mu)_-\|_T. \tag{3.10}$$

By the definition of asymptotic stability of Markov semigroups it follows that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that

$$\|P^t\nu - P^t\mu\|_T < \varepsilon \quad \text{for } t \ge t_\varepsilon. \tag{3.11}$$

Since $\nu(X) = \mu(X) = 1$, we have

$$0 = \nu(X) - \mu(X) = (\nu-\mu)(X) = (\nu-\mu)_+(X) - (\nu-\mu)_-(X),$$

and consequently

$$(\nu-\mu)_+(X) = (\nu-\mu)_-(X). \tag{3.12}$$

Hence the equality $\|\mu-\nu\|_T = (\nu-\mu)_+(X) + (\nu-\mu)_-(X)$ shows that

$$0 < \frac{1}{2}\|\mu-\nu\|_T = (\nu-\mu)_+(X) = (\nu-\mu)_-(X). \tag{3.13}$$

From inequality (3.11) it follows that

$$\left\| P^t\left(\frac{(\nu-\mu)_+}{\frac{1}{2}\|\mu-\nu\|_T}\right) - P^t\left(\frac{(\nu-\mu)_-}{\frac{1}{2}\|\mu-\nu\|_T}\right)\right\|_T < \frac{\varepsilon}{\frac{1}{2}\|\mu-\nu\|_T} \quad \text{for } t \ge t_\varepsilon. \tag{3.14}$$

On the other hand, by the definition of a Markov operator, conditions (3.10) and (3.13) show that

$$\left\| P^t\left(\frac{(\nu-\mu)_+}{\frac{1}{2}\|\mu-\nu\|_T}\right) - P^t\left(\frac{(\nu-\mu)_-}{\frac{1}{2}\|\mu-\nu\|_T}\right)\right\|_T = 2 \quad \text{for } t \ge t_\varepsilon. \tag{3.15}$$

Since ε was arbitrary, this contradicts (3.14).

Now assume that $(P_t)_{t\in T}$ is Lagrange stable and overlaps supports. We first show that the Markov semigroup $(P^t)_{t\in\mathbb{R}_+}$ is contracting with respect to the total variation norm. We only need to show that for all $\mu \in \mathcal{M}_{\mathrm{sig}}$ and $t \ge 0$,

$$\|P^t\mu\|_T \le \|\mu\|_T. \tag{3.16}$$

To see this, by the property of the Hahn decomposition of μ, and the definition of the total variation norm of P^t_μ, we have

$$\|P^t\mu\|_T \le P^t\mu_+(X) + P^t\mu_-(X) = \|\mu_+\|_T + \|\mu_-\|_T = \|\mu\|_T. \tag{3.17}$$

According to Theorem 3.4 the last inequality shows that the semigroup $(P^t)_{t\in\mathbb{R}_+}$ is strongly contracting with respect to the total variation norm. A straightforward application of Theorem 3.8 completes the proof. ∎

4. Applications. A generalized linear version of a Boltzmann type equation is considered. This research was stimulated by the problem of the stability of solutions of the following version of the Boltzmann equation in the Tjon–Wu form:

$$\frac{\partial u(t,x)}{\partial t} + u(t,x) = \int_x^\infty \frac{dy}{y} \int_0^y e^{-(y-z)} u(t,z)\,dz, \qquad t,x \ge 0, \tag{4.1}$$

which was derived by Tjon and Wu [TW], Lasota [L1], and Lasota and Mackey [LM].

The solution u of the problem has a simple physical interpretation, namely $u(t,\cdot)$ for fixed $t \ge 0$ is the probability distribution function of the energy of particles.

Due to its physical interpretation, equation (4.1) is considered with the additional conditions

$$\int_0^\infty u(t,x)\,dx = \int_0^\infty xu(t,x)\,dx = 1, \tag{4.2}$$

which describe the conservation of mass and energy.

Equation (4.1) may be considered as an evolution equation of the form

$$\frac{du}{dt} + u = Pu, \tag{4.3}$$

where

$$Pv(x) = \int_x^\infty \frac{dy}{y} \int_0^y e^{-(y-z)} v(z)\, dz. \tag{4.4}$$

Equation (4.3) was studied in the spaces $f \in L^p(\mathbb{R}_+)$ with $p = 1, 2$ and different weights.

The operator given by (4.4) has the following probabilistic interpretation. Assume there are three independent random variables ξ_1, ξ_2, η such that ξ_1 and ξ_2 have density distribution functions e^{-x} and v respectively and η is uniformly distributed on $[0, 1]$. Then Pv is the density distribution function of the random variable

$$\eta(\xi_1 + \xi_2). \tag{4.5}$$

Physically this means that the energies of the particles before a collision are independent quantities, and after collision a particle takes the η fraction of the sum of the energies of the colliding particles.

The assumption that η has density distribution function of the form $1_{[0,1]}$ is quite restrictive and can be relaxed.

Now we will consider a generalized version of (4.3) in the space $\mathcal{M}_{\mathrm{sig}}$ of all signed measures on $\mathbb{R}_+$. We define

$$D := \{\mu \in \mathcal{M}_1 : m_1(\mu) = 1\}, \quad \text{where} \quad m_1(\mu) = \int_0^\infty x\, \mu(dx). \tag{4.6}$$

The object of interest is the asymptotic behaviour of solutions of the equation

$$\frac{d\psi}{dt} + \psi = P\psi \qquad \text{for } t \geq 0 \tag{4.7}$$

with the initial condition

$$\psi(0) = \psi_0, \tag{4.8}$$

where $P : D \to D$ is a linear operator on measures analogous to (4.4), and $\psi_0 \in D$. In order to define P precisely we will introduce several notations.

Recall that the *convolution of measures* $\nu, \mu \in \mathcal{M}_{\mathrm{sig}}$ is a unique measure $\nu * \mu$ satisfying

$$(\nu * \mu)(A) := \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} 1_A(x + y)\, \nu(dx)\, \mu(dy) \qquad \text{for } A \in \mathcal{B}. \tag{4.9}$$

It is easy to verify that

$$\langle f, \nu * \mu \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} f(x+y)\, \nu(dx)\, \mu(dy) \tag{4.10}$$

for every Borel measurable $f\colon \mathbb{R}_+ \to \mathbb{R}$ such that $(x, y) \mapsto f(x + y)$ is integrable with respect to the product of the measures $|\nu|$ and $|\mu|$.

It is easy to verify that $\nu * \mu \in \mathcal{M}_1$ for every $\nu, \mu \in \mathcal{M}_1$. Moreover $\nu * \mu \in \mathcal{M}_1$ has a simple probabilistic interpretation. Namely, if ν and μ are the distributions of independent random variables ξ_1 and ξ_2 respectively, then $\nu * \mu$ is the distribution of $\xi_1 + \xi_2$.

For fixed $\nu \in \mathcal{M}_1$ we define a linear operator $P_{*2}\colon \mathcal{M}_{\mathrm{sig}} \to \mathcal{M}_{\mathrm{sig}}$ by

$$P_{*2}\mu := \nu * \mu \quad \text{for } \mu \in \mathcal{M}_{\mathrm{sig}}. \tag{4.11}$$

Another class of operators we are going to study is related to multiplication of random variables. The formal definition is as follows. Given $\varphi, \mu \in \mathcal{M}_{\mathrm{sig}}$, we define their *elementary product* $\varphi \circ \mu$ by

$$(\varphi \circ \mu)(A) := \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} 1_A(xy)\, \varphi(dx)\, \mu(dy) \quad \text{for } A \in \mathcal{B}. \tag{4.12}$$

It follows that

$$\langle f, \varphi \circ \nu \rangle = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} f(xy)\, \varphi(dx)\, \nu(dy) \tag{4.13}$$

for every Borel measurable $f\colon \mathbb{R}_+ \to \mathbb{R}$ such that $(x, y) \mapsto f(xy)$ is integrable with respect to the product of $|\varphi|$ and $|\mu|$. For fixed $\varphi \in \mathcal{M}_1$ we define a linear operator $P_\varphi\colon \mathcal{M}_{\mathrm{sig}} \to \mathcal{M}_{\mathrm{sig}}$ by

$$P_\varphi \mu := \varphi \circ \mu \quad \text{for } \mu \in \mathcal{M}_{\mathrm{sig}}. \tag{4.14}$$

Again, as in the case of convolution, $P_\varphi(\mathcal{M}_1) \subset \mathcal{M}_1$. For $\mu \in \mathcal{M}_1$ the measure $P_\varphi\mu$ has an immediate probabilistic interpretation. If φ and μ are the distributions of random variables ξ and η respectively, then $P_\varphi\mu$ is the distribution of the product $\xi\eta$.

Now we return to equation (4.7) and give a precise definition of P:

$$P(\mu) := P_\varphi P_{*2}(\mu) = \varphi \circ (\nu_0 * \mu) \quad \text{for } \mu \in \mathcal{M}_{\mathrm{sig}}, \tag{4.15}$$

where $\nu_0 \in D$ and $\varphi \in \mathcal{M}_1$ is a probability measure such that $m_1(\varphi) = 1/2$. From (4.15) it follows that $P(\mathcal{M}_1) \subset \mathcal{M}_1$. Further it is easy to verify that for $\mu \in D$,

$$m_1(P_{*2}\mu) = 2 \quad \text{and} \quad m_1(P_\varphi\mu) = 1/2. \tag{4.16}$$

We may summarize this discussion with the following observations:

(1) From condition (4.6) it follows immediately that D is a convex subset of $\mathcal{M}_{\mathrm{sig}}$.

(2) It is known that the set D with the total variation norm is a complete metric space.

(3) If $\nu_0, \varphi \in \mathcal{M}_1$ and $m_1(\varphi) = 1/2$, $m_1(\nu_0) = 1$, then P maps D into itself.

Remark 4.1. Evidently every fixed point of P is a stationary solution of equation (4.7).

Equation (4.7) together with the initial condition (4.8) may be considered in the convex subset D of the vector space of signed measures. From the observations (1), (2), (3) and the results of [C] it follows immediately that for every $\psi_0 \in D$ the initial value problem (4.7), (4.8) has exactly one solution ψ satisfying the integral equation

$$\psi(t) = e^{-t}\,\psi_0 + \int_0^t e^{-(t-s)} P\psi(s)\,ds \quad \text{for } t \in \mathbb{R}_+. \tag{4.17}$$

Corollary 4.2. *If $\varphi, \nu_0 \in \mathcal{M}_1$ and $m_1(\varphi) = 1/2$, $m_1(\nu_0) = 1$ then for every $\psi_0 \in D$ there exists a unique solution ψ of problem* (4.7), (4.8).

The solutions of (4.17) generate a semigroup $(P^t)_{t\ge 0}$ of Markov operators on D given by

$$\psi(t) = P^t\,\psi_0 \quad \text{for } t \in \mathbb{R}_+,\ \psi_0 \in D. \tag{4.18}$$

We have the following result concerning the asymptotic stability of $(P^t)_{t\ge 0}$.

Proposition 4.3. *Let P be the operator given by* (4.15). *Moreover, let $\varphi, \nu_0 \in \mathcal{M}_1$ and $m_1(\varphi) = 1/2$, $m_1(\nu_0) = 1$. If P has a fixed point $\psi_* \in D$ such that*

$$\operatorname{supp}\psi_* = \mathbb{R}_+, \tag{4.19}$$

then

$$\lim_{t\to\infty} \|\psi(t) - \psi_*\|_T = 0 \tag{4.20}$$

for every sequentially compact solution ψ of (4.7), (4.8) *given by* (4.18).

Proof. From (4.17) it follows immediately that

$$\|P^t\psi_0 - \psi_*\|_T \le e^{-t}\|\psi_0 - \psi_*\|_T + \int_0^t e^{-(t-s)}\|P^s\psi_0 - \psi_*\|_T\,ds \quad \text{for } \psi_0 \in D \text{ and } t > 0.$$

This may be rewritten in the form

$$\begin{aligned}\|P^t\,\psi_0 - \psi_*\|_T &\le e^{-t}\|\psi_0 - \psi_*\|_T + (1 - e^{-t})\|\psi_0 - \psi_*\|_T \\ &= \|\psi_0 - \psi_*\|_T \quad \text{for } \psi_0 \in D \text{ and } t > 0.\end{aligned} \tag{4.21}$$

Condition (4.19) is equivalent to the fact that the measures $P^t\psi_0, \psi_* \in D$ overlap supports for $t > 0$ and $\psi_0 \in D$. Consequently, in (4.21) we have a strict inequality. An application of Theorem 3.9 completes the proof. ■

REMARK 4.4. Observe that in the case of the classical linear Tjon–Wu type equation (4.1) the measure φ is absolutely continuous with density $\mathbf{1}_{[0,1]}$. Moreover, $u_*(t,x) := \exp(-x)$ is the density function of the stationary solution of (4.1). This is a simple illustration of the situation described by Proposition 4.3.

REMARK 4.5. The condition (4.19) is not particularly restrictive because in [LT3] it has been proven that the stationary solution ϕ_* of a more general equation has the following property: Either ψ_* is supported on one point or $\operatorname{supp}\psi_* = \mathbb{R}_+$.The first case holds if and only if $\varphi = \delta_{1/2}$. But this case can be ignored as a physical model of particle collisions because it is more restrictive than the model described by the classical Tjon–Wu equation.

References

[C] M. G. Crandall, *Differential equations on convex sets*, J. Math. Soc. Japan 22 (1970), 443–455.

[G1] H. Gacki, *On the Kantorovich–Rubinstein maximum principle for the Fortet–Mourier norm*, Ann. Polon. Math. 86 (2005), 107–121.

[G2] H. Gacki, *Applications of the Kantorovich–Rubinstein maximum principle in the theory of Markov semigroups*, Dissertationes Math. 448 (2007), 59 pp.

[GL] H. Gacki and A. Lasota, *A nonlinear version of the Kantorovich–Rubinstein maximum principle*, Nonlinear Anal. 52 (2003), 117–125.

[L1] A. Lasota, *Invariant principle for discrete time dynamical systems*, Univ. Iagell. Acta Math. 31 (1994), 111–127.

[L2] A. Lasota, *Asymptotic stability of some nonlinear Boltzmann-type equations*, J. Math. Anal. Appl. 268 (2002), 291–309.

[LM] A. Lasota and M. C. Mackey, *Chaos, Fractals, and Noise*, Springer, Berlin, 1994.

[LT1] A. Lasota and J. Traple, *An application of the Kantorovich–Rubinstein maximum principle in the theory of the Tjon–Wu equation*, J. Differential Equations 159 (1999), 578–596.

[LT2] A. Lasota and J. Traple, *Asymptotic stability of differential equations on convex sets*, J. Dynam. Differential Equations 15 (2003), 335–355.

[LT3] A. Lasota and J. Traple, *Properties of stationary solutions of a generalized Tjon–Wu equation*, J. Math. Anal. Appl. 335 (2007), 669 – 682.

[R] S. T. Rachev, *Probability Metrics and the Stability of Stochastic Models*, Wiley, New York, 1991.

[TW] J. A. Tjon and T. T. Wu, *Numerical aspects of the approach to a Maxwellian equation*, Phys. Rev. A 19 (1979), 883–888.

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