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A STUDY OF A UNILATERAL AND ADHESIVE CONTACT PROBLEM WITH NORMAL COMPLIANCE

Abstract. The aim of this paper is to study a quasistatic unilateral contact problem between an elastic body and a foundation. The constitutive law is nonlinear and the contact is modelled with a normal compliance condition associated to a unilateral constraint and Coulomb's friction law. The adhesion between contact surfaces is taken into account and is modelled with a surface variable, the bonding field, whose evolution is described by a first-order differential equation. We establish a variational formulation of the mechanical problem and prove an existence and uniqueness result in the case where the friction coefficient is small enough. The technique of proof is based on time-dependent variational inequalities, differential equations and the Banach fixed-point theorem. We also study a penalized and regularized problem which admits at least one solution and prove its convergence to the solution of the model when the penalization and regularization parameter tends to zero.

1. Introduction. Contact problems involving deformable bodies are quite frequent in industry as well as in daily life and play an important role in structural and mechanical systems. Contact processes involve complicated surface phenomena, and are modelled by highly nonlinear initial boundary value problems. Taking into account various contact conditions associated with more and more complex behavior laws leads to the introduction of new and nonstandard models, expressed with the aid of evolution variational inequalities. An early attempt to study contact problems within the framework of variational inequalities was made in [10]. The mathematical, mechanical and numerical state of the art can be found in [23] where we find detailed mathematical and numerical studies of adhesive contact prob-

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lems. Unilateral frictional contact problems involving Signorini's condition with or without adhesion were studied by several authors: see for instance the references in [1, 2, 5, 6, 8, 11, 17, 20, 27, 30, 32, 31].

In this paper, we study a mathematical model which describes a frictional unilateral contact problem with adhesion between a nonlinear elastic body and a deformable foundation. Following [16, 28] the contact is modelled with a normal compliance condition associated to unilateral constraint, where the penetration is limited. Recall that models for dynamic or quasistatic processes of frictionless adhesive contact between a deformable body and a foundation have been studied in [3, 4, 12, 17, 22, 23, 24, 28]. Also recently dynamic or quasistatic frictional contact problems with adhesion were studied in [7, 29]. Here as in [13, 14] we use the bonding field as an additional state variable β, defined on the contact surface of the boundary. The variable is restricted to values $0 \leq \beta \leq 1$; when $\beta = 0$ all the bonds are severed and there are no active bonds, when $\beta = 1$ all the bonds are active; when $0 < \beta < 1$ it measures the fraction of active bonds and partial adhesion takes place. We refer the reader to the extensive bibliography on the subject in [2, 12, 14, 20, 22, 23].

In this work we extend the result established in [27] to the unilateral contact problem with a normal compliance condition associated to Coulomb's friction law. We establish a variational formulation of the mechanical problem for which we prove the existence of a unique weak solution if the friction coefficient is small enough and obtain a partial regularity result for the solution. We also consider a penalized and regularized problem which has at least one solution and prove its convergence to the solution of the model when the parameter of penalization and regularization tends to zero.

The paper is structured as follows. In Section 2 we present some notations and give the variational formulation. In Section 3 we state and prove our main existence and uniqueness result, Theorem 3.1. In Section 4 we establish a convergence result, Theorem 4.6.

2. Problem statement and variational formulation. We consider a nonlinear elastic body which occupies a domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ and assume that its boundary Γ is regular and partitioned into three measurable and disjoint parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that $\operatorname{meas}(\Gamma_1) > 0$. The body is acted upon by a volume force of density φ_1 on Ω and a surface traction of density φ_2 on Γ_2. On Γ_3 the body is in unilateral and adhesive contact following Coulomb's friction law with a foundation.

Thus, the classical formulation of the mechanical problem is as follows.

PROBLEM P_1. Find a displacement $u : \Omega \times [0, T] \to \mathbb{R}^d$ and a bonding field $\beta : \Gamma_3 \times [0, T] \to \mathbb{R}$ such that

$$\text{(2.1)} \quad \operatorname{div}\sigma + \varphi_1 = 0 \quad \text{in } \Omega \times (0,T),$$
$$\text{(2.2)} \quad \sigma = F\varepsilon(u) \quad \text{in } \Omega \times (0,T),$$
$$\text{(2.3)} \quad u = 0 \quad \text{on } \Gamma_1 \times (0,T),$$
$$\text{(2.4)} \quad \sigma\nu = \varphi_2 \quad \text{on } \Gamma_2 \times (0,T),$$
$$\text{(2.5)} \quad \left.\begin{array}{l} u_\nu \le g,\ \sigma_\nu + p(u_\nu) - c_\nu \beta^2 R_\nu(u_\nu) \le 0 \\ (\sigma_\nu + p(u_\nu) - c_\nu\beta^2 R_\nu(u_\nu))(u_\nu - g) = 0 \end{array}\right\} \text{ on } \Gamma_3 \times (0,T),$$
$$\text{(2.6)} \quad \left.\begin{array}{l} |\sigma_\tau + c_\tau \beta^2 R_\tau(u_\tau)| \le \mu p(u_\nu) \\ |\sigma_\tau + c_\tau \beta^2 R_\tau(u_\tau)| < \mu p(u_\nu) \Rightarrow u_\tau = 0 \\ |\sigma_\tau + c_\tau \beta^2 R_\tau(u_\tau)| = \mu p(u_\nu) \Rightarrow \\ \exists \lambda \ge 0, \mu p(u_\nu) = -(\sigma_\tau + c_\tau \beta^2 R_\tau(u_\tau)) \end{array}\right\} \text{ on } \Gamma_3 \times (0,T),$$
$$\text{(2.7)} \quad \dot\beta = -[\beta(c_\nu(R_\nu(u_\nu))^2 + c_\tau |R_\tau(u_\tau)|^2) - \varepsilon_a]_+ \quad \text{on } \Gamma_3 \times (0,T),$$
$$\text{(2.8)} \quad \beta(0) = \beta_0 \quad \text{on } \Gamma_3.$$

Equation (2.1) represents the equilibrium equation. Equation (2.2) represents the elastic constitutive law of the material in which F is a given function and $\varepsilon(u)$ denotes the strain tensor; (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the Cauchy stress vector. The condition (2.5) represents the unilateral contact with adhesion in which p and $-c_\nu \beta^2 R_\nu(u_\nu)$ are the normal contact functions. Here c_ν is a given adhesion coefficient and R_ν is a truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which the bond has no additional traction (see [23]) and p is a normal compliance function which satisfies assumption (2.16) below. We denote by $g \ge 0$ the maximum value of the penetration. When $u_\nu < 0$, i.e. when there is separation between the body and the foundation, then the condition (2.5) combined with hypotheses (2.16) on the function p shows that $\sigma_\nu = c_\nu \beta^2 R_\nu(u_\nu)$ and by assumption (2.17) below, σ_ν does not exceed the value Lg. When $g > 0$, the body may interpenetrate into the foundation, but the penetration is limited, that is, $u_\nu \le g$. In this case of penetration (i.e. $u_\nu \ge 0$), when $0 \le u_\nu < g$ then $-\sigma_\nu = p(u_\nu)$, which means that the reaction of the foundation is uniquely determined by the normal displacement and $\sigma_\nu \le 0$. Since p is an increasing function, the reaction of the foundation increases with the penetration, and when $u_\nu = g$, then $-\sigma_\nu \ge p(g)$ and σ_ν is not uniquely determined. When $g > 0$ and $p = 0$, condition (2.5) becomes the Signorini contact condition with adhesion with a gap function,

$$u_\nu \le g, \quad \sigma_\nu - c_\nu \beta^2 R_\nu(u_\nu) \le 0, \quad (\sigma_\nu - c_\nu \beta^2 R_\nu(u_\nu))(u_\nu - g) = 0.$$

When $g = 0$, (2.5) combined with assumption (2.16) becomes the Signorini contact condition with adhesion with a zero gap function, given by

$$u_\nu \le 0, \quad \sigma_\nu - c_\nu \beta^2 R_\nu(u_\nu) \le 0, \quad (\sigma_\nu - c_\nu \beta^2 R_\nu(u_\nu))u_\nu = 0.$$

This contact condition was used in [6, 23, 24, 27, 28, 30]. The condition (2.6) represents the frictional contact in which $c_\tau \beta^2 R_\tau(u_\tau)$ is an adhesive where c_τ is an adhesion coefficient and R_τ is a truncation operator defined by

$$R_\tau(v) = \begin{cases} v & \text{if } |v| \le L, \\ L\dfrac{v}{|v|} & \text{if } |v| > L, \end{cases}$$

where $L > 0$ is the characteristic length of the bonds. Equation (2.7) is the ordinary differential equation which describes the evolution of the bonding field, where ε_a is an adhesion coefficient and $\beta_+ = \max(0, \beta)$. Since $\dot{\beta} \le 0$ on $\Gamma_3 \times (0, T)$, once debonding occurs bonding cannot be reestablished and, indeed, the adhesive process is irreversible. Also from [18] it must be pointed out that condition (2.7) does not allow for complete debonding in finite time. Finally, (2.8) is the initial condition, in which β_0 denotes the initial bonding field. In (2.7) a dot above a variable represents its derivative with respect to time. We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$), and $|\cdot|$ represents the Euclidean norm on $\mathbb{R}^d$ and S_d. Thus, for every $u, v \in \mathbb{R}^d$, $u.v = u_i v_i$, $|v| = (v.v)^{1/2}$, and for every $\sigma, \tau \in S_d$, $\sigma.\tau = \sigma_{ij}\tau_{ij}$, $|\tau| = (\tau.\tau)^{1/2}$. Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted.

Now, to proceed with the variational formulation, we need the following function spaces:

$$H = (L^2(\Omega))^d, \quad H_1 = (H^1(\Omega))^d, \quad Q = \{\sigma = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$
$$Q_1 = \{\sigma \in Q : \operatorname{div} \sigma \in H\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products

$$(u, v)_H = \int_\Omega u_i v_i \, dx, \quad (\sigma, \tau)_Q = \int_\Omega \sigma_{ij} \tau_{ij} \, dx.$$

The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)) = \frac{1}{2}(u_{i,j} + u_{j,i});$$

$\operatorname{div} \sigma = (\sigma_{ij,j})$ is the divergence of σ. For every $v \in H_1$ we denote by v_ν and v_τ the normal and tangential components of v on the boundary Γ given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu.$$

We also denote by σ_ν and σ_τ the normal and the tangential traces of a function $\sigma \in Q_1$, and when σ is a regular function then

$$\sigma_\nu = (\sigma\nu).\nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu\nu,$$

and the following Green formula holds:

$$(\sigma, \varepsilon(v))_Q + (\operatorname{div}\sigma, v)_H = \int_\Gamma \sigma\nu.v\,da \qquad \forall v \in H_1,$$

where da is the surface measure element.

Now, let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_1\},$$

and let the convex subset of admissible displacements be given by

$$K = \{v \in V : v_\nu \le g \text{ a.e. on } \Gamma_3\}.$$

Since $\operatorname{meas}(\Gamma_1) > 0$, the following Korn inequality holds [10]:

$$\|\varepsilon(v)\|_Q \ge c_\Omega \|v\|_{H_1} \qquad \forall v \in V, \tag{2.9}$$

where $c_\Omega > 0$ is a constant which depends only on Ω and Γ_1. We equip V with the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_Q$$

and $\|\cdot\|_V$ is the associated norm. It follows from (2.9) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on V. Thus $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which only depends on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \le d_\Omega \|v\|_V \qquad \forall v \in V. \tag{2.10}$$

For $p \in [1, \infty]$, we use the standard norm of $L^p(0, T; V)$. We also use the Sobolev space $W^{1,\infty}(0, T; V)$ equipped with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot{v}\|_{L^\infty(0,T;V)}.$$

For every real Banach space $(X, \|\cdot\|_X)$ and $T > 0$ we use the notation $C([0, T]; X)$ for the space of continuous functions from $[0, T]$ to X; recall that $C([0, T]; X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

We suppose that the body forces and surface tractions have the regularity

$$\varphi_1 \in W^{1,\infty}(0, T; H), \qquad \varphi_2 \in W^{1,\infty}(0, T; (L^2(\Gamma_2))^d), \tag{2.11}$$

and denote by $f(t)$ the element of V defined by

$$(f(t), v)_V = \int_\Omega \varphi_1(t).v\,dx + \int_{\Gamma_2} \varphi_2(t).v\,da, \qquad \forall v \in V,\ t \in [0, T]. \tag{2.12}$$

Using (2.11) and (2.12) yields

$$f \in W^{1,\infty}(0, T; V).$$

In the study of the mechanical problem P_1 we assume that the nonlinear elasticity operator $F : \Omega \times S_d \to S_d$ satisfies:

$$
(2.13) \quad \begin{cases}
\text{(a)} & \text{there exists } M > 0 \text{ such that} \\
& |F(x,\varepsilon_1) - F(x,\varepsilon_2)| \le M|\varepsilon_1 - \varepsilon_2|, \\
& \forall \varepsilon_1, \varepsilon_2 \in S_d, \text{ a.e. } x \in \Omega; \\
\text{(b)} & \text{there exists } m > 0 \text{ such that} \\
& (F(x,\varepsilon_1) - F(x,\varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \ge m|\varepsilon_1 - \varepsilon_2|^2, \\
& \forall \varepsilon_1, \varepsilon_2 \in S_d, \text{ a.e. } x \in \Omega; \\
\text{(c)} & \text{the mapping } x \mapsto F(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega, \\
& \text{for any } \varepsilon \in S_d; \\
\text{(d)} & F(x,0) = 0 \text{ for a.e. } x \in \Omega.
\end{cases}
$$

The adhesion coefficients are assumed to satisfy

$$
(2.14) \qquad c_\nu, c_\tau \in L^\infty(\Gamma_3), \quad \varepsilon_a \in L^2(\Gamma_3), \quad c_\nu, c_\tau, \varepsilon_a \ge 0 \quad \text{a.e. on } \Gamma_3,
$$

and the friction coefficient μ satisfies

$$
(2.15) \qquad \mu \in L^\infty(\Gamma_3) \quad \text{and} \quad \mu \ge 0 \text{ a.e. on } \Gamma_3.
$$

Next we define the functional $j_{\mathrm{ad}} : L^2(\Gamma_3) \times V \times V \to \mathbb{R}$ by

$$
j_{\mathrm{ad}}(\beta, u, v) = \int_{\Gamma_3} [(p(u_\nu) - c_\nu \beta^2 R_\nu(u_\nu))v_\nu + c_\tau \beta^2 R_\tau(u_\tau).v_\tau]\, da, \qquad \forall (\beta, u, v) \in L^2(\Gamma_3) \times V \times V,
$$

and the functional $j_{\mathrm{fr}} : V \times V \to \mathbb{R}_+$ by

$$
j_{\mathrm{fr}}(u, v) = \int_{\Gamma_3} \mu p(u_\nu)|v_\tau|\, da, \qquad \forall (u, v) \in V \times V.
$$

We assume that the normal compliance function $p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

$$
(2.16) \quad \begin{cases}
\text{(a)} & \text{There exists } L_p > 0 \text{ such that} \\
& |p(x,r_1) - p(x,r_2)| \le L_p|r_1 - r_2|, \\
& \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3; \\
\text{(b)} & (p(x,r_1) - p(x,r_2))(r_1 - r_2) \ge 0, \\
& \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3; \\
\text{(c)} & \text{the mapping } x \mapsto p_\nu(x,r) \text{ is Lebesgue measurable on } \Gamma_3, \\
& \text{for any } r \in \mathbb{R}; \\
\text{(d)} & p(x,r) = 0, \ \forall r \le 0, \text{ a.e. } x \in \Gamma_3.
\end{cases}
$$

We also assume that the initial bonding field satisfies

$$
(2.17) \qquad \beta_0 \in L^2(\Gamma_3), \quad 0 \le \beta_0 \le 1 \quad \text{a.e. on } \Gamma_3,
$$

and we introduce the set

$$B = \{\theta : [0,T] \to L^2(\Gamma_3);\ 0 \le \theta(t) \le 1,\ \forall t \in [0,T],\ \text{a.e. on } \Gamma_3\}.$$

Finally, assuming that the solution is sufficiently regular and applying Green's formula, we deduce the following variational formulation of the mechanical problem P_1.

PROBLEM P_2. Find a displacement field $u : [0,T] \to V$ and a bonding field $\beta : [0,T] \to L^2(\Gamma_3)$ such that

$$u(t) \in K,$$

$$\begin{aligned}(2.18)\quad &\big(F\varepsilon(u(t)), \varepsilon(v) - \varepsilon(u(t))\big)_Q + j_{\mathrm{ad}}\big(\beta(t), u(t), v - u(t)\big)\\ &+ j_{\mathrm{fr}}(u(t), v) - j_{\mathrm{fr}}(u(t), u(t)) \ge (f(t), v - u(t))_V,\ \forall v \in K,\ t \in [0,T],\end{aligned}$$

$$(2.19)\quad \dot\beta(t) = -\big[\beta(t)\big(c_\nu(R_\nu(u_\nu(t)))^2 + c_\tau|R_\tau(u_\tau(t))|^2\big) - \varepsilon_a\big]_+ \ \text{a.e. } t \in (0,T),$$

$$(2.20)\quad \beta(0) = \beta_0.$$

3. Existence and uniqueness result. Our main result is the following theorem.

THEOREM 3.1. *Let* (2.11) *and* (2.13)–(2.17) *hold. Then Problem* P_2 *has a unique solution, which satisfies*

$$(3.1)\qquad u \in W^{1,\infty}(0,T;V) \cap C([0,T];K)$$

and

$$(3.2)\qquad \beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap B,$$

if

$$\|\mu\|_{L^\infty(\Gamma_3)} < m/L_p d_\Omega^2.$$

The proof of Theorem 3.1 is carried out in several steps. In the first step, let $k > 0$ and consider the closed subset X of $C([0,T];L^2(\Gamma_3))$ defined as

$$X = \{\theta \in C([0,T];\ L^2(\Gamma_3)) \cap B : \theta(0) = \beta_0\},$$

where the Banach space $C([0,T];L^2(\Gamma_3))$ is endowed with the norm

$$\|\theta\|_X = \max_{t\in[0,T]} [\exp(-kt)\|\theta(t)\|_{L^2(\Gamma_3)}] \quad \text{for } \theta \in C([0,T];L^2(\Gamma_3)).$$

Next for a given $\beta \in X$, we consider the following variational problem.

PROBLEM $P_{1\beta}$. Find $u_\beta : [0,T] \to V$ such that

$$u_\beta(t) \in K,$$

$$\begin{aligned}(3.3)\quad &\big(F\varepsilon(u_\beta(t)), \varepsilon(v - u_\beta(t))\big)_Q + j_{\mathrm{ad}}\big(\beta(t), u_\beta(t), v - u_\beta(t)\big)\\ &+ j_{\mathrm{fr}}(u_\beta(t), v) - j_{\mathrm{fr}}(u_\beta(t), u_\beta(t)) \ge (f(t), v - u_\beta(t))_V\\ &\hspace{20em}\forall v \in K,\ t \in [0,T].\end{aligned}$$

We have the following result.

Proposition 3.2. *Problem $P_{1\beta}$ has a unique solution*

$$u_\beta \in C([0,T];K) \tag{3.4}$$

if

$$\|\mu\|_{L^\infty(\Gamma_3)} < m/L_p d_\Omega^2.$$

The proof of Proposition 3.2 will be established in several steps. In the first step for each $t \in [0,T]$ and a given $\eta \in K$, we consider the following intermediate problem.

Problem $P_{\beta\eta}$. Find $u_{\beta\eta}(t) \in K$ such that

$$\begin{aligned}(F\varepsilon(u_{\beta\eta}(t)), \varepsilon(v - u_{\beta\eta}(t)))_Q + j_{\mathrm{ad}}\big(\beta(t), u_{\beta\eta}(t), v - u_{\beta\eta}(t)\big) + j(\eta, v) \\ - j_{\mathrm{fr}}(\eta, u_{\beta\eta}(t)) \geq (f(t), v - u_{\beta\eta}(t))_V, \quad \forall v \in K.\end{aligned} \tag{3.5}$$

Lemma 3.3. *Problem $P_{\beta\eta}$ has a unique solution.*

Proof. Let the operator $A_{\beta(t)} : V \to V$ be defined by

$$(A_{\beta(t)}u, v)_V = (F\varepsilon(u), \varepsilon(v))_Q + j_{\mathrm{ad}}(\beta(t), u, v), \quad \forall u, v \in V.$$

We use (2.10), (2.13)(a), (2.13)(b), (2.16)(b) and (2.16)(c) to show that the operator $A_{\beta(t)}$ is strongly monotone and Lipschitz continuous.

Moreover, the functional $j(\eta, \cdot) : V \to \mathbb{R}_+$ is a continuous seminorm. Hence by a standard existence and uniqueness result for elliptic variational inequalities (see [25]), there exists a unique element $u_{\beta\eta}(t) \in K$ which satisfies (3.5) since K is a non-empty, closed convex subset of V. ■

Now, in the second step, for a fixed $t \in [0,T]$ we consider the map $T_t : K \to K$ defined as

$$T_t(\eta) = u_{\beta\eta}(t).$$

We have the following lemma.

Lemma 3.4. *The map T_t has a unique fixed point η^* and $u_{\beta\eta^*}(t)$ is a unique solution of the inequality* (3.3).

Proof. Let $\eta_1, \eta_2 \in K$. In (3.5) satisfied by $u_{\eta_1}(t)$ set $v = u_{\eta_2}(t)$, and also in the same inequality satisfied by $u_{\eta_2}(t)$ take $v = u_{\eta_1}(t)$. Using (2.10), (2.13)(c) and (2.16), it follows after adding the resulting inequalities that

$$\|T_t(\eta_1) - T_t(\eta_2)\|_V \leq \frac{\|\mu\|_{L^\infty(\Gamma_3)} L_p d_\Omega^2}{m} \|\eta_1 - \eta_2\|_V, \quad \forall \eta_1, \eta_2 \in K.$$

Then for $\|\mu\|_{L^\infty(\Gamma_3)} L_p d_\Omega^2 / m < 1$, the map T_t is a contraction; so it has a unique fixed point η^* and $u_{\beta\eta^*}(t)$ is a unique solution of inequality (3.3). Next, denote $u_{\beta\eta^*}(t) = u_\beta(t)$ for each $t \in [0,T]$. As in [31], to show that $u_\beta \in C([0,T];K)$, it suffices to see from (3.3) that there exists a constant

$c > 0$ such that

$$\text{(3.6)} \quad \|u_\beta(t_1) - u_\beta(t_2)\|_V \le \frac{c}{m - \|\mu\|_{L^\infty(\Gamma_3)} L_p d_\Omega^2} \big(\|f(t_1) - f(t_2)\|_V + \|\beta(t_1) - \beta(t_2)\|_{L^2(\Gamma_3)}\big)$$

for all $t_1, t_2 \in [0,T]$. Therefore, as $f \in C([0,T];V)$ and $\beta \in C([0,T];L^2(\Gamma_3))$, we immediately deduce (3.4). ■

In the second step, we consider the following problem.

PROBLEM $P_{2\beta}$. Find $\chi_\beta : [0,T] \to L^2(\Gamma_3)$ such that

$$\text{(3.7)} \quad \dot\chi_\beta(t) = -\big[\chi_\beta(t)\big(c_\nu(R_\nu(u_{\beta\nu}(t)))^2 + c_\tau|R_\tau(u_{\beta\tau}(t))|^2\big) - \varepsilon_a\big]_+ \quad \text{a.e. } t \in (0,T),$$

$$\text{(3.8)} \quad \chi_\beta(0) = \beta_0.$$

We obtain the following result.

LEMMA 3.5. *Problem $P_{2\beta}$ has a unique solution χ_β which satisfies*

$$\chi_\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap B.$$

Proof. Consider the mapping $F_\beta(t,\theta) : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_\beta(t,\theta) = -\big[\theta\big(c_\nu(R_\nu(u_{\beta\nu}(t)))^2 + c_\tau|R_\tau(u_{\beta\tau}(t))|^2\big) - \varepsilon_a\big]_+.$$

It follows from the properties of the truncation operators R_ν and R_τ, that F_β is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, for any $\theta \in L^2(\Gamma_3)$, the mapping $t \mapsto F_\beta(t,\theta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Then, from a version of the Cauchy–Lipschitz theorem, we deduce the existence of a unique fonction $\chi_\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$, which satisfies (3.7), (3.8). The regularity $\chi_\beta \in B$ follows from (3.7), (3.8) and (2.17) (see [22, 24, 26]). Therefore, from Lemma 3.5, we deduce that for all $\beta \in X$, the solution χ_β of Problem $P_{2\beta}$ belongs to X.

Next, we define the mapping $\Lambda : X \to X$ by

$$\Lambda\beta = \chi_\beta.$$

The third step consists of the following lemma.

LEMMA 3.6. *The mapping Λ has a unique fixed point β^*.*

Proof. We have

$$\Lambda\beta(t) = \beta_0 - \int_0^t \big[\chi_\beta(s)\big(c_\nu(R_\nu(u_{\beta\nu}(s)))^2 + c_\tau|R_\tau(u_{\beta\tau}(s))|^2\big) - \varepsilon_a\big]_+ ds,$$

where u_β is the solution of Problem $P_{1\beta}$. Then for β_1, $\beta_2 \in X$, using (2.19)(a) and the properties of R_ν and R_τ (see [23]), there exists a constant $c_1 > 0$ such that

$$|\chi_{\beta_1}(t) - \chi_{\beta_2}(t)| \le c_1 \int_0^t \big(|\chi_{\beta_1}(s) - \chi_{\beta_2}(s)| + |u_{\beta_1\tau}(s) - u_{\beta_2\tau}(s)|\big)\, ds.$$

Applying Gronwall's inequality and using (2.10) yields

$$\|\chi_{\beta_1}(t) - \chi_{\beta_2}(t)\|_{L^2(\Gamma_3)} \le c_2 \int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V \, ds$$

for some constant $c_2 > 0$. Now let $t \in [0, T]$. Then, using the inequalities (3.3), (2.13), (2.16) and $\|\mu\|_{L^\infty(\Gamma_3)} L_p d_\Omega^2 < m$, we deduce that there exists a constant $c_3 > 0$ (see [31]) such that

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V \le c_3 \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}.$$

Hence,

$$\|\Lambda\beta_1(t) - \Lambda\beta_2(t)\|_{L^2(\Gamma_3)} \le c_4 \int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\, ds, \qquad \forall t \in [0, T],$$

for some constant $c_4 > 0$. Therefore,

$$\|\Lambda\beta_1 - \Lambda\beta_2\|_X \le \frac{c_4}{k} \|\beta_1 - \beta_2\|_X, \qquad \forall \beta_1, \beta_2 \in X.$$

Thus, this previous inequality shows that for k sufficiently large, Λ is a contraction. Hence it has a unique fixed point β^* which satisfies (3.7) and (3.8). On the other hand, from (3.6) we deduce that $u_{\beta^*} \in W^{1,\infty}(0, T; V)$.

Proof of Theorem 3.1. Let $\beta = \beta^*$ and let u_{β^*} be the solution to Problem $P_{1\beta}$. We conclude by (3.3), (3.7) and (3.8) that (u_{β^*}, β^*) is a solution of Problem P_2. Now to prove the uniqueness of the solution, suppose that (u, β) is a solution of Problem P_2 which satisfies (2.18)–(2.20). It follows from (2.18) that u is a solution of Problem $P_{1\beta}$ and by Proposition 3.2 we get $u = u_\beta$. Taking $u = u_\beta$ in (2.18) and using the initial condition (2.20), we deduce that β is a solution of Problem $P_{2\beta}$. Finally, using Lemma 3.5, we obtain $\beta = \beta^*$ and so (u_{β^*}, β^*) is a unique solution to Problem P_2 which satisfies (3.1), (3.2).

4. A convergence result. In this section we consider a frictional contact problem with normal compliance and adhesion with unlimited penetration. The contact condition (2.5) is replaced by the contact condition

$$-\sigma_\nu = \frac{1}{\delta}(u_\nu - g)_+ + p(u_\nu) - c_\nu \beta^2 R_\nu(u_\nu) \quad \text{ on } \Gamma_3 \times (0, T).$$

We recall that $\delta > 0$ is the penalization and regularization parameter and $1/\delta$ is interpreted as the stiffness coefficient of the foundation. We understand that when δ is small, the reaction of the foundation to the penetration is important; also when δ is large then the reaction is weaker. We study the behavior of the solution as $\delta \to 0$ and prove that in the limit we obtain the solution of the adhesive frictionless contact problem with normal compliance and finite penetration. Next we define the functionals $j_{\mathrm{ad}\delta} : L^2(\Gamma_3) \times V \times V \to \mathbb{R}$ and $j_{\mathrm{fr}\delta} : V \times V \to \mathbb{R}$ by

$$j_{\mathrm{ad}\delta}(\beta, u, v) = \int_{\Gamma_3} \left(\frac{1}{\delta}(u_\nu - g)_+ + p(u_\nu) - c_\nu \beta^2 R_\nu(u_\nu) \right) v_\nu \, da, \qquad \forall (\beta, u, v) \in L^2(\Gamma_3) \times V \times V,$$

$$j_{\mathrm{fr}\delta}(u, v) = \int_{\Gamma_3} \mu p(u_\nu) \frac{u_\tau}{\sqrt{u_\tau^2 + \delta^2}} v_\tau \, da, \qquad \forall u, v \in V.$$

With this notation, the variational formulation of the penalized and regularized problem with frictional contact and adhesion is the following.

Problem P_δ. Find $u_\delta : [0, T] \to V$ and $\beta_\delta : [0, T] \to [0, 1]$ such that

$$(4.1) \qquad (F\varepsilon(u_\delta(t)), \varepsilon(v))_Q + j_{\mathrm{ad}\delta}(\beta_\delta(t), u_\delta(t), v) + j_{\mathrm{fr}\delta}(u_\delta(t), v) = (f(t), v)_V, \qquad \forall v \in V,\ t \in [0, T],$$

$$(4.2) \qquad \dot\beta_\delta(t) = -\left[\beta_\delta(t)\left(c_\nu (R_\nu(u_{\delta\nu}))^2 + c_\tau |R_\tau(u_{\delta\tau})|^2\right) - \varepsilon_a\right]_+ \text{ on } \Gamma_3 \times (0, T),$$

$$(4.3) \qquad \beta_\delta(0) = \beta_0.$$

We have the following result.

Theorem 4.1. *Problem P_δ has a solution which satisfies*

$$u_\delta \in L^\infty(0, T; V), \qquad \beta_\delta \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap B.$$

Proof. As in [24], the proof is similar to the proof of Theorem 3.1 and it is carried out in several steps.

(i) For any $\beta \in X$, we consider the problem below.

Problem $P_{1\delta}$. Find $u_\delta : [0, T] \to V$ such that

$$(4.4) \qquad \left(F\varepsilon(u_\delta(t)), \varepsilon(v)\right)_Q + j_{\mathrm{ad}\delta}(\beta(t), u_\delta(t), v) + j_{\mathrm{fr}\delta}\left(u_\delta(t), v\right) = (f(t), v)_V, \qquad \forall v \in V,\ t \in [0, T].$$

We have the following lemma.

Lemma 4.2. *Problem $P_{1\delta}$ has a solution $u_\delta \in L^\infty(0, T; V)$.*

Proof. As in [11], to prove Lemma 4.2 we use the theorem on pseudomonotone operators. For this, let us state some properties and abstract results for these operators, obtained in [33]. We start with the following definition.

DEFINITION 4.3. Let X be a reflexive Banach space and X' be its dual space. An operator $A : X \to X'$ is called *pseudomonotone* if

$$u_k \to u \text{ weakly in } X, \limsup_{k\to\infty} \langle Au_k, u_k - u\rangle_{X'\times X} \le 0 \\ \Rightarrow \langle Au, u-w\rangle_{X'\times X} \le \liminf_{k\to\infty} \langle Au_k, u_k - w\rangle_{X'\times X}, \ \forall w \in X.$$

In order to decide whether a given operator is pseudomonotone or not, we need Proposition 4.4 and Theorem 4.5 below.

PROPOSITION 4.4. *Let X be a reflexive Banach space, $A : X \to X'$ be strongly monotone and $B : X \to X'$ be completely continuous. Then the sum $C = A + B$ is pseudomonotone.*

A nonlinear operator $C : X \to X'$ is called *coercive* if

$$\lim_{\|u\|_X\to\infty} \frac{\langle Au, u\rangle_{X'\times X}}{\|u\|_X} = +\infty.$$

The theorem on pseudomonotone operators that we use is the following.

THEOREM 4.5. *Let X be a real, reflexive and separable Banach space and $C : X \to X'$ be pseudomonotone, bounded and coercive. Then for each $L \in X'$ the equation*

$$Cu = L$$

has a solution.

In order to apply this theorem to our situation, we define the operators $A_t, P : V \to V'$ by

$$\langle A_t u, v\rangle_{V'\times V} = \langle F\varepsilon(u), \varepsilon(v)\rangle_Q + j_{\mathrm{ad}\delta}(\beta(t), u, v),$$
$$\langle Pu, v\rangle_{V'\times V} = \int_{\Gamma_3} \mu p(u_\nu) \frac{u_\tau}{\sqrt{u_\tau^2 + \delta^2}} v_\tau \, da, \qquad \forall u, v \in V.$$

We use (2.13), (2.14) and (2.16) and the definition of the operator R_ν to see that A_t is strongly monotone and Lipschitz continuous. We also use (2.15) and (2.16), the compact embedding $H^{1/2}(\Gamma_3) \hookrightarrow L^2(\Gamma_3)$ and the Lebesgue dominated convergence theorem to show that the operator P is completely continuous. We have

$$|\langle Pu, v\rangle_{V'\times V}| \le \|\mu\|_{L^\infty(\Gamma_3)} L_p d_\Omega^2 \|u\|_V \|v\|_V, \quad \forall u, v \in V.$$

Moreover,

$$\langle Pv, v\rangle_{V'\times V} \geq 0, \qquad \forall v \in V.$$

Thus the operator $C_t = A_t + P$ is pseudomonotone, bounded and coercive. Consequently, by applying Theorem 4.5 we deduce that equation (4.4) has solution $u_\delta(t) \in V$. To prove that $u_\delta \in L^\infty(0,T;V)$, it suffices to take $v = u_\delta(t)$ in (4.4); since

$$j_{\mathrm{ad}\delta}\big(\beta_\delta(t), u_\delta(t), u_\delta(t)\big) \geq 0, \qquad j_{\mathrm{fr}\delta}(u_\delta(t), u_\delta(t)) \geq 0,$$

it follows from (4.4) that

$$\big(F\varepsilon(u_\delta(t)), \varepsilon(u_\delta(t))\big)_Q \leq (f(t), u_\delta(t))_V.$$

This inequality together with the assumption (2.13)(b) implies that

$$\|u_\delta(t)\|_V \leq \|f(t)\|_V/m. \tag{4.5}$$

Hence using the regularity $f \in C([0,T];V)$, we immediately conclude from (4.5) that $u_\delta \in L^\infty(0,T;V)$.

(ii) There exists a unique β_δ such that

(4.6) $\beta_\delta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap B$,

(4.7) $\dot\beta_\delta(t) =$

$$- \big[\beta_\delta(t)\big(c_\nu(R_\nu(u_{\delta\nu}(t)))^2 + c_\tau|R_\tau(u_{\delta\tau}(t))|^2\big) - \varepsilon_a\big]_+ \text{ a.e. } t \in (0,T),$$

(4.8) $\beta_\delta(0) = \beta_0$.

(iii) Let β_δ be as defined in (ii) and denote again by u_δ the function obtained in step (i) for $\beta = \beta_\delta$. Then, by using (4.4), (4.7) and (4.8) it is easy to see that (u_δ, β_δ) is a solution to Problem P_δ and it satisfies

$$(u_\delta, \beta_\delta) \in L^\infty(0,T;V) \times W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap B.$$

Now, as in [23, 24], we study the convergence of the solution (u_δ, β_δ) as $\delta \to 0$ in the following theorem.

THEOREM 4.6. *Assume that* (2.13)–(2.16) *hold. Then*

$$\lim_{\delta\to 0}\|u_\delta(t) - u(t)\|_V = 0 \quad \textit{for all } t \in [0,T], \tag{4.9}$$

$$\lim_{\delta\to 0}\|\beta_\delta(t) - \beta(t)\|_{L^2(\Gamma_3)} = 0 \quad \textit{for all } t \in [0,T]. \tag{4.10}$$

The proof is carried out in several steps. In the first step, we show the following lemma.

LEMMA 4.7. *For each $t \in [0,T]$, there exists $\bar u(t) \in K$ such that after passing to a subsequence still denoted $(u_\delta(t))$ we have*

$$u_\delta(t) \to \bar u(t) \quad \textit{weakly in } V. \tag{4.11}$$

Proof. Let $t \in [0,T]$. Setting $v = u_\delta(t)$ in (4.1), we have

$$\big(F\varepsilon(u_\delta(t)), \varepsilon(u_\delta(t))\big)_Q + j_{\mathrm{ad}\delta}\big(\beta_\delta(t), u_\delta(t), u_\delta(t)\big) + j_{\mathrm{fr}\delta}(u_\delta(t), u_\delta(t)) = (f(t), u_\delta(t))_V. \tag{4.12}$$

Then it is easy to see from (4.12) that estimate (4.5) holds and so there exists an element $\bar{u}(t) \in V$ and a subsequence still denoted $u_\delta(t)$ such that

$$u_\delta(t) \to \bar{u}(t) \quad \text{weakly on } V.$$

On the other hand, from (4.12) we also have

$$\int_{\Gamma_3} \left(\frac{u_{\delta\nu}(t) - g}{\delta}\right)_+ (u_{\delta\nu}(t) - g)\, da \le (f(t), u_\delta(t))_V,$$

which implies that

$$\|(u_{\delta\nu}(t) - g)_+\|^2_{L^2(\Gamma_3)} \le \delta \|f(t)\|^2_V / m.$$

Then by (4.11), it follows that

$$\|(\bar{u}_\nu(t) - g)_+\|_{L^2(\Gamma_3)} \le \liminf_{\delta\to 0} \|(u_{\delta\nu}(t) - g)_+\|_{L^2(\Gamma_3)} = 0. \tag{4.13}$$

Therefore we conclude by (4.13) that $(\bar{u}_\nu(t) - g)_+ = 0$, i.e. $\bar{u}_\nu(t) \le g$ a.e. on Γ_3, which shows that $\bar{u}(t) \in K$. ∎

Now, we state the following problem.

Problem P_3. Find $\beta : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot\beta(t) = -\big[\beta(t)\big(c_\nu(R_\nu(\bar{u}_\nu(t)))^2 + c_\tau|R_\tau(\bar{u}_\tau(t))|^2\big) - \varepsilon_a\big]_+ \text{ a.e. } t \in (0,T),$$

$$\beta(0) = \beta_0.$$

As in [24, Lemma 3.2] we have the following result.

Lemma 4.8. *Problem P_3 has a unique solution $\beta \in W^{1,\infty}(0,T; L^2(\Gamma_3)) \cap B$.*

We also show the following convergence result.

Lemma 4.9. *Let β be the solution to Problem P_3. Then*

$$\lim_{\delta\to 0} \|\beta_\delta(t) - \beta(t)\|_{L^2(\Gamma_3)} = 0 \quad \text{for all } t \in [0,T]. \tag{4.14}$$

Proof. As in [24, Lemma 3.2], we have

$$\|\beta_\delta(t) - \beta(t)\|_{L^2(\Gamma_3)} \le c\int_0^t (\|u_{\delta\nu}(s) - \bar{u}_\nu(s)\|_{L^2(\Gamma_3)} + \|u_{\delta\tau}(s) - \bar{u}_\tau(s)\|_{(L^2(\Gamma_3))^d})\, ds, \tag{4.15}$$

for some $c > 0$. Using (4.11) we deduce that $u_{\delta\nu}(t) \to \bar{u}_\nu(t)$ strongly in $L^2(\Gamma_3)$ and $u_{\delta\tau}(t) \to \bar{u}_\tau(t)$ strongly in $(L^2(\Gamma_3))^d$ as $\delta \to 0$. On the other hand, we have

$$\|u_{\delta\nu}(t) - \bar{u}_\nu(t)\|_{L^2(\Gamma_3)} + \|u_{\delta\tau}(s) - \bar{u}_\tau(s)\|_{(L^2(\Gamma_3))^d} \le 2d_\Omega \|u_\delta(t) - \bar{u}(t)\|_V \le d_\Omega\bigg(\frac{\|f(t)\|_V}{m} + \|\bar{u}(t)\|_V\bigg),$$

which implies that there exists a constant $c_1 > 0$ such that

$$\|u_{\delta\nu}(t) - \bar{u}_\nu(t)\|_{L^2(\Gamma_3)} + \|u_{\delta\tau}(s) - \bar{u}_\tau(s)\|_{(L^2(\Gamma_3))^d} \le c_1.$$

Then it follows from Lebesgue's convergence theorem that

$$\lim_{\delta\to 0}\int_0^t (\|u_{\delta\nu}(s) - \bar{u}_\nu(s)\|_{L^2(\Gamma_3)} + \|u_{\delta\tau}(s) - \bar{u}_\tau(s)\|_{(L^2(\Gamma_3))^d})\, ds = 0. \tag{4.16}$$

The convergence result is now a consequence of (4.15) and (4.16). ■

Next, we prove the following lemma.

Lemma 4.10. *We have $\bar{u}(t) = u(t)$ for all $t \in [0, T]$.*

Proof. Let $v \in K$ and take $v - u_\delta(t)$ in (4.1) to obtain

$$\begin{aligned}\big(F\varepsilon(u_\delta(t)), \varepsilon(v - u_\delta(t))\big)_Q &+ j_{\mathrm{ad}\delta}\big(\beta_\delta(t), u_\delta(t), v - u_\delta(t)\big)\\ &+ j_{\mathrm{fr}\delta}(u_\delta(t), v - u_\delta(t)) \ge (f(t), v - u_\delta(t))_V, \qquad \forall v \in K.\end{aligned} \tag{4.17}$$

Since

$$\begin{aligned}&j_{\mathrm{ad}\delta}\big(\beta_\delta(t), u_\delta(t), v - u_\delta(t)\big)\\ &\quad= \int_{\Gamma_3}\bigg(\bigg(\frac{u_{\delta\nu}(t) - g}{\delta}\bigg)_+ + p(u_{\delta\nu}(t)) - c_\nu\beta_\delta^2 R_\nu(u_{\delta\nu}(t))\bigg)(v_\nu - u_{\delta\nu}(t))\, da\\ &\quad\le \int_{\Gamma_3}\big(p(u_{\delta\nu}(t)) - c_\nu\beta_\delta^2 R_\nu(u_{\delta\nu}(t))(v_\nu - u_{\delta\nu}(t))\big)\, da,\end{aligned}$$

we use (2.16), (4.10), (4.12), the properties of R_ν and the compact imbedding $H^{1/2}(\Gamma_3) \hookrightarrow L^2(\Gamma_3)$ to see that

$$\int_{\Gamma_3}\big(p(u_{\delta\nu}(t)) - c_\nu\beta_\delta^2 R_\nu(u_{\delta\nu}(t))\big)(v_\nu - u_{\delta\nu}(t))\, da \to j_{\mathrm{ad}}\big(\beta(t), \bar{u}(t), v - \bar{u}(t)\big)$$

as $\delta \to 0$ and

$$\lim_{\delta\to 0} j_{\mathrm{fr}\delta}(u_\delta(t), v - u_\delta(t)) \le j_{\mathrm{fr}}(\bar{u}(t), v) - j_{\mathrm{fr}}(\bar{u}(t), \bar{u}(t)).$$

Therefore, passing to the limit in (4.17) as $\delta \to 0$, we obtain

$$\begin{aligned}(4.18)\qquad &\big(F\varepsilon(\bar{u}(t)),\varepsilon(v-\bar{u}(t))\big)_Q + j_{\mathrm{ad}}\big(\beta(t),\bar{u}(t),v-\bar{u}(t)\big) + j_{\mathrm{fr}}(\bar{u}(t),v)\\ &\qquad - j_{\mathrm{fr}}(\bar{u}(t),\bar{u}(t)) \ge (f(t),v-\bar{u}(t))_V, \quad \forall v \in K.\end{aligned}$$

Now, setting $v = u(t)$ in (4.18) and $v = \bar{u}(t)$ in (2.18) and adding the resulting inequalities, we obtain, by using the assumption (2.13)(b) on F,

$$\begin{aligned}m\|\bar{u}(t)-u(t)\|_V^2 &\le j_{\mathrm{ad}}\big(\beta(t),\bar{u}(t),u(t)-\bar{u}(t)\big) + j_{\mathrm{ad}}\big(\beta(t),u(t),\bar{u}(t)-u(t)\big)\\ &\quad + j_{\mathrm{fr}}(\bar{u}(t),u(t)) - j_{\mathrm{fr}}(\bar{u}(t),\bar{u}(t)) + j_{\mathrm{fr}}(u(t),\bar{u}(t)) - j_{\mathrm{fr}}(u(t),u(t)).\end{aligned}$$

Moreover using (2.16) and the properties of R_ν and R_τ we see that

$$j_{\mathrm{ad}}\big(\beta(t),\bar{u}(t),u(t)-\bar{u}(t)\big) + j_{\mathrm{ad}}\big(\beta(t),u(t),\bar{u}(t)-u(t)\big) \le 0,$$

which implies that

$$\begin{aligned}&m\|\bar{u}(t)-u(t)\|_V^2\\ &\quad \le j_{\mathrm{fr}}(\bar{u}(t),u(t)) - j_{\mathrm{fr}}(\bar{u}(t),\bar{u}(t)) + j_{\mathrm{fr}}(u(t),\bar{u}(t)) - j_{\mathrm{fr}}(u(t),u(t)).\end{aligned}$$

On the other hand using (2.10) and (2.16) we have

$$\begin{aligned}&j_{\mathrm{fr}}(\bar{u}(t),u(t)) - j_{\mathrm{fr}}(\bar{u}(t),\bar{u}(t)) + j_{\mathrm{fr}}(u(t),\bar{u}(t)) - j_{\mathrm{fr}}(u(t),u(t))\\ &\qquad \le L_p d_\Omega^2 \|\bar{u}(t)-u(t)\|_V^2.\end{aligned}$$

Hence we get

$$(m - L_p d_\Omega^2 \|\mu\|_{L^\infty(\Gamma_3)})\|\bar{u}(t)-u(t)\|_V^2 \le 0,$$

and so as $m - L_p d_\Omega^2 \|\mu\|_{L^\infty(\Gamma_3)} > 0$, we obtain

$$(4.19)\qquad \bar{u}(t) = u(t).$$

Now, we have all the ingredients to prove Theorem 4.6. Indeed, from (4.19), we deduce immediately (4.10). To prove (4.9), we set $v = u(t)$ in (4.17) to obtain

$$\begin{aligned}&m\|u_\delta(t)-u(t)\|_V^2\\ &\quad \le j_{\mathrm{ad}}\big(\beta_\delta(t),u_\delta(t),u(t)-u_\delta(t)\big) - j_{\mathrm{ad}}\big(\beta(t),u(t),u(t)-u_\delta(t)\big)\\ &\qquad + j_{\mathrm{fr}\delta}(u_\delta(t),u(t)-u_\delta(t)) - j_{\mathrm{fr}}\big(u(t),u(t)-u_\delta(t)\big)\\ &\qquad + \big(F\varepsilon(u(t)),\varepsilon(u(t)-u_\delta(t))\big)_Q + (f(t),u_\delta(t)-u(t))_V.\end{aligned}$$

Passing to the limit as $\delta \to 0$ and using the convergences

$$j_{\rm ad}\big(\beta_\delta(t), u_\delta(t), u(t) - u_\delta(t)\big) - j_{\rm ad}\big(\beta(t), u(t), u(t) - u_\delta(t)\big) \to 0,$$
$$j_{{\rm fr}\delta}\big(u_\delta(t), u(t) - u_\delta(t)\big) - j_{\rm fr}\big(u(t), u(t) - u_\delta(t)\big) \to 0,$$
$$\big(F\varepsilon(u(t)), \varepsilon(u(t) - u_\delta(t))\big)_Q + \big(f(t), u_\delta(t) - u(t)\big)_V \to 0,$$

we obtain the strong convergence

$$\|u_\delta(t) - u(t)\|_V \to 0 \quad \text{for all } t \in [0, T].$$

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