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# THE SUCCESSIVE APPROXIMATION METHOD FOR THE DIRICHLET PROBLEM IN A PLANAR DOMAIN

*Abstract.* The Dirichlet problem for the Laplace equation for a planar domain with piecewise-smooth boundary is studied using the indirect integral equation method. The domain is bounded or unbounded. It is not supposed that the boundary is connected. The boundary conditions are continuous or $p$-integrable functions. It is proved that a solution of the corresponding integral equation can be obtained using the successive approximation method.

**1. Introduction.** The integral equation method is a classical tool for the study of the Dirichlet problem for the Laplace equation. For a bounded domain with connected smooth boundary and a smooth boundary condition the solution of the Dirichlet problem has been looked for in the form of a double layer potential. In 1919 J. Radon studied the Dirichlet problem on a bounded planar domain whose boundary is a curve with bounded rotation (see [18], [19]). He proved the existence of a classical solution of the Dirichlet problem for continuous boundary conditions. In the second half of the 20th century the classical Dirichlet problem was studied using the integral equation method on domains with nonsmooth boundary in general Euclidean space (see [9], [11]). Later this method was used to study a generalized solution of the Dirichlet problem on a bounded domain with connected Lipschitz boundary and boundary conditions from $L^p$, $p \geq 2$ (see [7]). If the domain is convex, the boundary condition is continuous and we look for a solution in the form of a double layer potential then the solution of the corresponding integral equation can be calculated by the Neumann series (see [9]).

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This result was proved for more general bounded domains with connected boundary in 1998 (see [12]).

If $G$ is a domain with bounded boundary and $G$ is not bounded or if $\partial G$ is not connected then the solution of the Dirichlet problem is not a double layer potential in general. This problem was overcome for $G \subset \mathbb{R}^m$ with $m > 2$ (see [14]). D. Medková looked for a solution of the Dirichlet problem in the form of a sum of a single layer potential and a double layer potential with the same density. For a wide class of domains the existence of a solution of the corresponding integral equation was proved and the solution of this equation was expressed by the Neumann series. This method cannot be directly used for planar domains. The first reason is that the single layer potential is not bounded in the planar case. Secondly, if $G \subset \mathbb{R}^m$ with $m > 2$ and the single layer potential with density $\varphi$ vanishes on $\partial G$ then $\varphi \equiv 0$. This is not true for planar domains. So, we must modify the method and look for a solution of the Dirichlet problem in a slightly different form.

This paper is devoted to the Dirichlet problem for the Laplace equation for a planar domain $G$ with boundary formed by finitely many curves with bounded rotation. This domain might be bounded or unbounded. The classical solution and also the scale of strong solutions with $L^p$ boundary condition are studied. The solution is looked for in the form

$$u = \mathcal{D}Mf + a\mathcal{S}Mf + \frac{1}{\mathcal{H}_1(\partial G)} \int_{\partial G} f \, d\mathcal{H}_1,$$

where

$$Mf = f - \frac{1}{\mathcal{H}_1(\partial G)} \int_{\partial G} f \, d\mathcal{H}_1,$$

$$\mathcal{D}\varphi(x) = \frac{1}{2\pi} \int_{\partial G} \frac{n(y) \cdot (y - x)}{|x - y|^2} \varphi(y) \, d\mathcal{H}_1(y)$$

is the double layer potential with density $\varphi$,

$$\mathcal{S}\varphi(x) = \frac{1}{2\pi} \int_{\partial G} \varphi(y) \ln \frac{1}{|x - y|} \, d\mathcal{H}_1(y)$$

is the single layer potential with density $\varphi$, and $a$ is a nonnegative constant. (If $G$ is bounded then $u = \mathcal{D}f + a\mathcal{S}Mf$. If $G$ is unbounded then

$$u = \mathcal{D}f + a\mathcal{S}Mf + \frac{1}{\mathcal{H}_1(\partial G)} \int_{\partial G} f \, d\mathcal{H}_1.)$$

This leads to the integral equation $T_a f = g$, where $g$ is a boundary condition. Here

$$T_a f(x) = KMf(x) + d_G(x)Mf(x) + a\mathcal{S}Mf(x) + \frac{1}{\mathcal{H}_1(\partial G)} \int_{\partial G} f \, d\mathcal{H}_1$$

where

$$K\varphi(x) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int\limits_{\{y \in \partial G;\, |x-y|>\varepsilon\}} \frac{n(y)\cdot(y-x)}{|x-y|^2}\, \varphi(y)\, d\mathcal{H}_1(y),$$

and

$$d_G(x) = \lim_{r\to 0_+} \frac{\mathcal{H}_2(\Omega_r(x)\cap G)}{\mathcal{H}_2(\Omega_r(x))}$$

is the density of $G$ at $x$.

Fix $R > \operatorname{diam} G$ and define

$$c_R = \sup_{x\in\partial G} \frac{1}{2\pi} \int\limits_{\partial G} \ln \frac{R}{|x-y|}\, d\mathcal{H}_1(y).$$

(If $\partial G$ is formed by segments $C_1,\dots,C_k$ of lengths $l_1,\dots,l_k$ then $c_R \le 2\pi^{-1}\sum l_j[1-\ln(l_j/2R)]$.) The main result of the paper is the following theorem:

THEOREM. *Put*

$$p_0 = 1 + \sup_{x\in\partial G} |1-2d_G(x)|.$$

*Fix* $\gamma > (1+ac_R)/2$, $p\in(p_0;\infty)$. *The operator* $T_a$ *is continuously invertible in* $L^p(\partial G)$ *and in* $\mathcal{C}(\partial G)$ *and*

$$T_a^{-1} = \gamma^{-1}\sum_{n=0}^{\infty}(I-\gamma^{-1}T_a)^n.$$

*If* $g\in L^p(\partial G)$ *then there is a unique* $L^p$*-solution* $u$ *of the Dirichlet problem*

$$\begin{aligned} \Delta u &= 0 \quad \text{in } G,\\ u &= g \quad \text{on } \partial G. \end{aligned}$$

(*If* $g\in\mathcal{C}(\partial G)$ *then* $u$ *is a classical solution of the problem.*) *If we put* $f = T_a^{-1} g$ *then*

$$u = \mathcal{D}Mf + a\mathcal{S}Mf + \frac{1}{\mathcal{H}_1(\partial G)}\int\limits_{\partial G} f\, d\mathcal{H}_1.$$

*Fix* $f_0\in L^p(\partial G)$ *and put*

$$f_{n+1} = \gamma^{-1}g + (I-\gamma^{-1}T_a)f_n \quad \text{for } n\ge 0.$$

*Then* $f_n$ *converges to* $f$ *in* $L^p(\partial G)$ *and* $\|f_n - f\| \le Mq^n/(1-q)$, *where* $M$ *is a constant depending on* $G$, $a$, $\gamma$, $p$, $g$, $f_0$ *and* $q\in(0;1)$ *is a constant depending on* $G$, $a$, $p$, $\gamma$. *The same is true in* $\mathcal{C}(\partial G)$.

**2. Formulation of the problem.** Let $S$ be a rectifiable curve in $\mathbb{R}^2$ and let $s$ denote arc length on $S$ ($0\le s\le l$). If the angle $\theta(s)$ made by the positively oriented tangent and the $x$-axis is a function of bounded variation on $[0;l]$, the curve $S$ is said to be a *curve with bounded rotation.* Note that piecewise $C^{1+\alpha}$ bounded curves with $\alpha > 0$ and the boundary of a convex bounded set are curves with bounded rotation.

Denote by $\mathcal{H}_k$ the $k$-dimensional Hausdorff measure normalized so that $\mathcal{H}_k$ is the Lebesgue measure on $\mathbb{R}^k$. If $x \in \mathbb{R}^2$ and $r > 0$, define $\Omega_r(x) = \{y \in \mathbb{R}^2;\ |x-y| < r\}$. If $G \subset \mathbb{R}^2$ is a measurable set and $x \in \mathbb{R}^2$, denote by

$$d_G(x) = \lim_{r\to 0_+} \frac{\mathcal{H}_2(\Omega_r(x) \cap G)}{\mathcal{H}_2(\Omega_r(x))}$$

the *density* of $G$ at $x$.

Let $G$ be a domain in $\mathbb{R}^2$ with bounded nonempty boundary $\partial G$. Suppose that $\partial G$ is formed by finitely many disjoint Jordan curves with bounded rotation. Suppose moreover that $G$ has no cusps, i.e.

$$0 < \inf_{x\in\partial G} d_G(x) \le \sup_{x\in\partial G} d_G(x) < 1. \tag{1}$$

We remark that $\partial G$ is locally the graph of a Lipschitz function. Let $S$ be a fixed Jordan curve of $\partial G$. Suppose that $S = \{(x(s), y(s));\ 0 \le s \le l\}$, where the parameter $s$ is the arc length on $S$. If $\theta(s)$ is the angle made by the positively oriented tangent and the $x$-axis, then

$$x(s) = x(0) + \int_0^s \cos\theta(t)\, dt, \qquad y(s) = y(0) + \int_0^s \sin\theta(t)\, dt$$

(see [19, p. 1126]). Fix $s_0 \in (0; l)$. Since $\theta$ is a function of bounded variation, it follows that $\theta_+(s_0)$, the limit from the right of $\theta$ at $s_0$, and $\theta_-(s_0)$, the limit from the left of $\theta$ at $s_0$, both exist. We can choose a coordinate system so that $-\pi/2 < \theta_-(s_0) \le 0 \le \theta_+(s_0) < \pi/2$. This implies that there are positive constants $\alpha$, $s_1$, $s_2$ such that $0 < s_1 < s_0 < s_2 < l$ and $-\pi/2 < -\alpha < \theta(s) < \alpha < \pi/2$ for each $s \in (s_1; s_2)$. Put $S_1 = \{(x(s), y(s));\ s_1 < s < s_2)\}$. Since $x(s)$ is an increasing function in $(s_1; s_2)$ we can express $y(s)$ as a function of $x(s)$ for $S_1$. If $s_1 < s < \tau < s_2$ then

$$|y(\tau) - y(s)| \le \tau - s \le \int_s^\tau \frac{\cos\theta(t)}{\cos\alpha}\, dt = \frac{1}{\cos\alpha}\, |x(\tau) - x(s)|,$$

because $\tau - s$ is the length of the part of $S_1$ between $(x(s), y(s))$ and $(x(\tau), y(\tau))$. So, $S_1$ is the graph of a Lipschitz function.

If $g \in \mathcal{C}(\partial G)$ we say that $u$ is a classical solution of the Dirichlet problem for the Laplace equation

$$\Delta u = 0 \quad \text{in } G, \tag{2}$$

$$u = g \quad \text{on } \partial G \tag{3}$$

if $u \in \mathcal{C}^2(G) \cap \mathcal{C}(\operatorname{cl} G)$ is bounded and satisfies (2), (3). (Here $\operatorname{cl} G$ denotes the closure of $G$.)

We will also study the scale of strong solutions of the Dirichlet problem (2), (3) for $g \in L^p(\partial G)$.

For $x \in \partial G$ let

$$\Gamma(x) = \Gamma_\alpha(x) = \{y \in G;\ |x-y| < (1+\alpha)\operatorname{dist}(y,\partial G)\}$$

denote the *nontangential approach region* of opening $\alpha$ corresponding to $G$ and $x$, where $\alpha > 0$ is taken large enough depending on the Lipschitz constant associated with $G$. Here $\operatorname{dist}(y, M)$ denotes the distance of the point $y$ from the set $M$. If $u$ is a function on $G$ we denote by

$$N(u)(x) = \sup\{|u(y)|;\ y \in \Gamma(x)\}$$

the *nontangential maximal function* of $u$ with respect to $G$. If $x \in \partial G$ and

$$c = \lim_{y \to x,\, y \in \Gamma(x)} u(y)$$

we say that $c$ is the *nontangential limit* of $u$ at $x$ with respect to $G$.

Let $1 < p < \infty$ and $g \in L^p(\partial G)$. We say that $u$ is an *$L^p$-solution* of the Dirichlet problem (2), (3) if $u \in \mathcal{C}^2(G)$ satisfies (2), $N(u) \in L^p(\partial G)$ and $g(x)$ is the nontangential limit of $u$ at $x$ with respect to $G$ for $\mathcal{H}_1$-a.a. $x \in \partial G$.

**3. Potentials.** Since $G$ has locally Lipschitz boundary there is an outward unit normal $n(y)$ at almost all $y \in \partial G$. For $f \in L^p(\partial G)$, $1 < p < \infty$, define

$$\mathcal{D}f(x) = \frac{1}{2\pi} \int_{\partial G} \frac{n(y)\cdot(y-x)}{|x-y|^2} f(y)\, d\mathcal{H}_1(y)$$

the *double layer potential* with density $f$, and

$$\mathcal{S}f(x) = \frac{1}{2\pi} \int_{\partial G} f(y) \ln \frac{1}{|x-y|}\, d\mathcal{H}_1(y)$$

the *single layer potential* with density $f$. Then $\mathcal{D}f$, $\mathcal{S}f$ are harmonic functions in $G$, $N(\mathcal{D}f) \in L^p(\partial G)$, $N(|\nabla \mathcal{S}f|) \in L^p(\partial G)$ and the nontangential limits of $\mathcal{D}f$, $\nabla\mathcal{S}f$ with respect of $G$ exist at almost all points of $\partial G$ (see [7, Theorem 2.2.13]). The single layer potential is well defined for all $x \in \mathbb{R}^n$ and $\mathcal{S}f \in \mathcal{C}(\mathbb{R}^n)$ (see [9, Lemma 2.18] or [16, Lemma 3.1]). If

$$\int_{\partial G} f\, d\mathcal{H}_1 = 0 \tag{4}$$

then $\mathcal{S}f(x) \to 0$ as $|x| \to \infty$ and $N(\mathcal{S}f) \in L^p(\partial G)$. If (4) does not hold and $G$ is unbounded then $|\mathcal{S}f(x)| \to \infty$ as $|x| \to \infty$ and $N(\mathcal{S}f) \equiv \infty$ on $\partial G$.

For $\varepsilon > 0$ and $x \in \partial G$ denote

$$K_\varepsilon f(x) = \frac{1}{2\pi} \int_{\{y \in \partial G;\, |x-y| > \varepsilon\}} \frac{n(y)\cdot(y-x)}{|x-y|^2} f(y)\, d\mathcal{H}_1(y).$$

For almost all $x \in \partial G$ we have

$$Kf(x) = \lim_{\varepsilon \to 0} K_\varepsilon f(x)$$

and $\frac{1}{2} f(x) + Kf(x)$ is the nontangential limit of $\mathcal{D}f$ at $x$ (see [7, Theorem 2.2.13]). If $f \in \mathcal{C}(\partial G)$ then $Kf(x)$ makes sense for each $x \in \partial G$ and

$$\lim_{y \in G,\, y \to x} \mathcal{D}f(y) = d_G(x) f(x) + Kf(x)$$

(see [9, Theorem 2.19, Lemma 2.15, Proposition 2.8 and Lemma 2.9]). Observe that $d_G(x) = 1/2$ for almost all $x \in \partial G$ because $G$ has locally Lipschitz boundary.

If $G$ is unbounded then $\mathcal{D}1 = 0$ in $G$ and $d_G(x) + K1(x) = 0$ on $\partial G$. (This is an easy consequence of Green's formula.) If $G$ is bounded then $\mathcal{D}1 = 1$ in $G$ and $d_G(x) + K1 = 1$ on $\partial G$.

For $f \in L^2(\partial G)$, $\varepsilon > 0$ and $y \in \partial G$ set

$$K_\varepsilon^* f(y) = \frac{1}{2\pi} \int\limits_{\{y \in \partial G;\, |x-y| > \varepsilon\}} \frac{n(y) \cdot (y-x)}{|x-y|^2} f(y)\, d\mathcal{H}_1(x).$$

For almost all $y \in \partial G$ we have

$$K^* f(y) = \lim_{\varepsilon \to 0} K_\varepsilon^* f(y),$$

and $\frac{1}{2} f(y) + K^* f(y)$ is the nontangential limit of $-n(y) \cdot \nabla \mathcal{S} f$ with respect to $\mathbb{R}^2 \setminus \operatorname{cl} G$ at $y$ (see [7, Theorem 2.2.13]).

**4. Reduction of the problem.** Define

$$Mf = f - \frac{1}{\mathcal{H}_1(\partial G)} \int\limits_{\partial G} f\, d\mathcal{H}_1 \tag{5}$$

for $f \in L^1(\partial G)$.

Fix $a \geq 0$. We look for a solution of the problem (2)–(3) in the form

$$u = \mathcal{D}Mf + a\mathcal{S}Mf + \frac{1}{\mathcal{H}_1(\partial G)} \int\limits_{\partial G} f\, d\mathcal{H}_1. \tag{6}$$

Here $f \in \mathcal{C}(\partial G)$ if $g \in \mathcal{C}(\partial G)$ and if we look for a classical solution; $f \in L^p(\partial G)$ if $g \in L^p(\partial G)$ and if we look for an $L^p$-solution. Note that $u = \mathcal{D}f + a\mathcal{S}Mf$ for $G$ bounded, and

$$u = \mathcal{D}f + a\mathcal{S}Mf + \frac{1}{\mathcal{H}_1(\partial G)} \int\limits_{\partial G} f\, d\mathcal{H}_1$$

for $G$ unbounded.

For $f \in L^p(\partial G)$ and $x \in \partial G$ define

$$T_a f(x) = KMf(x) + d_G(x) Mf(x) + a\mathcal{S}Mf(x) + \frac{1}{\mathcal{H}_1(\partial G)} \int\limits_{\partial G} f\, d\mathcal{H}_1 \tag{7}$$

if the expression makes sense. If $f, g \in \mathcal{C}(\partial G)$ then $u$ given by (6) is a classical solution of the Dirichlet problem (2), (3) if and only if $T_a f = g$. If $f, g \in L^p(\partial G)$, $1 < p < \infty$, then $u$ given by (6) is an $L^p$-solution of the Dirichlet problem (2), (3) if and only if $T_a f = g$.

If $G$ is bounded then

$$T_a f(x) = Kf(x) + d_G(x)f(x) + a\mathcal{S}Mf(x).$$

If $G$ is unbounded then

$$T_a f(x) = Kf(x) + d_G(x)f(x) + a\mathcal{S}Mf(x) + \frac{1}{\mathcal{H}_1(\partial G)} \int_{\partial G} f \, d\mathcal{H}_1.$$

**5. Properties of the integral operator.** Let $X$ be a real Banach space. Denote by $\operatorname{compl} X$ the complexification of $X$, i.e. $\operatorname{compl} X = \{x + iy;\ x, y \in X\}$. If $T$ is a linear operator on $X$ extend $T$ onto $\operatorname{compl} X$ by setting $T(x+iy) = Tx+iTy$. In particular, the complexifications of the spaces $\mathcal{C}(D)$ and $L^p(D)$ of real-valued functions are the corresponding spaces $\mathcal{C}_{\mathbb{C}}(D)$ and $L^p_{\mathbb{C}}$ of complex-valued functions.

The bounded linear operator $T$ on the Banach space $X$ is called *Fredholm* if $\alpha(T)$, the dimension of the kernel of $T$, is finite, the range $T(X)$ of $T$ is a closed subspace of $X$, and $\beta(T)$, the codimension of $T(X)$, is finite. The number $i(T) = \alpha(T) - \beta(T)$ is the *index* of $T$.

Let $X$ be a complex Banach space and $T$ be a bounded linear operator in $X$. Denote by $\sigma(T)$ the spectrum of $T$, $r(T) = \sup\{|\lambda|;\ \lambda \in \sigma(T)\}$ the spectral radius of $T$, and $r_e(T) = \sup\{|\lambda|;\ \lambda I - T$ is not a Fredholm operator with index $0\}$ the essential spectral radius of $T$. Here $I$ denotes the identity operator on $X$.

LEMMA 5.1. *The operator $T_a$ is a bounded linear operator in $\mathcal{C}_{\mathbb{C}}(\partial G)$ and in $L^p_{\mathbb{C}}(\partial G)$. Moreover, $r_e(T_a - \frac{1}{2}I) < \frac{1}{2}$ in $\mathcal{C}_{\mathbb{C}}(\partial G)$. Put*

$$p_0 = 1 + \sup_{x \in \partial G} |1 - 2d_G(x)|. \tag{8}$$

*Then $1 \le p_0 < 2$. If $p \in (1, \infty)$ then $T_a$ is a Fredholm operator with index $0$ in $L^p(\partial G)$ if and only if $p > 1+|1-2d_G(x)|$ for each $x \in \partial G$. If $p_0 < p < \infty$ then $r_e(T_a - \frac{1}{2}I) < \frac{1}{2}$ in $L^p_{\mathbb{C}}(\partial G)$.*

*Proof.* Since $G$ has no cusps, (1) implies that $1 \le p_0 < 2$.

Define $\widetilde{T} f(x) = d_G(x)f(x) + Kf(x)$. The operator $\widetilde{T}$ is a bounded linear operator on $\mathcal{C}(\partial G)$ (see [9, §2] or [19]). J. Radon proved that $r_e(\widetilde{T} - \frac{1}{2}I) < \frac{1}{2}$ in $\mathcal{C}_{\mathbb{C}}(\partial G)$ for $\partial G$ connected (see [19, p. 1149]). Denote by $C_1, \ldots, C_k$ all components of $\partial G$. Let $G_1, \ldots, G_k$ be open sets such that $G = \bigcap_{j=1}^k G_j$ and

$C_j = \partial G_j$ for $j = 1, \dots, k$. For $f \in L^p_{\mathbb{C}}(C_j)$, $1 < p < \infty$, set

$$\widehat{T}_j f(x) = d_{G_j}(x) f(x) + \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int\limits_{\{y \in C_j;\, |x-y| > \varepsilon\}} \frac{n(y) \cdot (y-x)}{|x-y|^2} f(y)\, d\mathcal{H}_1(y).$$

Then $r_e(\widehat{T}_j - \frac{1}{2} I) < \frac{1}{2}$ in $\mathcal{C}_{\mathbb{C}}(\partial G_j)$. For $f \in L^p_{\mathbb{C}}(\partial G)$ define $\widehat{T} f(x) = \widehat{T}_j f(x)$ for $x \in C_j$, $j = 1, \dots, k$. Let now $\lambda \in \mathbb{C}$, $|\lambda| \geq 1/2$. If $f \in \mathcal{C}_{\mathbb{C}}(\partial G)$ then $(\widehat{T} - \frac{1}{2} I - \lambda I) f = 0$ if and only if $(\widehat{T}_j - \frac{1}{2} I - \lambda I) f = 0$ for $j = 1, \dots, k$ and thus

$$\alpha(\widehat{T} - \tfrac{1}{2} I - \lambda I) = \alpha(\widehat{T}_1 - \tfrac{1}{2} I - \lambda I) + \dots + \alpha(\widehat{T}_k - \tfrac{1}{2} I - \lambda I).$$

Moreover, for a given $g \in \mathcal{C}_{\mathbb{C}}(\partial G)$ there is $f \in \mathcal{C}_{\mathbb{C}}(\partial G)$ such that $(\widehat{T} - \frac{1}{2} I - \lambda I) f = g$ if and only if there are $f_j \in \mathcal{C}_{\mathbb{C}}(C_j)$ such that $(\widehat{T}_j - \frac{1}{2} I - \lambda I) f_j = g$ on $C_j$. Therefore

$$\beta(\widehat{T} - \tfrac{1}{2} I - \lambda I) = \beta(\widehat{T}_1 - \tfrac{1}{2} I - \lambda I) + \dots + \beta(\widehat{T}_k - \tfrac{1}{2} I - \lambda I).$$

Since $\widehat{T}_j - \frac{1}{2} I - \lambda I$ are Fredholm operators in $\mathcal{C}_{\mathbb{C}}(C_j)$ with index 0 for $j = 1, \dots, k$, we conclude that $\widehat{T} - \frac{1}{2} I - \lambda I$ is a Fredholm operator in $\mathcal{C}_{\mathbb{C}}(\partial G)$ with index 0. (Compare also [22, Proposition 1].) Since

$$(\widetilde{T} - \widehat{T}) f(x) = \int\limits_{\partial G \setminus C_j} \frac{n(y) \cdot (y-x)}{|x-y|^2} f(y)\, d\mathcal{H}_1(y)$$

for $x \in C_j$, we see that $\widetilde{T} - \widehat{T}$ is a bounded linear operator from $\mathcal{C}_{\mathbb{C}}(\partial G)$ into $\mathcal{C}^1_{\mathbb{C}}(\partial G)$. The compact embedding of $\mathcal{C}^1(\partial G)$ into $\mathcal{C}(\partial G)$ implies that $\widetilde{T} - \widehat{T}$ is a compact linear operator in $\mathcal{C}_{\mathbb{C}}(\partial G)$. Since $\widetilde{T} - \frac{1}{2} I - \lambda I = (\widehat{T} - \frac{1}{2} I - \lambda I) + (\widetilde{T} - \widehat{T})$, $\widehat{T} - \frac{1}{2} I - \lambda I$ is a Fredholm operator with index 0, and $\widetilde{T} - \widehat{T}$ is a compact linear operator, we conclude that $\widetilde{T} - \frac{1}{2} I - \lambda I$ is a Fredholm operator with index 0 (cf. [20, Theorem 5.10]). Hence $r_e(\widetilde{T} - \frac{1}{2} I) < \frac{1}{2}$ in $\mathcal{C}_{\mathbb{C}}(\partial G)$.

Easy calculation yields

$$(T_a - \widetilde{T}) f = a \mathcal{S} M f + \left[ \frac{1}{\mathcal{H}_1(\partial G)} \int\limits_{\partial G} f\, d\mathcal{H}_1 \right] (1 - \widetilde{T} 1). \tag{9}$$

We now show that $T_a - \widetilde{T}$ is a compact linear operator in $\mathcal{C}_{\mathbb{C}}(\partial G)$. Since $\widetilde{T} 1 = 0$ for $G$ unbounded and $\widetilde{T} 1 = 1$ for $G$ bounded (see above), we have

$$(T_a - \widetilde{T}) f = a \mathcal{S} M f + \frac{1}{\mathcal{H}_1(\partial G)} \int\limits_{\partial G} f\, d\mathcal{H}_1$$

for $G$ unbounded and

$$(T_a - \widetilde{T}) f = a \mathcal{S} M f$$

for $G$ bounded. The operator $f \mapsto \mathcal{S}f$ is a bounded linear operator from $L^2(\partial G)$ to $\mathcal{C}(\partial G)$ (see [16, Lemma 3.1]). The compact embedding of $\mathcal{C}(\partial G)$ into $L^2(\partial G)$ gives that $\mathcal{S}$ is a bounded compact operator in $\mathcal{C}(\partial G)$. Since the composition of a bounded operator and a compact operator is a compact operator, the finite-dimensional operator is a compact operator and the sum of two compact operators is a compact operator (see [24, Chapter X, §2]), we infer that $T_a - \widetilde{T}$ is a bounded compact linear operator in $\mathcal{C}_{\mathbb{C}}(\partial G)$. Since $T_a - \widetilde{T}$ is a bounded linear operator in $\mathcal{C}_{\mathbb{C}}(\partial G)$, we find that $T_a$ is a bounded linear operator in $\mathcal{C}_{\mathbb{C}}(\partial G)$. Fix now $\lambda \in \mathbb{C}$, $|\lambda| \geq 1/2$. Since $\widetilde{T} - \frac{1}{2}I - \lambda I$ is a Fredholm operator with index 0 and $T_a - \widetilde{T}$ is a compact operator in $\mathcal{C}_{\mathbb{C}}(\partial G)$, we deduce that $T_a - \frac{1}{2}I - \lambda I$ is a Fredholm operator with index 0 in $\mathcal{C}_{\mathbb{C}}(\partial G)$ (cf. [20, Theorem 5.10]). Therefore, $r_e(T_a - \frac{1}{2}I) < \frac{1}{2}$ in $\mathcal{C}_{\mathbb{C}}(\partial G)$.

Fix $1 < p < \infty$. Then $K$ is a bounded linear operator in $L^p_{\mathbb{C}}(\partial G)$ (see [7, Theorem 2.2.13]). Since $G$ has locally Lipschitz boundary we have $d_G(x) = \frac{1}{2}$ for almost all $x \in \partial G$. Thus $\widetilde{T} = \frac{1}{2}I + K$ is a bounded linear operator in $L^p_{\mathbb{C}}(\partial G)$. The operator $\mathcal{S}$ is a compact linear operator in $L^p_{\mathbb{C}}(\partial G)$ (see [16, Lemma 3.1]). Using (9) and the same reasoning as above we prove that $T_a - \widetilde{T}$ is a bounded compact linear operator in $L^p_{\mathbb{C}}(\partial G)$. Therefore $T_a$ is a bounded linear operator in $L^p_{\mathbb{C}}(\partial G)$. In the same way as above we prove that $\widetilde{T} - \widehat{T}$ is a compact linear operator in $L^p_{\mathbb{C}}(\partial G)$. Fix now $\lambda \in C$. Since $T_a - \widehat{T}$ is a compact linear operator in $L^p_{\mathbb{C}}(\partial G)$ the operator $T_a - \lambda I$ is a Fredholm operator in $L^p_{\mathbb{C}}(\partial G)$ if and only if $\widehat{T} - \lambda I$ is a Fredholm operator in $L^p_{\mathbb{C}}(\partial G)$ and $i(T_a - \lambda I) = i(\widehat{T} - \lambda I)$ (see [20, Theorem 5.10]). If $f \in L^p_{\mathbb{C}}(\partial G)$ then $(\widehat{T} - \lambda I)f = 0$ if and only if $(\widehat{T}_j - \lambda I)f = 0$ for $j = 1, \dots, k$ and thus $\alpha(\widehat{T} - \lambda I) = \alpha(\widehat{T}_1 - \lambda I) + \dots + \alpha(\widehat{T}_k - \lambda I)$. Moreover, for a given $g \in L^p_{\mathbb{C}}(\partial G)$ there is $f \in L^p_{\mathbb{C}}(\partial G)$ such that $(\widehat{T} - \lambda I)f = g$ if and only if there are $f_j \in L^p_{\mathbb{C}}(C_j)$ such that $(\widehat{T}_j - \lambda I)f_j = g$ on $C_j$. Therefore $\beta(\widehat{T} - \lambda I) = \beta(\widehat{T}_1 - \lambda I) + \dots + \beta(\widehat{T}_k - \lambda I)$. Hence $\widehat{T} - \lambda I$ is a Fredholm operator in $L^p_{\mathbb{C}}(\partial G)$ if and only if $\widehat{T}_j - \lambda I$ is a Fredholm operator in $L^p_{\mathbb{C}}(C_j)$ for $j = 1, \dots, k$, and $i(\widehat{T} - \lambda I) = i(\widehat{T}_1 - \lambda I) + \dots + i(\widehat{T}_k - \lambda I)$. (Cf. also [22, Proposition 1].) According to [22, Theorem 5], $\widehat{T}_j$ is a Fredholm operator in $L^p_{\mathbb{C}}(C_j)$ if and only if $p \neq 1 + |1 - 2d_G(x)|$ for each $x \in \partial C_j$. Moreover, $i(\widehat{T}_j) \geq 0$, and $i(\widehat{T}_j) = 0$ if and only if $p > 1 + |1 - 2d_G(x)|$ for each $x \in \partial C_j$. Altogether, $T_a$ is a Fredholm operator with index 0 in $L^p(\partial G)$ if and only if $p > 1 + |1 - 2d_G(x)|$ for each $x \in \partial G$.

Let $p_0 < p < \infty$, $\lambda \in \mathbb{C}$, $|\lambda| \geq \frac{1}{2}$. Then $\widehat{T}_j - \frac{1}{2}I - \lambda I$ is a Fredholm operator with index 0 in $L^p(C_j)$ for $j = 1, \dots, k$ by [22, Theorem 6] (see also [21, Theorem 4]). We have shown that $T_a - \frac{1}{2}I - \lambda I$ is a Fredholm operator with index 0 in $L^p(\partial G)$. Thus $r_e(T_a - \frac{1}{2}I) < \frac{1}{2}$ in $L^p_{\mathbb{C}}(\partial G)$.

Corollary 5.2. *For $f \in L^2_{\mathbb{C}}(\partial G)$ and $x \in \partial G$ define*

$$T^*_a f(x) = MK^* f(x) + \frac{1}{2} Mf(x) + aM\mathcal{S}f(x) + \frac{1}{\mathcal{H}_1(\partial G)} \int_{\partial G} f \, d\mathcal{H}_1 \tag{10}$$

*whenever it makes sense. Then the operator $T^*_a$ is bounded in $L^2_{\mathbb{C}}(\partial G)$ and it is the adjoint operator of $T_a$ in $L^2_{\mathbb{C}}(\partial G)$.*

*Proof.* Easy calculation gives that $K^*_\varepsilon$ is the adjoint operator of $K_\varepsilon$. If $f \in L^2(\partial G)$ then $\|K_\varepsilon f\| \le C\|f\|$, $\|K^*_\varepsilon f\| \le C\|f\|$ where the constant $C$ does depend on $f$ and $\varepsilon$ (see [23, Lemma 1.2]). This, the definitions of $K^*$, $K$ and the Lebesgue lemma imply that $K^*$ is a bounded operator in $L^2_{\mathbb{C}}(\partial G)$ which is the adjoint operator of $K$ in $L^2_{\mathbb{C}}(\partial G)$. Since the operator $f \mapsto \mathcal{S}f$ is a bounded linear operator in $L^2_{\mathbb{C}}(\partial G)$ (see [16, Lemma 3.1]) we infer that $T^*_a$ is a bounded operator in $L^2_{\mathbb{C}}(\partial G)$. Since $d_G(x) = \frac{1}{2}$ for almost all $x \in \partial G$, Fubini's theorem shows that $T^*_a$ is the adjoint operator of $T_a$ in $L^2_{\mathbb{C}}(\partial G)$.

Lemma 5.3. *Denote by $L^2_0(\partial G)$ the set of all $f \in L^2(\partial G)$ satisfying* (4). *Fix $R \ge \operatorname{diam} \partial G$, where* $\operatorname{diam} \partial G$ *denotes the diameter of $\partial G$. Define*

$$c_R = \sup_{x \in \partial G} \frac{1}{2\pi} \int_{\partial G} \ln \frac{R}{|x-y|} \, d\mathcal{H}_1(y). \tag{11}$$

*Let $f = f_1 + i f_2 \in L^2_{0,\mathbb{C}}(\partial G)$, $0 < \int |f|^2 \le 1$. Denote by $\overline{f} = f_1 - i f_2$ the complex conjugate of $f$. Then*

$$0 < \int_{\mathbb{R}^2 \setminus \partial G} |\nabla \mathcal{S}f|^2 \, d\mathcal{H}_2 = \int_{\partial G} f \mathcal{S}\overline{f} \, d\mathcal{H}_1 \le c_R < \infty.$$

*Proof.* Denote by $\mathcal{H}$ the restriction of $\mathcal{H}_1$ onto $\partial G$. Since $\mathcal{S}f_j$ is bounded on $\partial G$ and $f_j \in L^2_0(\partial G)$ the real measure $f_j\mathcal{H}$ has finite energy (see [10, Chapter I, §4]). According to [10, Chapter I, Theorem 1.16] we have

$$\int_{\partial G} f_j \mathcal{S} f_j \, d\mathcal{H}_1 \ge 0,$$

and equality holds if and only if $f_j = 0$ a.e. on $\partial G$. Moreover, [10, Chapter I, Theorem 1.20] shows that

$$\int_{\mathbb{R}^2 \setminus \partial G} |\nabla \mathcal{S} f_j|^2 \, d\mathcal{H}_2 = \int_{\partial G} f_j \mathcal{S} f_j \, d\mathcal{H}_1.$$

Fubini's theorem gives

$$\begin{aligned} \int_{\mathbb{R}^2 \setminus \partial G} |\nabla \mathcal{S}f|^2 \, d\mathcal{H}_2 &= \int_{\mathbb{R}^2 \setminus \partial G} [|\nabla \mathcal{S}f_1|^2 + |\nabla \mathcal{S}f_2|^2] \, d\mathcal{H}_2 \\ &= \int_{\partial G} [f_1 \mathcal{S} f_1 + f_2 \mathcal{S} f_2] \, d\mathcal{H}_1 = \int_{\partial G} f \mathcal{S}\overline{f} \, d\mathcal{H}_1. \end{aligned}$$

Since $0 < \int |f|^2$ we obtain

$$0 < \int\limits_{\partial G} [f_1 \mathcal{S} f_1 + f_2 \mathcal{S} f_2]\, d\mathcal{H}_1 = \int\limits_{\partial G} f \mathcal{S} \overline{f}\, d\mathcal{H}_1.$$

Since $f \in L^2_{0,\mathbb{C}}(\partial G)$ we have

$$\mathcal{S}\overline{f}(x) = \frac{1}{2\pi} \int\limits_{\partial G} \overline{f}(y) \ln \frac{R}{|x-y|}\, d\mathcal{H}_1(y).$$

Using Hölder's inequality and Fubini's theorem we obtain

$$\int\limits_{\partial G} f \mathcal{S} \overline{f}\, d\mathcal{H}_1 \le \int\limits_{\partial G} |f(x)| \sqrt{\int\limits_{\partial G} |f(y)|^2 \frac{1}{2\pi} \ln \frac{R}{|x-y|}\, d\mathcal{H}_1(y)} \sqrt{c_R}\, d\mathcal{H}_1(x)$$

$$\le \sqrt{c_R} \sqrt{\int\limits_{\partial G} |f(x)|^2\, d\mathcal{H}_1(x)} \sqrt{\int\limits_{\partial G} \int\limits_{\partial G} |f(y)|^2 \frac{1}{2\pi} \ln \frac{R}{|x-y|}\, d\mathcal{H}_1(y)\, d\mathcal{H}_1(x)}$$

$$\le \sqrt{c_R} \sqrt{\int\limits_{\partial G} |f(y)|^2 \int\limits_{\partial G} \frac{1}{2\pi} \ln \frac{R}{|x-y|}\, d\mathcal{H}_1(x)\, d\mathcal{H}_1(y)} \le c_R.$$

LEMMA 5.4. *Fix $R \ge \operatorname{diam}(\partial G)$. If $f$ is a nontrivial function from the space $L^2_{0,\mathbb{C}}(\partial G)$ then*

$$0 < \int\limits_{\partial G} (M\mathcal{S} f)(\mathcal{S}\overline{f})\, d\mathcal{H}_1 \le c_R \int\limits_{\partial G} f \mathcal{S} \overline{f}\, d\mathcal{H}_1,$$

*where $c_R$ is given by* (11).

*Proof.* Hölder's inequality gives

$$\Big| \int\limits_{\partial G} \mathcal{S} f\, d\mathcal{H}_1 \Big|^2 \le \mathcal{H}_1(\partial G) \int\limits_{\partial G} |\mathcal{S} f|^2\, d\mathcal{H}_1$$

and the equality holds if and only if $\mathcal{S} f = c$ a.e. on $\partial G$, where $c$ is constant. In that case

$$0 = \int\limits_{\partial G} \overline{f} \mathcal{S} f\, d\mathcal{H}_1,$$

which contradicts Lemma 5.3. Thus

$$0 < \int\limits_{\partial G} |\mathcal{S} f|^2\, d\mathcal{H}_1 - \frac{1}{\mathcal{H}_1(\partial G)} \Big| \int\limits_{\partial G} \mathcal{S} f\, d\mathcal{H}_1 \Big|^2 = \int\limits_{\partial G} (M\mathcal{S} f)(\mathcal{S}\overline{f})\, d\mathcal{H}_1.$$

For $\phi, \psi \in L^2_{0,\mathbb{C}}(\partial G)$ define

$$[\phi, \psi] = \int\limits_{\partial G} \phi \mathcal{S} \overline{\psi}\, d\mathcal{H}_1.$$

Lemma 5.3 implies that $[\phi, \psi]$ is a scalar product on $L^2_{0,\mathbb{C}}(\partial G)$. Define $\Lambda = \{\phi \in L^2_{0,\mathbb{C}}(\partial G); \int |\phi|^2 \le 1\}$. Since $M\mathcal{S} f \in L^2_{0,\mathbb{C}}(\partial G)$, using Schwarz's inequality (see [4, Theorem I.7.4]) we get

$$\begin{aligned}\int_{\partial G} (M\mathcal{S}f)(\mathcal{S}\bar{f})\, d\mathcal{H}_1 &\le \sup_{\phi\in\Lambda} [\phi, f] \|M\mathcal{S}f\|_{L^2(\partial G)} \\ &\le \sup_{\phi\in\Lambda} |[\phi, f]| \sup_{\psi\in\Lambda} \Big| \int_{\partial G} \psi M\mathcal{S}f\, d\mathcal{H}_1 \Big| = \sup_{\phi\in\Lambda} |[\phi, f]| \sup_{\psi\in\Lambda} \Big| \int_{\partial G} \psi \mathcal{S}\bar{f}\, d\mathcal{H}_1 \Big| \\ &\le \sup_{\phi\in\Lambda} \sqrt{[\phi,\phi]}\sqrt{[f,f]} | \sup_{\psi\in\Lambda} \sqrt{[f,f]}\sqrt{[\psi,\psi]} \le c_R[f,f]\end{aligned}$$

by Lemma 5.3.

Proposition 5.5. *Let $a \ge 0$, $f \in L^2_{\mathbb{C}}(\partial G)$ be nontrivial, and $T_a^* f = \lambda f$ where $\lambda$ is a complex number. If $f \notin L^2_{0,\mathbb{C}}(\partial G)$ then $\lambda = 1$. If $f \in L^2_{0,\mathbb{C}}(\partial G)$ then $0 \le \lambda \le 1 + ac_R$. If $a > 0$ or $\partial G$ is connected then $\lambda > 0$.*

*Proof.* Suppose first that $f \notin L^2_{0,\mathbb{C}}(\partial G)$. Since $T_a^*$ is the adjoint operator of $T_a$ we have

$$\lambda \int_{\partial G} f\, d\mathcal{H}_1 = \int_{\partial G} 1 \cdot T_a^* f\, d\mathcal{H}_1 = \int_{\partial G} f \cdot T_a 1\, d\mathcal{H}_1 = \int_{\partial G} f\, d\mathcal{H}_1.$$

Let now $f \in L^2_{0,\mathbb{C}}(\partial G)$. Then

$$\begin{aligned}\lambda \int_{\partial G} f\mathcal{S}\bar{f}\, d\mathcal{H}_1 &= \int_{\partial G} [T_a^* f][\mathcal{S}\bar{f}]\, d\mathcal{H}_1 \\ &= \int_{\partial G} (K^* f + \tfrac{1}{2} f)\mathcal{S}\bar{f}\, d\mathcal{H}_1 + a \int_{\partial G} (M\mathcal{S}f)(\mathcal{S}\bar{f})\, d\mathcal{H}_1 \\ &= \int_{\mathbb{R}^2\setminus G} |\nabla \mathcal{S}f|^2\, d\mathcal{H}_1 + a \int_{\partial G} (M\mathcal{S}f)(\mathcal{S}\bar{f})\, d\mathcal{H}_1\end{aligned}$$

by [12, Lemma 7]. Using Lemmas 5.3 and 5.4 we get

$$0 \le \frac{\int_{\mathbb{R}^2\setminus \mathrm{cl}\, G} |\nabla \mathcal{S}f|^2\, d\mathcal{H}_2}{\int_{\mathbb{R}^2\setminus \partial G} |\nabla \mathcal{S}f|^2\, d\mathcal{H}_2} + a\, \frac{\int_{\partial G} (M\mathcal{S}f)(\mathcal{S}\bar{f})\, d\mathcal{H}_1}{\int_{\partial G} f\mathcal{S}\bar{f}\, d\mathcal{H}_1} = \lambda \le 1 + ac_R.$$

If $a > 0$ then $\lambda > 0$ by Lemma 5.4. Suppose now that $a = 0$ and $\partial G$ is connected. If $\lambda = 0$ then $\nabla \mathcal{S}f = 0$ in $\mathbb{R}^2 \setminus \mathrm{cl}\, G$. Since $R^2 \setminus \mathrm{cl}\, G$ is connected there is a constant $c$ such that $\mathcal{S}f = c$ in $\mathbb{R}^2 \setminus \mathrm{cl}\, G$. Since $\mathcal{S}f \in \mathcal{C}(\mathbb{R}^2)$ we obtain $\mathcal{S}f = c$ on $\partial G$. Since $f \in L^2_{0,\mathbb{C}}(\partial G)$ we have

$$\int_{\partial G} f\mathcal{S}\bar{f}\, d\mathcal{H}_1 = \int_{\partial G} f\bar{c}\, d\mathcal{H}_1 = 0.$$

Lemma 5.3 yields $f \equiv 0$, which is a contradiction.

Corollary 5.6. *Let $a \ge 0$. If $\partial G$ is not connected, suppose that $a > 0$. Fix $R > \mathrm{diam}(\partial G)$ and $\gamma > (1 + ac_R)/2$. Then $\sigma(T_a) \subset \{\lambda \in \mathbb{C};\ |\lambda - \gamma| < \gamma\}$ in $\mathcal{C}_{\mathbb{C}}(\partial G)$ and in $L^p_{\mathbb{C}}(\partial G)$ for $p_0 < p < \infty$, where $p_0$ is given by (8).*

*Proof.* Let $\lambda \in \sigma(T_a)$ in $\mathcal{C}_{\mathbb{C}}(\partial G)$ or in $L^p_{\mathbb{C}}(\partial G)$, where $p_0 < p < \infty$. Since $\{\lambda \in \mathbb{C};\ |\lambda - 1/2| < 1/2\} \subset \{\lambda \in \mathbb{C};\ |\lambda - \gamma| < \gamma\}$, we can suppose that $|\lambda - 1/2| \geq 1/2$. Since $T_a - \lambda I$ is a Fredholm operator with index 0 by Lemma 5.1, the complex number $\lambda$ must be an eigenvalue of $T_a$ in the corresponding space. Since $T_a - \lambda I$ is a Fredholm operator with index 0 in $L^2_{\mathbb{C}}(\partial G)$, the number $\lambda$ is an eigenvalue of $T_a$ in $L^2_{\mathbb{C}}(\partial G)$ (see [17, Lemma 2.1]). Since $\lambda \in \sigma(T_a)$ in $L^2_{\mathbb{C}}(\partial G)$ we have $\lambda \in \sigma(T_a^*)$ in $L^2_{\mathbb{C}}(\partial G)$ (see [24, Chapter VIII, §6, Theorem 2]). Since $T_a^* - \lambda I$ is a Fredholm operator with index 0 (see [20, Theorem 7.22]), we deduce that $\lambda$ is an eigenvalue of $T_a^*$. Proposition 5.5 implies that $\lambda \in (0; 1 + ac_r] \subset \{\beta \in \mathbb{C};\ |\beta - \gamma| < \gamma\}$.

Theorem 5.7. *Let $a \geq 0$. If $\partial G$ is not connected, suppose that $a > 0$. Fix $R > \operatorname{diam}(\partial G)$ and $\gamma > (1 + ac_R)/2$. Let $p_0 < p < \infty$. Then there are positive constants $M$, $M_p$ and $q, q_p \in (0,1)$ such that*

$$\|(I - \gamma^{-1} T_a)^n\|_{L^p(\partial G)} \leq M_p q_p^n, \tag{12}$$

$$\|(I - \gamma^{-1} T_a)^n\|_{\mathcal{C}(\partial G)} \leq M q^n \tag{13}$$

*for each nonnegative integer $n$. The operator $T_a$ is continuously invertible in $L^p_{\mathbb{C}}(\partial G)$ and in $\mathcal{C}_{\mathbb{C}}(\partial G)$ and*

$$T_a^{-1} = \gamma^{-1} \sum_{n=0}^{\infty} (I - \gamma^{-1} T_a)^n. \tag{14}$$

*Proof.* Corollary 5.6 and the spectral mapping theorem (see [20, Theorem 9.5]) give that $\sigma(I - \gamma^{-1}T_a) \subset \{\lambda \in \mathbb{C};\ |\lambda| < 1\}$. Since $r(I - \gamma^{-1}T_a) < 1$ and $r(I - \gamma^{-1}T_a) = \lim \|(I - \gamma^{-1}T_a)^n\|^{1/n}$ as $n \to \infty$ (see [24, Chapter VIII, §2]), we deduce that there are positive constants $M$, $M_p$ and $q, q_p \in (0,1)$ such that (12), (13) hold. Since $T_a = \gamma[I - (I - \gamma^{-1}T_a)]$, easy calculation gives (14).

## 6. Solution of the problem

Lemma 6.1. *Let $0 < R < \infty$ and $u$ be a bounded harmonic function in $V = \{x \in \mathbb{R}^2;\ |x| > R\}$. Then $|\nabla u(x)| = O(|x|^{-2})$ as $|x| \to \infty$.*

*Proof.* Set $\widetilde{V} = \{x \in \mathbb{R}^2;\ |x| < 1/R\}$ and $v(x) = u(x/|x|^2)$ for $x \in V \setminus \{(0,0)\}$. Then $v$, the Kelvin transform of $u$, is a function harmonic in $\widetilde{V} \setminus \{(0,0)\}$ (see [2, Corollary 1.6.4]). Since $v$ is a bounded harmonic function in $\widetilde{V} \setminus \{(0,0)\}$, we can define $v$ at the point $(0,0)$ in such a way that $v$ is a harmonic function in $V$ (see [2, Corollary 5.2.3]). Since $v \in \mathcal{C}^1(\widetilde{V})$ there is a positive constant $M$ such that $|\nabla v(x)| \leq M$ for each $x$ with $|x| \leq 1/2R$. If $|x| > 2R$ then $u(x) = v(x/|x|^2)$ and $|\partial_j u(x)| = |\sum \partial_i v(x/|x|^2) \partial_j (x_i/|x|^2))| \leq 3M/|x|^2$.

THEOREM 6.2. *Let $a \geq 0$. If $\partial G$ is not connected, suppose that $a > 0$. Fix $R >$ diam$(\partial G)$ and $\gamma > (1 + ac_R)/2$. If $g \in \mathcal{C}(\partial G)$ then there is a unique classical solution $u$ of the Dirichlet problem* (2), (3) *with the boundary condition $g$. This solution is given by* (6), *where $f = T_a^{-1}g$ and $T_a^{-1}$ is given by* (14). *Let $p_0 < p < \infty$. If $g \in L^p(\partial G)$ then there is a unique $L^p$-solution $u$ of the Dirichlet problem* (2), (3) *with the boundary condition $g$. This solution is given by* (6), *where $f = T_a^{-1}g$ and $T_a^{-1}$ is given by* (14).

*Proof.* If $T_a^{-1}$ is given by (14), $f = T_a^{-1}g$ and $u$ is given by (6) then $u$ is a solution of the problem by §4 and Theorem 5.7. Therefore it suffices to prove the uniqueness of an $L^p$-solution for $p_0 < p \leq 2$.

Let $u$ be an $L^p$-solution of the Dirichlet problem (2), (3) with the boundary condition $g \equiv 0$, $p_0 < p \leq 2$. If $G$ is bounded then $u \equiv 0$ (see [7, Theorem 2.3.15] or [8]). Let now $G$ be unbounded. Fix $R > 0$ such that $\partial G \subset \Omega_R(0)$. Put $\widetilde{G} = G \cap \Omega_R(0)$ and $g = u$ on $\partial \Omega_R(0)$. Since $g \in W^{1,p}(\partial G)$ there is a unique $L^p$-solution $v$ of the Dirichlet problem for the Laplace equation on $G_R$ with the boundary condition $g$ such that $N(|\nabla v|) \in L^p(\partial G_R)$ (see [6, Theorem 5.6]). Uniqueness for bounded domains gives $u = v$. Since $N(|\nabla u|) \in L^p(\partial G_R)$ the nontangential limit of $\nabla u$ exists at almost all $x \in \partial G_R$ (see [5, Theorem] and [3, Theorem 1]). Set $h(x) = n(x) \cdot \nabla u(x)$. Then $h \in L^p(\partial G_R)$ and $u$ is an $L^p$-solution of the Neumann problem for the Laplace equation in $G_R$ with the boundary condition $h$. This means that there is $\varphi \in L^p(\partial G_R)$ and a constant $c$ such that $u = \mathcal{S}\varphi + c$ (see [16, Theorem 8.7]). This shows that $u \in \mathcal{C}(\mathrm{cl}\, G)$. Since $u$ is bounded, $|\nabla u(x)| = O(|x|^{-2})$ as $|x| \to \infty$ by Lemma 6.1. Since $u = 0$ on $\partial G$, we get for $R \to \infty$ using Green's formula (cf. [15, p. 229])

$$\int_G |\nabla u|^2 \, d\mathcal{H}_2 = \lim_{R\to\infty} \int_{G_R} |\nabla u|^2 \, d\mathcal{H}_2 = \lim_{R\to\infty} \int_{\partial G_R} u \frac{\partial u}{\partial n} \, d\mathcal{H}_1 = 0.$$

Since $\nabla u = 0$ in $G$ the function $u$ is constant. Thus $u \equiv 0$.

REMARK 6.3. Let $a \geq 0$. If $\partial G$ is not connected, suppose that $a > 0$. Fix $R >$ diam$(\partial G)$ and $\gamma > (1 + ac_R)/2$. (If $\partial G$ is formed by segments $C_1, \ldots, C_k$ of lengths $l_1, \ldots, l_k$ then we have the following estimate of $c_R$:

$$c_R \leq \frac{1}{2\pi} \sum_{j=1}^{k} l_j [1 - \ln(l_j/2R)]$$

by [16, Example 11.1]. Let $g \in \mathcal{C}(\partial G)$ or $g \in L^p(\partial G)$, where $p_0 < p < \infty$. According to Theorem 6.2 there is a unique solution $u$ of the Dirichlet problem (2), (3) with the boundary condition $g$, which is given by (6), where $f$ is a solution of the integral equation $T_a f = g$. We want to find $f$ by successive approximations. We modify the equation $T_a f = g$ as $f = \gamma^{-1}g + (I - \gamma^{-1}T_a)f$.

Fix arbitrary $f_0$ and put

$$f_{n+1} = \gamma^{-1} g + (I - \gamma^{-1} T_a) f_n$$

for each nonnegative integer $n$. Since the spectral radius of the operator $I - \gamma^{-1} T_a$ is smaller than 1 by Corollary 5.6 and the spectral mapping theorem (see [20, Theorem 9.5]), the series $f_n$ converges to the solution $f$ (see [1, Chapter V, §5]). Since

$$f_{n+1} - f_n = (I - \gamma^{-1} T_a)^n (f_1 - f_0),$$

Theorem 5.7 yields constants $q \in (0, 1)$, $M$ depending on $G$, $g$ and $f_0$ such that $\|f_{n+1} - f_n\| \le M q^n$ and thus $\|f_n - f\| \le M q^n / (1 - q)$.

## References


[1] G. P. Akilov and L. V. Kantorovich, *Functional Analysis*, Nauka, Moscow, 1977 (in Russian).

[2] D. H. Armitage and S. J. Gardiner, *Classical Potential Theory*, Springer, London, 2001.

[3] B. E. J. Dahlberg, *Estimates of harmonic measure*, Arch. Ration. Mech. Anal. 65 (1977), 272–288.

[4] S. Goldberg, *Unbounded Linear Operators. Theory and Applications*, McGraw-Hill, 1966.

[5] R. Hunt and R. L. Wheeden, *On the boundary value of harmonic functions*, Trans. Amer. Math. Soc. 132 (1968), 307–322.

[6] D. Jerison and C. E. Kenig, *The inhomogeneus Dirichlet problem in Lipschitz domains*, J. Funct. Anal. 130 (1995), 161–219.

[7] C. E. Kenig, *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems*, Amer. Math. Soc., Providence, RI, 1994.

[8] C. Kenig and J. Pipher, *Hardy spaces and the Dirichlet problem on Lipschitz domains*, Rev. Mat. Iberoamer. 3 (1987), 191–248.

[9] J. Král, *Integral Operators in Potential Theory*, Lecture Notes in Math. 823, Springer, Berlin, 1980.

[10] N. L. Landkof, *Fundamentals of Modern Potential Theory*, Nauka, Moscow, 1966 (in Russian).

[11] V. G. Maz'ya, *Boundary Integral Equations*, in: Analysis IV, Encyclopaedia Math. 27, Springer, New York, 1991, 127–222.

[12] D. Medková, *Solution of the Neumann problem for the Laplace equation*, Czechoslovak Math. J. 48 (1998), 768–784.

[13] —, *Solution of the Robin problem for the Laplace equation*, Appl. Math. 43 (1998), 133–155.

[14] —, *Solution of the Dirichlet problem for the Laplace equation*, ibid. 44 (1999), 143-168.

[15] —, *The oblique derivative problem for the Laplace equation in a plain domain*, Integral Equations Operator Theory 48 (2004), 225–248.

[16] —, *The third problem for the Laplace equation on a planar cracked domain with modified jump conditions on cracks*, J. Integral Equations Appl. 18 (2006), 471–507.

[17] D. Mitrea, *The method of layer potentials for non-smooth domains with arbitrary topology*, Integral Equations Operator Theory 29 (1997), 320–338.

[18] J. Radon, *Über Randwertaufgaben beim logarithmischen Potential*, Sitz. Akad. Wiss. Wien 128 (1919), 1123–1167.
[19] —, *Über Randwertaufgaben beim logarithmischen Potential*, in: Collected Works, Vol. 1, Birkhäuser, Vienna, 1987.
[20] M. Schechter, *Principles of Functional Analysis*, Amer. Math. Soc., Providence, RI, 2002.
[21] V. Yu. Shelepov, *On the index of an integral operator of the type of a potential in the space $L_p$*, Dokl. Akad. Nauk SSSR 186 (1969), 1266–1268 (in Russian).
[22] —, *On the index of an integral operator of the type of a potential along Radon's curves*, Mat. Sb. 181 (1990), 751–778 (in Russian).
[23] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal. 59 (1984), 572–611.
[24] K. Yosida, *Functional Analysis*, Springer, Berlin, 1965.

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