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LARGE GAMES WITH ONLY SMALL PLAYERS AND STRATEGY SETS IN EUCLIDEAN SPACES

Abstract. The games of type considered in the present paper (LSE-games) extend the concept of LSF-games studied by Wieczorek in [2004], both types of games being related to games with a continuum of players. LSE-games can be seen as anonymous games with finitely many types of players, their action sets included in Euclidean spaces and payoffs depending on a player's own action and finitely many integral characteristics of distributions of the players' (of all types) actions. We prove the existence of equilibria and present a minimization problem and a complementarity problem (both nonlinear) whose solutions are exactly the same as equilibria in the given game. Examples of applications include a model of social adaptation and a model of economic efficiency enforced by taxation.

1. The basic concepts. The object of our study, large games with only small players and strategy sets in Euclidean spaces (if necessary, we refer to them as LSE-games; L for *large*, S for *small* and E for *Euclidean*) is more general than that considered by Wieczorek in [2004] (referred to there as LSF-games), but the present paper offers results of different kind. The LSE-games deal with situations involving a large number of anonymous players who independently choose their actions in a set included in a Euclidean space and whose payoffs depend on finitely many integral characteristics of distributions of the players' actions. This concept generalizes that of LSE-games in Wieczorek [2004], where the sets of actions were assumed finite. Not

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surprisingly, LSE-games can be found as a mathematical skeleton implicitly appearing in the framework of some games with a continuum of players as well as some anonymous games (Balder [1990, 1993], Khan [1985, 1989], Khan and Rustichini [1991, 1993], Mas-Colell [1984], Flåm and Wieczorek [1995]; cf. also Schmeidler [1973] and Rath [1992]).

In the subsequent sections we define and prove the existence of equilibrated distributions in LSE-games, we suggest some ways to search for such distributions and we discuss relations of such games to games with a continuum of players. Finally, in Section 5, we sketch some areas of applications of LSE-games, all of them involving a large (infinite) number of agents, which include models of market behavior, a social adaptation model and a model of inefficient economic behavior moderated by an appropriate taxation procedure driven by infinitely many "efficiency oriented" market forces.

An LSE-game is determined by a specification of a positive integer n, nonempty bounded sets $W^1, \ldots, W^n$ such that W^i is included in a k^i-dimensional Euclidean space and real functions $\Psi^1, \ldots, \Psi^n$ such that Ψ^i is defined on the product $W^i \times \operatorname{co} W^1 \times \operatorname{co} W^2 \times \cdots \times \operatorname{co} W^n$; so an LSE-game can be identified with a system

$$\gamma = (n; W^1, \ldots, W^n; \Psi^1, \ldots, \Psi^n).$$

The numbers $1, \ldots, n$ represent the *types* (*of players*); for each type i, elements of W^i are *actions* available to the type i. Probability measures on (Borel subsets of) W^i are *distributions of actions* of type i; an n-tuple $(\mathbf{m}^1, \ldots, \mathbf{m}^n)$ of such measures is an overall distribution of actions. For any such measure $\mathbf{m}^i$ and $j = 1, \ldots, k^i$ we denote its *characteristics* by

$$m_j^i := \int_{W^i} x_j \, \mathbf{m}^i(dx) \quad \text{and} \quad m^i = (m_1^i, \ldots, m_{k^i}^i).$$

A natural interpretation of what is going on in an LSE-game is that the number $\Psi^i(x^i; m^1, \ldots, m^n)$ is the payoff of any player of type i when he decides to use his action $x^i \in W^i$ while, for $i = 1, \ldots, n$, the prevailing distribution of actions $\mathbf{m} = (\mathbf{m}^1, \ldots, \mathbf{m}^n)$ undertaken by the (possibly infinitely many) players has the characteristics $m = (m^1, \ldots, m^n)$.

We say that an LSE-game is *continuous* whenever all functions $\Psi^1, \ldots$ $\ldots, \Psi^n$ are continuous.

Comment. There are many situations where one is also interested in other characteristics of the distributions than just marginal expectations. Often they can still be studied in the present setup. For instance, in the case of $W^i = [0, 1]$, for a distribution $\mathbf{m}$ on W^i, the expectation m_1 is formally the only available characteristic; however, information about the variance of $\mathbf{m}$ can be included by replacing W^i by $\widetilde{W^i} := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_1 \le 1,$

$x_2 = x_1^2\}$. The distributions on W^i and $\widetilde{W^i}$ are naturally identified (by a projection) and then the *variance* of $\mathbf{m}$ is simply expressed as

$$\int_0^1 x_1^2\,\mathbf{m}(dx) - \Big[\int_0^1 x_1\,\mathbf{m}(dx)\Big]^2 = m_2 - (m_1)^2.$$

2. Equilibria. A distribution of actions $\mathbf{m} = (\mathbf{m}^1, \ldots, \mathbf{m}^n)$ in an LSE-game is called an *equilibrium* (or *equilibrated* distribution) whenever, for some numbers $C^1, \ldots, C^n$ and each $i = 1, \ldots, n$,

$$\Psi^i(x^i; m^1, \ldots, m^n) \le C^i \quad \text{for all } x^i \text{ in } W^i$$

and

$$\Psi^i(x^i; m^1, \ldots, m^n) = C^i \quad \text{for } \mathbf{m}^i\text{-almost all } x^i \text{ in } W^i.$$

This number $C^i = C^i(\mathbf{m})$ is called the *payoff level* of type i at $\mathbf{m}$.

So the problem of finding **all** equilibrated distributions for an LSE-game reduces to the following *problem* (**E**):

Given (for $i = 1, \ldots, n$) nonempty bounded sets $W^1 \subset \mathbb{R}^{k^1}, \ldots, W^n \subset \mathbb{R}^{k^n}$ and functions $\Psi^i : W^i \times \operatorname{co} W^1 \times \operatorname{co} W^2 \times \cdots \times \operatorname{co} W^n \to \mathbb{R}$, find numbers $C^1, \ldots, C^n$ and probability measures: $\mathbf{m}^1$ on Borel subsets of $W^1, \ldots,$ $\mathbf{m}^n$ on Borel subsets of W^n such that

$$\Psi^i(x^i; m^1, \ldots, m^n) \le C^i \quad \text{for all } x^i \text{ in } W^i$$

and

$$\Psi^i(x^i; m^1, \ldots, m^n) = C^i \quad \text{for } \mathbf{m}^i\text{-almost all } x^i \text{ in } W^i.$$

We shall denote by D^i, for $i = 1, \ldots, n$, the set of all probability measures on W^i, to be regarded as a topological space with the weak* topology. As is well known (see e.g. K. Parthasarathy [1967]), D^i is metrizable, it is compact whenever W^i is compact and it can be regarded as a convex subset of a locally convex topological vector space.

2.1. Theorem. (a) *If all sets W^i are compact while all functions Ψ^i, for $i = 1, \ldots, n$, are continuous then problem* (**E**) *has a solution, hence every continuous LSE-game with compact sets of actions has an equilibrium.*

(b) *If, moreover, for indices i in some set $N_0 \subseteq \{1, \ldots, n\}$, the sets W^i are convex, while, for each distribution $\mathbf{m} = (\mathbf{m}^1, \ldots, \mathbf{m}^n)$, the set*

$$B^i(\mathbf{m}) := \operatorname*{Argmax}_{x^i \in W^i} \Psi^i(x^i; m^1, \ldots, m^n)$$

is acyclic (*e.g. homeomorphic to a compact convex set*) *then the game has an equilibrium such that for each $i \in N_0$, $\mathbf{m}^i$ is concentrated at a point.*

Proof. (a) Define a correspondence H from $D := D^1 \times \cdots \times D^n$ to subsets of D letting, for $i = 1, \ldots, n$,

$$H^i(\mathbf{m}^1, \ldots, \mathbf{m}^n) := \{\mathbf{n}^i \in D^i \mid \operatorname{supp} \mathbf{n}^i \subseteq B^i(\mathbf{m}^1, \ldots, \mathbf{m}^n)\}$$

and

$$H(\mathbf{m}^1, \ldots, \mathbf{m}^n) := H^1(\mathbf{m}^1, \ldots, \mathbf{m}^n) \times \cdots \times H^n(\mathbf{m}^1, \ldots, \mathbf{m}^n).$$

Since D is nonempty compact and convex while H is upper semicontinuous (we check this below) and its values are nonempty, convex and compact (Parthasarathy [1967, Theorem 6.7]), it follows from the Ky Fan–Glicksberg Theorem that H has a fixed point $\overline{\mathbf{m}} = (\overline{\mathbf{m}}^1, \ldots, \overline{\mathbf{m}}^n)$. This $\overline{\mathbf{m}}$ together with immediately calculated constants C^i gives a solution to (**E**).

We now check that H is upper semicontinuous. If not, there would exist, for some i, a sequence $(\mathbf{n}^{(r)})$ of probability measures on W^i weak* converging to a probability measure $\mathbf{n}^{(0)}$ and another sequence $(\overline{\mathbf{n}}^{(r)})$ of probability measures on W^i weak* converging to a probability measure $\overline{\mathbf{n}}^{(0)}$ such that for all $r = 1, 2, \ldots$, $\overline{\mathbf{n}}^{(r)}$ is concentrated at $Y^{(r)} = \operatorname{Argmax}_{x \in W^i} \Psi^i(x; n^{(r)})$ but $\overline{\mathbf{n}}^{(0)}$ is not concentrated at $Y^{(0)} = \operatorname{Argmax}_{x \in W^i} \Psi^i(x; n^{(0)})$, which means that there exists a compact set $K \subset W^i$ disjoint from $Y^{(0)}$ and such that $\overline{\mathbf{n}}^{(0)}(K) > 0$. Let $K' \subset W^i$ be any compact set, also disjoint from $Y^{(0)}$ and including K in its interior.

By a known theorem (Berge [1959], p. 122), the correspondence $\Xi^i : \operatorname{co} W^i \to 2^{W^i}$ defined by $\Xi^i(m) := \operatorname{Argmax}_{x \in W^i} \Psi^i(x; m)$ is upper semicontinuous. Since, in particular, it is u.s.c. at $\mathbf{n}^{(0)}$, we find that for sufficiently large r, all sets $Y^{(r)}$ are disjoint from K'. Now, let g be a continuous function on W^i into $\mathbb{R}_+$ vanishing on the complement of K' and positive on K. We then have

$$\int_{W^i} g(x)\, \overline{\mathbf{n}}^{(r)}(dx) = 0 \quad \text{for } r = 1, 2, \ldots,$$

but also

$$\int_{W^i} g(x)\, \overline{\mathbf{n}}^{(0)}(dx) > 0,$$

which contradicts the hypothesis of the weak* convergence of the sequence $(\overline{\mathbf{n}}^{(r)})$ to $\overline{\mathbf{n}}^{(0)}$.

The minor details dropped in the reasoning above can be easily reconstructed with the help of Parthasarathy [1967] or any other textbook in metric measure theory.

To prove (b), it is sufficient to modify the correspondence H, as defined in the proof of (a), by replacing, for $i \in N_0$, D^i by W^i, H^i by B^i as defined in the formulation of (b), and applying, instead of the Ky Fan–Glicksberg Theorem, the Eilenberg-Montgomery [1946] Theorem to the modified correspondence to obtain its fixed point which gives a required distribution. ∎

Further extensions of Theorem 2.1(b) are possible by applying more advanced fixed point results (see e.g. Górniewicz and Rozpłoch-Nowakowska [1996] for a survey).

For computational reasons, the problem of finding all equilibria in an LSE-game $\gamma = (n; W^1, \ldots, W^n; \Psi^1, \ldots, \Psi^n)$ can be formulated in a complementarity-like form (**CP**):

Find measures $\mathbf{m}^1, \ldots, \mathbf{m}^n$ and real numbers $C^1, \ldots, C^n$ such that

$$\Psi^i(x^i; m^1, \ldots, m^n) - C^i \le 0 \quad \text{for all } x^i \text{ in } W^i \text{ and } i = 1, \ldots, n$$

and

$$\int_{W^i} [\Psi(x^i; m^1, \ldots, m^n) - C^i]\, \mathbf{m}^i(dx) = 0 \quad \text{for } i = 1, \ldots, n.$$

A (variational) minimization problem (**MP**) yielding all equilibrated distributions is the following:

Find measures $\mathbf{m}^1, \ldots, \mathbf{m}^n$ minimizing the value of the expression

$$\Theta_\gamma(\mathbf{m}) := \sum_{i=1}^{n} \Big[\sup_{x \in W^i} \Psi^i(x; m^1, \ldots, m^n) - \int_{W^i} \Psi^i(x^i; m^1, \ldots, m^n)\, \mathbf{m}^i(dx) \Big].$$

Still another problem equivalent to finding all equilibrated distributions for γ is to solve the equation

$$\Theta_\gamma(\mathbf{m}) = 0 \quad \text{for the measures } \mathbf{m} = (\mathbf{m}^1, \ldots, \mathbf{m}^n).$$

The last three problems extend their counterparts formulated by Wieczorek in [2004] or [1996] in the case of LSF-games. All of them are difficult to solve in the general case but special cases can often be solved by methods specific to a particular problem.

3. Relations to LSF-games. LSF-games have been defined by Wieczorek in [2004] (reported earlier in [1996]) by a specification of positive integers $n, k^1, \ldots, k^n$, and real functions $\Phi^1, \ldots, \Phi^n$ such that Φ^i is defined on $V^i \times \Delta_{k^1} \times \cdots \times \Delta_{k^n}$ (where $V^i = \{1, \ldots, k^i\}$ and Δ_k denotes the standard simplex in $\mathbb{R}^k$, with vertices corresponding to integers $1, \ldots, k$, i.e. one of dimension $k-1$).

A distribution of actions in an LSF-game is called *equilibrated* whenever there exist real numbers $C^1, \ldots, C^n$ such that

$$\Phi^i(j; \mathbf{p}^1, \ldots, \mathbf{p}^n) \le C^i \quad \text{for } i = 1, \ldots, n \text{ and all } j \text{ in } V^i$$

and

$$\Phi^i(j; \mathbf{p}^1, \ldots, \mathbf{p}^n) = C^i \quad \text{for all } x^i \text{ in } \operatorname{supp} \mathbf{p}^i.$$

The simplest representation of an LSF-game $\gamma = (n; k^1, \ldots, k^n; \Phi^1, \ldots, \Phi^n)$ as an LSE-game seems to be the following:

$$\overline{\gamma} = (n; W^1, \ldots, W^n; \Psi^1, \ldots, \Psi^n),$$

where, for $i = 1, \ldots, n$, W^i denotes the set of all unit versors $\{\mathbf{e}_1, \ldots, \mathbf{e}_{k^i}\}$ in $\mathbb{R}^{k^i}$ and the functions Ψ^i are defined by $\Psi^i(\mathbf{e}_j; \cdot) := \Phi^i(j; \cdot)$. Clearly, there is a one-to-one correspondence between equilibria for γ and $\overline{\gamma}$.

4. Relations to games with a continuum of players. Since LSE-games are interpreted as involving an infinite population of players, one might expect a formal link between LSE-games and some games with a continuum of players falling into finitely many types with the set of available strategies included in a Euclidean space, namely those in which every player's payoff depends only on his own strategy and finitely many integrated characteristics of distributions of the other players' choice of strategies. This will be done in this section.

Theorem 2.1 of this paper can also be derived, with some auxiliary constructions, from quite general results of Mas-Colell [1984] and Balder [1993]; however, the proof given in Section 2 does not involve any elements redundant in the present context and it may also directly suggest computational procedures to get an equilibrated distribution.

A *game with a continuum of players* or, more properly, a *game with a measure space of players* is given by a specification of the players, usually identified with elements of a normed measure space $(T, \mathcal{T}, \mu)$, the players' nonempty *strategy sets* S^t, $t \in T$, assumed to be all included in some set S (usually—rather for technical reasons—equipped with a σ-field Σ) and the players' *payoff functions*. The payoff function of player t, $u^t(\sigma^t, s)$, depends on the player's own choice of strategy $\sigma^t \in S^t$ and the entire *strategy profile* s, $s(t) \in S^t$; we assume that all *admissible* strategy profiles are measurable with respect to μ and that $u^t(\sigma^t, s) = u^t(\sigma^t, s')$ whenever s and s' are measure equivalent. Hence, a game with a measure space of players is identified with a system

$$\Gamma = ((T, \mathcal{T}, \mu), (S^t \mid t \in T), (S, \Sigma), (u^t \mid t \in T)).$$

Measurable sets of players carrying the measure zero are referred to as *negligible*. A strategy profile s is said to form a *Cournot–Nash equilibrium* if the set of all players t who have a strategy $\sigma^t \in S^t$ such that $u^t(\sigma^t, s) > u^t(s(t), s)$, is negligible.

We say that two players have the *same type* whenever they have the same strategy sets and payoff functions.

4.1. Theorem (equal treatment). *At a Cournot–Nash equilibrium the payoff of almost all players of the same type is equal* (*even though they may use different strategies*).

Suppose that, possibly except for a negligible set of players, there are only finitely many types of players and that players of each of these types form a measurable set. Let us enumerate those players by $1, \dots, n$ and let $\mu(T^i) > 0$ be the measure of the set of all players of type i. Denote the type i players' strategy set by S^i; we assume that it is equipped with a σ-field Σ^i. For $i = 1, \dots, n$ and $j = 1, \dots, k^i$, let f_j^i be a bounded real-valued function defined on S^i. We assume that, for any player of type i, his payoff function has the form

$$u^t(\sigma^t, s) := \Psi^i(f_1^i(\sigma^t), \dots, f_{k^i}^i(\sigma^t); G_1^1, \dots, G_{k^1}^1; \dots; G_1^n, \dots, G_{k^n}^n),$$

where (just for typographical reasons) G_j^i abbreviates $\int_{T^i} f_j^i(s(t)\,\mu(dt)$, for $i = 1, \dots, n$, $j = 1, \dots, k^i$.

If Γ has such a form, we shall say that it is *simple*. We also define, for $i = 1, \dots, n$,

$$W^i := \{f_1^i(\sigma), \dots, f_{k^i}^i(\sigma) \mid \sigma \in S^t\} \subset \mathbb{R}^{k^i}.$$

We define the LSE-game γ *associated* with a simple game Γ with a measure space of players, to be

$$\gamma = (n; W^1, \dots, W^n; \Psi^1, \dots, \Psi^n).$$

4.2. Theorem. *Let $\gamma = (n; W^1, \dots, W^n; \Psi^1, \dots, \Psi^n)$ be an LSE-game associated with a simple game Γ with a measure space of players and let s be a strategy profile for Γ. Let a distribution of actions $\mathbf{m} = (\mathbf{m}^1, \dots, \mathbf{m}^n)$ for γ be the image of s under the f's, i.e. for any $i = 1, \dots, n$ and any measurable set $B^i \subseteq W^i$,*

$$\mathbf{m}^i(B^i) = \mu(\{t \in T \mid f_1^i(s(t)), \dots, f_{k^i}^i(s(t)) \in B^i\}).$$

Then s is a Cournot–Nash equilibrium for Γ if and only if $\mathbf{m}$ is an equilibrium for γ.

Proof. In view of Theorem 4.1, this is a routine verification. ∎

5. Applications in economics and social sciences. Generally, LSE-games are applicable to various situations involving a large number of anonymous actors in which every actor's payoff depends only on his own action and on (finitely many) average characteristics of the other actors' actions.

A. *Production-consumption models.* A model of production and consumption, presented by Wieczorek in [2004] and also studied by Ekes (Roman) and Wieczorek [1999] and Ekes [2003], deals with infinitely many agents who first face the choice among k activities (the choice of the jth activity by an agent yields the production of a fixed amount of the jth good only) and then consume the goods jointly produced. The existence of a competitive equilibrium has been proven in Wieczorek [2004] by a reduction of the model to an LSF-game. The model itself can be easily extended to the more general case where the production process has more complex character, i.e. production abilities of each type of agents are given in the form of their general production sets. One could expect that in such a case the model might be represented as an LSE-game. This is really so, but the existence of competitive equilibria can be obtained in an analogous manner only in the case where the demand functions of each type of agents are affine (which is hardly an acceptable assumption), since the aggregated demand should only depend on the current prices and the mean supply of each good by respective types of agents.

B. *Social adaptation.* Let us consider a society composed of a large (modelled as infinite) number of individuals, falling into n types. The characteristic of an individual of type i is given by an element of a nonempty set $W^i \subset \mathbb{R}^k$, for $i = 1, \ldots, n$; note that k is the same for all types, so the sets W^i may coincide, be disjoint or just intersect. For each type there is an "ideal" characteristic $\bar{x}^i \in W^i$. The individual payoff is a function of the distance of one's own characteristic from the ideal characteristic point and its distance from the mean characteristics of all respective types. So, for each type i, the corresponding payoff has the form

$$\Psi^i(x^i; m^1, \ldots, m^n) = \psi^i(\|x^i - \overline{x}^i\|, \|x^i - m^1\|, \ldots, \|x^i - m^n\|).$$

By Theorem 2.1, *if all sets W^i of individual characteristics are compact and all payoff functions Ψ^i are continuous, then there exists an equilibrium*, i.e. a distribution of characteristics that only agents forming a *negligible* set may be willing to change in order to increase their satisfaction.

Suppose, for instance, that there is just one type of individuals whose characteristics fall into the unit ball $B = \{x \in \mathbb{R}^k \mid \|x\| \leq 1\}$ in a k-dimensional Euclidean space, the ideal point is $\mathbf{0}$ and the payoff function has the form

$$\psi(x, m) := -\alpha\|x\| + \beta\|x - m\| - \gamma\|x - m\|^2$$

for some positive constants α, β and γ; so the payoff of an individual is influenced by the proximity of the ideal $\mathbf{0}$ (the term $\alpha\|x\|$) and by the satisfaction of keeping a reasonable distance from the others (the term $\beta\|x - m\|$), moderated by the discomfort of being unduly isolated (the term $\gamma\|x - m\|^2$).

It is easy to check the following: if $\beta \leq \alpha$ then the distribution concentrated at the origin $\mathbf{0}$ is the only equilibrium. Otherwise, every distribution concentrated on the sphere of radius $r = \min\left\{\frac{1}{2}\gamma^{-1}(\beta - \alpha), 1\right\}$ around $\mathbf{0}$ and having all marginal expectations equal to 0 is an equilibrium and there are no other equilibria. Obviously, the *uniform* distribution on the sphere satisfies those conditions but there are also distributions with finite supports with this property, e.g. in the case $k = 2$ such is the distribution concentrated at the points $r[\cos(2\pi j/h) + i\sin(2\pi j/h)], j = 1, \ldots, h$ (i.e. the hth roots of unity multiplied by r) with mass h^{-1} assigned to each of them, $h \geq 2$.

The questions breifly reported above have been considered in more detail in Wieczorek and Wiszniewska [1999].

C. *Efficiency via taxation.* Let us consider a uniform (i.e. with one type of agents) infinite society whose members have the choice of *intensity* x of their productive activities: this intensity is a number in $I = [0, 1]$. Individual choice generates a distribution of intensities, which is a Borel measure $\mathbf{m}$ on I and the *average* intensity then becomes $m = \int_0^1 x\,\mathbf{m}(dx)$. Suppose that the *income* of an individual depends on his own action x and the total average m, i.e. it is equal to some number $f(x, m)$. We know, by Theorem 2.1, that, whenever f is continuous, there exists an equilibrium and the case is rather trivial so far. However, in most specific cases of some economic meaning, $f(x, m)$ is increasing in x and quite often it is, at least on a part of the domain, decreasing in m. In such a case, an equilibrium may occur at $\mathbf{m}$ concentrated at 1 and it may be socially inefficient in the sense that it does not maximize the value of the total income $T(m) := \int_0^1 f(x, m)\,\mathbf{m}(dx)$ (for instance, this is a very natural case if the social activity is interpreted as extraction of a common resource).

Taxation is often a means to enforce efficiency (cf. e.g. Mirrlees [1986] and the literature quoted therein). Suppose that a (linear) tax t has been imposed per unit activity (extraction) and then the revenue is uniformly distributed among all individuals. The net income of an individual active at x and facing the overall distribution of activities $\mathbf{m}$ becomes now

$$F^t(x, m) := f(x, m) - t(x - m).$$

We assume that $f(x, m)$ is concave in x and the function $g(m) := f(m, m)$ is quasi-concave and continuously differentiable on I (these assumptions could be made weaker at the cost of some extra technical effort).

Under these circumstances *there exists a taxation level* $0 \leq t \leq 1$ *and a distribution* $\mathbf{m}$ *which is an equilibrium for the modified payoff function* F^t *and which is efficient with respect to the original payoff function* f.

To prove the above statement we construct an *auxiliary* LSE-game with two types of players: type 1 are the individuals previously considered, with the action space $W^1 = I$; type 2 are infinitely many small economic forces in charge of taxation, which have the choice among lobbying for *low* taxation ($z = -1$) and *high* taxation ($z = 1$), hence $W^2 = \{-1, 1\}$. Given a distribution $\mathbf{m}^2$ on W^2, the *actual rate of taxation* t is the percentage of high taxation lobbyists, i.e.

$$\tfrac{1}{2}[(-1) \cdot \mathbf{m}^2(\{-1\}) + 1 \cdot \mathbf{m}^2(\{1\})] + \tfrac{1}{2};$$

in our previous notation we have $t(m^2) = \frac{1}{2}m^2 + \frac{1}{2}$ (we skip here the subscript 1 as m has just one coordinate; the same applies to subsequent notation). The payoff functions for the auxiliary game are defined by

$$\Psi^1(x; m^1, m^2) := f(x, m^1) - t(m^2)(x - m^1),$$
$$\Psi^2(z; m^1, m^2) := -z \cdot \frac{dg}{dm}(m^1),$$

for any $x \in W^1$, $z \in W^2$ and any $m^1 \in \operatorname{co} W^1 = W^1$, $m^2 \in \operatorname{co} W^2 = [-1, 1]$.

By Theorem 2.1, the auxiliary game has an equilibrium $(\mathbf{m}^1, \mathbf{m}^2)$. It is not difficult to check that the assumptions imposed on f imply that the distribution $\mathbf{m}^1$ and the taxation level $t(m^2)$ are as required in our statement.

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