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MORREY REGULARITY AND CONTINUITY RESULTS FOR ALMOST MINIMIZERS OF ASYMPTOTICALLY CONVEX INTEGRALS

Abstract. In a recent paper [Forum Math., 2008] the authors established some global, up to the boundary of a domain $\Omega \subset \mathbb{R}^n$, continuity and Morrey regularity results for almost minimizers of functionals of the form $\mathbf{u} \mapsto \int_\Omega g(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x}$. The main assumptions for these results are that g is asymptotically convex and that it satisfies some growth conditions. In this article, we present a specialized but significant version of this general result. The primary purpose of this paper is provide several applications of this simplified result.

1. Introduction. We present several applications for a recently proved Morrey regularity result for minimizers of functionals of the general form

$$\mathbf{u} \mapsto \int_\Omega g(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x}))\, d\mathbf{x}, \tag{1}$$

where Ω is an open, bounded subset of $\mathbb{R}^n$ and $\mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^N)$, with $n, N \geq 1$. In addition to a growth assumption, the primary structural assumption on the integrand g is that there is a $p \in (1, \infty)$ such that for each $(\mathbf{x}, \mathbf{u}) \in \mathbb{R}^n \times \mathbb{R}^N$, the function $\mathbf{F} \mapsto g(\mathbf{x}, \mathbf{u}, \mathbf{F})$ behaves like $\mathbf{F} \mapsto \|\mathbf{F}\|^p$ whenever $\|\mathbf{F}\|$ is sufficiently large. Integrands with this property are called asymptotically convex.

To be more precise, we introduce a few definitions regarding asymptotic convexity. Other notation and definitions are collected in Section 2.

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Definition 1. Two measurable functions $f_1, f_2 : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^* := \mathbb{R} \cup \{+\infty\}$ are $L^{p,\kappa} \times L^\infty_{\mathrm{loc}}$-*asymptotically related to the order* p if for each $\varepsilon > 0$ there is a $\sigma_\varepsilon \in L^{p,\kappa}(\Omega)$ and a $\tau_\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ such that for a.e. $(\mathbf{x}, \mathbf{u}) \in \Omega \times \mathbb{R}^N$,

$$\sup_{\|\mathbf{F}\| > \sigma_\varepsilon(\mathbf{x}) + \tau_\varepsilon(\mathbf{u})} \|\mathbf{F}\|^{-p} |f_1(\mathbf{x}, \mathbf{u}, \mathbf{F}) - f_2(\mathbf{x}, \mathbf{u}, \mathbf{F})| < \varepsilon.$$

For convenience, when p is understood, let us just say that f_1 and f_2 are $L^{p,\kappa}$-*asymptotically related* if they are $L^{p,\kappa} \times L^\infty_{\mathrm{loc}}$-asymptotically related to the order p. It may be quickly verified that Definition 1 can be used to produce an equivalence relation in the class of measurable functions on $\Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$. Roughly speaking, two functions $f_1, f_2 : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^*$ are in the same equivalence class if for a.e. $(\mathbf{x}, \mathbf{u}) \in \Omega \times \mathbb{R}^N$, the functions $\mathbf{F} \mapsto f_1(\mathbf{x}, \mathbf{u}, \mathbf{F})$ and $\mathbf{F} \mapsto f_2(\mathbf{x}, \mathbf{u}, \mathbf{F})$ have the same asymptotic growth, relative to $\|\mathbf{F}\|^p$. Next, we define a special class of representatives for some of these equivalence classes.

Definition 2. A function $f \in \mathscr{C}^2(\mathbb{R}^{N\times n})$ has a p-*Uhlenbeck structure* if there are $\Lambda_*, \Lambda^* > 0$ and an $\widetilde{f} \in \mathscr{C}^2([0,\infty))$ such that for every $\mathbf{F}, \boldsymbol{\xi} \in \mathbb{R}^{N\times n}$ the following hold:

(i) $f(\mathbf{F}) = \widetilde{f}(\|\mathbf{F}\|^2)$;
(ii) $\Lambda_*(1 + \|\mathbf{F}\|^2)^{p/2} \le f(\mathbf{F}) \le \Lambda^*(1 + \|\mathbf{F}\|^2)^{p/2}$;
(iii) $\left\|\frac{\partial}{\partial \mathbf{F}} f(\mathbf{F})\right\| \le \Lambda^*(1 + \|\mathbf{F}\|^2)^{(p-2)/2} \|\mathbf{F}\|$;
(iv) $\left\|\frac{\partial^2}{\partial \mathbf{F}^2} f(\mathbf{F})\right\| \le \Lambda^*(1 + \|\mathbf{F}\|^2)^{(p-2)/2}$;
(v) $\frac{\partial^2}{\partial \mathbf{F}^2} f(\mathbf{F}) :: [\boldsymbol{\xi} \otimes \boldsymbol{\xi}] \ge \Lambda_*(1 + \|\mathbf{F}\|^2)^{(p-2)/2} \|\boldsymbol{\xi}\|^2$.

For convenience, we will set

$$\mathscr{U}^p := \{f \in \mathscr{C}^2(\mathbb{R}^{N\times n}) : f \text{ has a } p\text{-Uhlenbeck structure}\}.$$

Definition 3. A function $g : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is $L^{p,\kappa}$-*asymptotically convex* if there is an $f \in \mathscr{U}^p$, for some $p \in (1,\infty)$, such that f and g are $L^{p,\kappa}$-asymptotically related.

Thus $g : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is $L^{p,\kappa}$-asymptotically convex if, as the modulus of its third argument becomes large, the function g becomes relatively close to a function with p-Uhlenbeck structure. This does not imply that $\mathbf{F} \mapsto g(\mathbf{x}, \mathbf{u}, \mathbf{F})$ is convex on any open subset of $\mathbb{R}^{N\times n}$. As an example, consider the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(F) := |F|^p - |F|\chi_{\mathbb{Q}}$, where $\chi_{\mathbb{Q}}$ is the characteristic function of the set of rational numbers. Clearly g is $L^{p,n}$-asymptotically convex with $\sigma_\varepsilon = \varepsilon^{-1/(p-1)}$, yet g is nowhere convex. Nevertheless, one can show that an $L^{p,\kappa}$-asymptotically convex function does, in some sense, behave like a convex function at infinity. In particular, suppose that g is $L^{p,\kappa}$-asymptotically convex and $\mathbf{F} \mapsto g(\mathbf{x}, \mathbf{u}, \mathbf{F})$ is continuous for a.e. $(\mathbf{x}, \mathbf{u}) \in \Omega \times \mathbb{R}^N$. Then it can be shown that for each $\varepsilon > 0$, there is an $R_\varepsilon : \Omega \times \mathbb{R}^N \to \mathbb{R}$ such that for a.e. $(\mathbf{x}, \mathbf{u}) \in \Omega \times \mathbb{R}^N$ the following is true: if $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{N\times n}$ and the line-segment $\{\lambda\mathbf{A} + (1-\lambda)\mathbf{B} : \lambda \in [0,1]\}$ is contained in $\mathbb{R}^{N\times n} \setminus \overline{\mathcal{B}}_{R(\mathbf{x},\mathbf{u})}$, then

$$g(\mathbf{x}, \mathbf{u}, \lambda\mathbf{A} + (1-\lambda)\mathbf{B}) \le \lambda(1+\varepsilon)\, g(\mathbf{x}, \mathbf{u}, \mathbf{A}) + (1-\lambda)(1+\varepsilon)\, g(\mathbf{x}, \mathbf{u}, \mathbf{B})$$

for each $\lambda \in [0,1]$. Thus, we recover the usual convexity inequality along "lines at infinity".

The focus of this paper is to present some applications of a recent regularity result established by M. Foss, A. Passarelli di Napoli & A. Verde in [12]. This result provides Morrey regularity for the gradient of a minimizer, if one exists, for functionals of the general form (1), under the assumption that the function g is $L^{1,\kappa}$-asymptotically convex. This is a broad generalization, and unification, of Morrey regularity results proved in [4, 10, 19, 20]. In each of these papers, the growth of g, with respect to the third argument, is assumed to be either superquadratic or subquadratic, while both the superquadratic and subquadratic cases are treated in [12]. Moreover, the hypotheses assumed in [12] allow more flexibility regarding how the integrand depends on its arguments than previously permitted. In the setting where the integrand has no explicit dependence on $\mathbf{u}$ itself, K. Fey & M. Foss [9] have recently proved analogous regularity results while only requiring the integrand to satisfy a p-q growth condition. As in [10], the results in [12] yield regularity up to the boundary, provided that the boundary data is sufficiently regular. Rather than restating this regularity result in its most general form, in Section 3 we state a simpler version that is still flexible enough to have broad implications. In the final section, several applications are presented: the existence of Morrey regular minimizing sequences, Morrey regularity for minimizers of relaxed functionals, regularity for an optimal design problem, an alternative characterization for Sobolev–Morrey spaces (Definition 5), regularity for solutions to a class of systems of partial differential equations, and regularity for solutions to obstacle problems.

2. Notation. We use C to denote a generic constant which may change from occurrence to occurrence. Given $p \in [1, \infty]$, we use p^* to denote the Sobolev conjugate exponent defined by

$$p^* := \begin{cases} \dfrac{np}{n-p}, & 1 \le p < n, \\ +\infty, & p \ge n. \end{cases}$$

Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an open set. For each non-negative integer k and each $\gamma \in (0,1]$, the Hölder space $\mathscr{C}^{k,\gamma}(\overline{\mathcal{U}})$ consists of k-times continuously differentiable functions, defined on $\overline{\mathcal{U}}$, with kth order derivatives that are uniformly Hölder continuous with exponent γ. The space of k-times differentiable functions on $\overline{\mathcal{U}}$ with uniformly continuous kth order derivatives is denoted by

$\mathscr{C}^{k,0}(\overline{\mathcal{U}})$. To denote the open ball of radius ϱ centered at $\mathbf{x}_0 \in \mathbb{R}^n$, we use $\mathcal{B}_{\mathbf{x}_0,\varrho}$. We say that $\mathcal{U} \subset \mathbb{R}^n$ has a $\mathscr{C}^{1,0}$-boundary $\partial\mathcal{U}$ if for each $\mathbf{x}_0 \in \partial\mathcal{U}$, there is a $r_{\mathbf{x}_0} > 0$ and a $\mathscr{C}^{1,0}$-diffeomorphism that "straightens out" $\partial\mathcal{U} \cap \mathcal{B}_{\mathbf{x}_0,r_{\mathbf{x}_0}}$.

Next, we define the Morrey and Sobolev–Morrey spaces and recall their relation to the Lebesgue and Hölder spaces. Additional material on these spaces can be found, for example, in [2, 14, 15].

DEFINITION 4. For each $p \in [1, +\infty)$ and $\kappa \in [0, n]$, we define the *Morrey space*

$$L^{p,\kappa}(\mathcal{U}; \mathbb{R}^N) := \Big\{ \mathbf{u} \in L^p(\mathcal{U}; \mathbb{R}^N) : \sup_{\substack{\mathbf{x}_0 \in \mathcal{U} \\ \varrho > 0}} \frac{1}{\varrho^\kappa} \int_{\mathcal{U} \cap \mathcal{B}_{\mathbf{x}_0,\varrho}} \|\mathbf{u}(\mathbf{x})\|^p \, d\mathbf{x} < \infty \Big\}.$$

DEFINITION 5. For each $p \in [1, +\infty)$ and $\kappa \in [0, n]$, we define the *Sobolev–Morrey space*

$$W^{1,(p,\kappa)}(\mathcal{U}; \mathbb{R}^N) := \left\{ \mathbf{u} \in W^{1,p}(\mathcal{U}; \mathbb{R}^N) : \begin{array}{l} \mathbf{u} \in L^{p,\kappa}(\mathcal{U}; \mathbb{R}^N), \\ \nabla\mathbf{u} \in L^{p,\kappa}(\mathcal{U}; \mathbb{R}^{N\times n}) \end{array} \right\}.$$

Each of these spaces is a Banach space, with an appropriately defined norm. Given $p \in [1, \infty)$, one finds $L^{p,n} \cong L^\infty$. If $\mathcal{U}$ has finite measure, then Hölder's inequality immediately shows that $L^q \subseteq L^{p,n(q-p)/q}$ for each $q \in [p, \infty]$. In general, however, if $q > p$, then $L^{p,\kappa} \not\subseteq L^q$ for any $\kappa \in [0, n)$. Thus, the Morrey spaces provide an interpolation between L^p and L^∞ that is distinct from the one produced by the Lebesgue spaces. An imbedding that we will repeatedly recall is that if $\mathcal{U}$ has a $\mathscr{C}^{1,0}$-boundary and $\kappa \in (n-p, n)$, then $W^{1,(p,\kappa)}(\mathcal{U}; \mathbb{R}^N) \subseteq \mathscr{C}^{0,1-(n-\kappa)/p}(\overline{\mathcal{U}}; \mathbb{R}^N)$. We also note that it can be shown that $W^{1,(p,\kappa)} \subseteq L^{p,p+\kappa}$ if $\kappa \in [0, n-p)$.

Finally, we introduce a generalized notion of an almost minimizer. Though not as general as those used in [12], these types of minimizers are suitable for our purposes.

DEFINITION 6. Let $K : W^{1,1}(\Omega; \mathbb{R}^N) \to \mathbb{R}^*$ be given. Let $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega)$ be given, and suppose that $\omega \in \mathscr{C}^0([0, +\infty))$ is a non-decreasing function satisfying $\omega(0) = 0$. We will say that $\mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^N)$ is a $(K, \omega, \{\nu_\varepsilon\})$-*minimizer at* $\mathbf{x}_0$ if $K[\mathbf{u}] < +\infty$ and for each $\varepsilon > 0$ and every $\varrho > 0$,

$$K[\mathbf{u}] \le K[\mathbf{u} + \boldsymbol{\varphi}] + (\omega(\varrho) + \varepsilon) \int_{\Omega \cap \mathcal{B}_{\mathbf{x}_0,\varrho}} (1 + \|\nabla\mathbf{u}\|^p + \|\nabla\boldsymbol{\varphi}\|^p) \, d\mathbf{x} + \int_{\Omega \cap \mathcal{B}_{\mathbf{x}_0,\varrho}} |\nu_\varepsilon(\mathbf{x})| \, d\mathbf{x} \tag{2}$$

for all $\boldsymbol{\varphi} \in W_0^{1,1}(\Omega \cap \mathcal{B}_{\mathbf{x}_0,\varrho}; \mathbb{R}^N)$. If a mapping $\mathbf{u}$ is a $(K, \omega, \{\nu_\varepsilon\})$-minimizer at each $\mathbf{x}_0 \in \Omega$, then we will simply call it a $(K, \omega, \{\nu_\varepsilon\})$-*minimizer*.

Observe that we do not require the variation $\boldsymbol{\varphi}$ to have compact support in Ω, since we are also interested in regularity up to the boundary of Ω. If $\mathbf{u}$ is a $(K,\omega,\{0\})$ minimizer, then our definition reduces to one that is very similar to E. Giusti's definition of ω-minimizers in [15].

3. Main result. In this section, we state and discuss a simplified version of a regularity result proved in [12]. This is the main result that we will be using for the applications. We also recall some other results that will be needed.

The main regularity result that we will be using is

THEOREM 1. *Suppose $\Omega \subset \mathbb{R}^n$ is an open bounded set with a $\mathscr{C}^{1,0}$-boundary. Let $\kappa \in [0,n)$, $p \in (1,\infty)$ and $s \in [0,p^*)$ be given. With $\alpha \in L^{1,\kappa}(\Omega)$, $\{\Sigma_\varepsilon\}_{\varepsilon>0} \subset [0,\infty)$ and $\beta \in [0,\infty)$, suppose that $g : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^*$ satisfies*

(i) *for each $(\mathbf{x},\mathbf{u},\mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$,*

$$|g(\mathbf{x},\mathbf{u},\mathbf{F})| \le \alpha(\mathbf{x}) + \beta(\|\mathbf{u}\|^s + \|\mathbf{F}\|^p);$$

(ii) *g is $L^{p,\kappa}$-asymptotically convex, with $\tau_\varepsilon(\mathbf{u}) = \Sigma_\varepsilon\|\mathbf{u}\|^{s/p}$ for each $\varepsilon > 0$.*

Let $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}(\Omega)$ and $\omega \in \mathscr{C}^0([0,\infty))$, a non-decreasing function satisfying $\omega(0) = 0$, be given. Define the functional $K : W^{1,1}(\Omega;\mathbb{R}^N) \to \mathbb{R}^*$ by

$$K[\mathbf{u}] := \int_\Omega g(\mathbf{x},\mathbf{u}(\mathbf{x}),\boldsymbol{\nabla}\mathbf{u}(\mathbf{x}))\,d\mathbf{x}.$$

If $\mathbf{u}\in W^{1,1}(\Omega;\mathbb{R}^N)$ is a $(K,\omega,\{\nu_\varepsilon\})$-minimizer such that $\mathbf{u}-\overline{\mathbf{u}}\in W_0^{1,1}(\Omega;\mathbb{R}^N)$ for some $\overline{\mathbf{u}} \in W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$, then $\mathbf{u} \in W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$.

REMARK 1. To conclude that $\mathbf{u} \in W^{1,(p,\kappa)}$, it is actually not necessary for $\mathbf{u}$ to satisfy (2) for all $\varepsilon > 0$: it is sufficient to have (2) for all $0 < \varepsilon \le \varepsilon_0$, where ε_0 depends only upon n, N, p, $1/(n-\kappa)$, Λ^*/Λ_* and $\partial\Omega$. This observation can be deduced by examining the constants in the proof for the general result in [12] (the proof for Lemma 5.1 in particular). We will use this later to prove the existence of regular minimizing sequences.

REMARK 2. It is worth explicitly noting that if $p > n-\kappa$, then $\mathbf{u}$ is uniformly Hölder continuous with exponent $1-(n-\kappa)/p$. Thus Theorem 1 can be viewed as providing conditions under which a minimizer possesses some additional lower-order regularity. In a very thorough survey [18], G. Mingione points out that while higher-order regularity has been studied by many researchers, a lower-order regularity theory for vector-valued minimizers has remained largely undeveloped. There has, however, been some recent progress [11], and Theorem 1 can be seen as a contribution to this effort.

REMARK 3. For simplicity, we are presenting only the global version of this regularity result. One can define a local analogue of a $(K, \omega, \{\nu_\varepsilon\})$-minimizer and state a local version of Theorem 1. Of course, the local version does not require any regularity of the boundary data.

REMARK 4. We wish to emphasize that the integrand g is not assumed to be continuous on its domain. From hypothesis (ii), however, one can deduce that $(\mathbf{x}, \mathbf{u}) \mapsto g(\mathbf{x}, \mathbf{u}, \mathbf{F})$ is in some sense close to being continuous when $\mathbf{F}$ is sufficiently large.

Rather than providing a detailed proof for Theorem 1, we attempt to convey the underlying strategy. First we describe the method for establishing the interior Morrey regularity. Then we comment on how this is modified to establish the regularity at the boundary. For the interior regularity, we split each $\mathcal{B}_{\mathbf{x}_0,R} \subset \Omega$ into two regions: one region $\mathcal{A} \subseteq \mathcal{B}_{\mathbf{x}_0,R}$, where $\|\nabla \mathbf{u}\|$ is larger than an appropriate function $\zeta \in L^{1,\kappa}$; and the complementing region $\mathcal{B} := \mathcal{B}_{\mathbf{x}_0,R} \backslash \mathcal{A}$. Within $\mathcal{B}$, the mapping $\mathbf{u}$ has a Morrey regular gradient. Within $\mathcal{A}$, we may use the $L^{p,\kappa}$-asymptotic convexity of g to ultimately compare $\mathbf{u}$ to the minimizer $\mathbf{v} \in W^{1,1}(\mathcal{B}_{\mathbf{x}_0,R}; \mathbb{R}^N)$ of the functional

$$\mathbf{v} \mapsto \int_{\mathcal{B}_{x_0,R}} f(\nabla \mathbf{v}(\mathbf{x}))\, d\mathbf{x}$$

satisfying $\mathbf{v} - \mathbf{u} \in W_0^{1,1}(\mathcal{B}_{\mathbf{x}_0,R}; \mathbb{R}^N)$. The regularity results of E. Acerbi & N. Fusco [1] and K. Uhlenbeck [21] show that $\mathbf{v} \in W_{\mathrm{loc}}^{1,\infty}(\mathcal{B}_{\mathbf{x}_0,R}; \mathbb{R}^N)$ and provide L^∞ estimates for $\nabla \mathbf{v}$. Some of the strength of these estimates is lost when transferring them to $\nabla \mathbf{u}$, so we only obtain estimates for the Morrey regularity of $\nabla \mathbf{u}$ on $\mathcal{A}$. Thus, we conclude that $\nabla \mathbf{u} \in L_{\mathrm{loc}}^{1,\kappa}(\mathcal{B}_{\mathbf{x}_0,R}; \mathbb{R}^{N\times n})$. This yields the local Morrey regularity of $\nabla \mathbf{u}$ in Ω. The main obstacle to extending this result to the boundary is that the regularity results in [1, 21] are essentially local results; they have only been established up to the boundary when the minimizer satisfies homogeneous (constant) boundary conditions. For the Morrey regularity within neighborhoods of boundary points, we first transform a neighborhood of a boundary point $\mathbf{x}_0 \in \partial\Omega$ to a unit half-disk $\mathcal{B}^+$. As before, the half-disk is decomposed into two regions. The key observation is that on the set where $\nabla \mathbf{u}$ is large, we may compare $\mathbf{u}$ to the minimizer $\mathbf{v} \in W^{1,1}(\mathcal{B}^+; \mathbb{R}^N)$ of the functional

$$\mathbf{v} \mapsto \int_{\mathcal{B}^+} f(\nabla \mathbf{v}(\mathbf{x}))\, d\mathbf{x}$$

satisfying $\mathbf{v} + \overline{\mathbf{u}} - \mathbf{u} \in W_0^{1,1}(\mathcal{B}^+; \mathbb{R}^N)$. Since $\mathbf{v} + \overline{\mathbf{u}} - \mathbf{u}$ has zero trace on the straight portion $\mathcal{D}$ of $\partial\mathcal{B}^+$, the L^∞ estimates for $\nabla \mathbf{v}$ are still available and are valid up to $\mathcal{D}$. In the same way that the local Morrey regularity was established, we obtain the Morrey regularity of $\nabla \mathbf{u}$ up to the boundary $\partial\Omega$. This describes the basic idea for the proof of Theorem 1.

4. Applications. The main purpose of this article is to present several applications of Theorem 1. Throughout this section $\kappa \in [0, n)$ and $p \in (1, \infty)$ are fixed and $\Omega \subset \mathbb{R}^n$ is an open bounded set with a $\mathscr{C}^{1,0}$-boundary. We again point out that if $p > n - \kappa$, then $W^{1,(p,\kappa)}(\Omega; \mathbb{R}^N) \subset \mathscr{C}^{0,1-(n-\kappa)/p}(\overline{\Omega}; \mathbb{R}^N)$.

4.1. *Regularity for minimizing sequences and relaxed functionals.* Our first application establishes the existence of Morrey regular minimizing sequences for functionals that need not be sequentially weakly lower semicontinuous. The study of regular minimizing sequences has been carried out by several authors. In particular, G. Dolzmann & J. Kristensen have demonstrated the existence of minimizing sequences with higher integrability properties [6]. Our hypotheses are different from those assumed in [6], but the common feature is that the integrands are assumed to be asymptotically convex. We also provide some conditions under which a minimizer for the relaxed functional is Morrey regular.

Let $\overline{\mathbf{u}} \in W^{1,(p,\kappa)}(\Omega; \mathbb{R}^N)$ be given, and set

$$\mathscr{A} := \{\mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^N) : \mathbf{u} - \overline{\mathbf{u}} \in W_0^{1,1}(\Omega; \mathbb{R}^N)\}.$$

With $g : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^*$ a measurable function, define the functional $K : W^{1,1}(\Omega; \mathbb{R}^N) \to \mathbb{R}^*$ by

$$K[\mathbf{u}] := \int_\Omega g(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x}. \tag{3}$$

The relaxed functional $\overline{K} : W^{1,1}(\Omega; \mathbb{R}^N) \to \mathbb{R}^*$ is given by

$$\overline{K}[\mathbf{u}] := \inf\Big\{\liminf_{k\to\infty} K[\mathbf{u}_k] : \{\mathbf{u}_k\}_{k=1}^\infty \subset W^{1,p}(\Omega; \mathbb{R}^N) \text{ and } \mathbf{u}_k \rightharpoonup \mathbf{u} \text{ in } W^{1,1}\Big\}.$$

While K may not be sequentially weakly lower semicontinuous in $W^{1,1}$, the functional $\overline{K}$ is. In fact, if $\{\mathbf{u}_k\}_{k=1}^\infty \subset \mathscr{A}$ is a minimizing sequence for K, i.e. $\lim_{k\to\infty} K[\mathbf{u}_k] = \inf_{\mathbf{v}\in\mathscr{A}} K[\mathbf{v}]$ and if there is a $\mathbf{u} \in \mathscr{A}$ such that $\mathbf{u}_k \rightharpoonup \mathbf{u}$ in $W^{1,1}$, then $\mathbf{u}$ is a minimizer for $\overline{K}$ over $\mathscr{A}$.

Now suppose that $\overline{K}$ has an integral representation, so there is a function $\overline{g} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^*$ such that

$$\overline{K}[\mathbf{u}] = \int_\Omega \overline{g}(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x}. \tag{4}$$

Further suppose that g satisfies the hypotheses of Theorem 1. Then one can conclude that $\overline{K}$ is coercive in $W^{1,p}$, i.e. $\lim_{\|\mathbf{u}\|_{1,p}\to+\infty} \overline{K}[\mathbf{u}] = +\infty$. This and the sequential weak lower semicontinuity of $\overline{K}$ allow us to conclude that there is actually a minimizer $\mathbf{u} \in \mathscr{A}$ for $\overline{K}$ over $\mathscr{A}$. Moreover, Theorem 1 implies that $\mathbf{u} \in W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$. However, since the original functional K may not be sequentially weakly lower semicontinuous in $W^{1,1}$, we cannot conclude that $\mathbf{u}$ is a minimizer, or that a minimizer in $\mathscr{A}$ even exists, for K itself. Nevertheless, we will show that there is a minimizing sequence $\{\mathbf{u}_k\}_{k=1}^{\infty} \subset \mathscr{A} \cap W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$ for K over $\mathscr{A}$. As mentioned above, we conclude this section with some conditions on g that ensure the relaxed functional has an integral representation as in (4) and the integrand $\overline{g}$ satisfies the hypotheses of Theorem 1.

To prove the existence of a regular minimizing sequence, we recall the following version of Ekeland's variational principle (see for example [15]).

THEOREM 2. *Let $(\mathcal{V}, d)$ be a complete metric space, and let $J : \mathcal{V} \to \mathbb{R}^*$ be a lower semicontinuous functional that is finite at some point in $\mathcal{V}$. Let $\varepsilon > 0$ be given and suppose that there is a $v \in \mathcal{V}$ such that*

$$J[v] \le \inf_{w\in\mathcal{V}} J[w] + \varepsilon^2.$$

Then there is a point $u \in \mathcal{V}$ such that

$$d(u,v) \le \varepsilon \quad \textit{and} \quad J[u] \le J[w] + \varepsilon d(u,w) \quad \textit{for all } w \in \mathcal{V}.$$

We have the following

THEOREM 3. *With $\{\Sigma_\varepsilon\}_{\varepsilon>0} \subset [0,\infty)$, $\alpha \in L^{1,\kappa}(\Omega)$, $\beta \in [0,\infty)$ and $s \in [0,p^*)$, suppose that $g : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^*$ satisfies*

(i) *$g(\cdot,\mathbf{u},\mathbf{F})$ is measurable for each $(\mathbf{u},\mathbf{F}) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$;*
(ii) *$g(\mathbf{x},\cdot,\cdot)$ is continuous for a.e. $\mathbf{x} \in \Omega$;*
(iii) *for each $(\mathbf{x},\mathbf{u},\mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$,*

$$|g(\mathbf{x},\mathbf{u},\mathbf{F})| \le \alpha(\mathbf{x}) + \beta(\|\mathbf{u}\|^s + \|\mathbf{F}\|^p);$$

(iv) *g is $L^{p,\kappa}$-asymptotically convex, with $\tau_\varepsilon(\mathbf{u}) = \Sigma_\varepsilon\|\mathbf{u}\|^{s/p}$ for each $\varepsilon > 0$.*

Define $K : W^{1,1}(\Omega;\mathbb{R}^N) \to \mathbb{R}^*$ by (3). If $\{\mathbf{v}_k\}_{k=1}^{\infty} \subset \mathscr{A}$ is a minimizing sequence for K over $\mathscr{A}$, then there is $\{\mathbf{u}_k\}_{k=1}^{\infty} \subset \mathscr{A} \cap W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$ that is also a minimizing sequence for K over $\mathscr{A}$ such that $\lim_{k\to\infty}\|\mathbf{u}_k - \mathbf{v}_k\|_{1,q} = 0$ for each $q \in [1,p)$.

Proof. Since $\overline{\mathbf{u}} \in \mathscr{A}$ and $K[\overline{\mathbf{u}}] < \infty$ by (iii), we deduce $\inf_{\mathbf{u}\in\mathscr{A}} K[\mathbf{u}] < \infty$ and we may assume that $K[\mathbf{v}_k] < \infty$ for each k. Hypotheses (iii) and (iv) imply that K is coercive in $W^{1,p}$ (see Lemma 4.5 in [12]). Without loss of generality, we may assume that $\{\mathbf{v}_k\}_{k=1}^{\infty} \subset \mathscr{A} \cap W^{1,p}(\Omega;\mathbb{R}^N)$, and there is an $M < \infty$ such that $\|\mathbf{v}_k\|_{1,p} \le M$ for each k. For each $k = 1, 2, \ldots$, put $\varepsilon_k := (K[\mathbf{v}_k] - \inf_{\mathbf{v}\in\mathscr{A}} K[\mathbf{v}])^{1/2}$, so $\lim_{k\to\infty}\varepsilon_k = 0$.

Using hypotheses (i) and (ii) and Fatou's lemma, we conclude that K is lower semicontinuous on $\mathscr{A}$ with the metric d given by

$$d(\mathbf{u},\mathbf{v}) := \|\mathbf{u}-\mathbf{v}\|_{1,1}.$$

We notice that $(\mathscr{A}, d)$ is a complete metric space, so Theorem 2 implies the existence of $\{\mathbf{u}_k\}_{k=1}^{\infty} \subset \mathscr{A}$ such that $\|\mathbf{u}_k - \mathbf{v}_k\|_{1,1} \le \varepsilon_k$ and for every $\varrho > 0$ and $\mathbf{x}_0 \in \Omega$,

$$K[\mathbf{u}_k] \le K[\mathbf{u}_k + \boldsymbol{\varphi}] + \varepsilon_k \|\boldsymbol{\varphi}\|_{1,1} \tag{5}$$

for each $\boldsymbol{\varphi} \in W_0^{1,1}(\Omega \cap \mathcal{B}_{\mathbf{x}_0,\varrho};\mathbb{R}^N)$.

Let $\varrho > 0$, $\mathbf{x}_0 \in \Omega$ and $\boldsymbol{\varphi} \in W_0^{1,1}(\Omega \cap \mathcal{B}_{\mathbf{x}_0,\varrho};\mathbb{R}^N)$ be given. By Hölder's and Sobolev's inequalities, there is a constant $C < \infty$ such that for each k we may write

$$K[\mathbf{u}_k] \le K[\mathbf{u}_k + \boldsymbol{\varphi}] + (\varepsilon_k + C\varrho) \int\limits_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \|\nabla\boldsymbol{\varphi}\|\,d\mathbf{x},$$

and Young's inequality yields

$$K[\mathbf{u}_k] \le K[\mathbf{u}_k + \boldsymbol{\varphi}] + (\varepsilon_k + C\varrho) \int\limits_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} (1 + \|\nabla\boldsymbol{\varphi}\|^p)\,d\mathbf{x}.$$

Now, we may apply Theorem 1 and Remark 1 to conclude that there is a $k_0 < \infty$ such that for each $k > k_0$ we have $\mathbf{u}_k \in \mathscr{A} \cap W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$. Setting $\mathbf{u}_k = \bar{\mathbf{u}}$ for $k = 1,\dots,k_0$ yields the desired Morrey regular minimizing sequence for K.

With $q \in [1,p)$, it remains to show that $\lim_{k\to\infty} \|\mathbf{u}_k - \mathbf{v}_k\|_{1,q} = 0$. Also, we may assume that $\|\mathbf{u}_k\|_{1,p} \le M$ because of the coercivity in $W^{1,p}$. We see that $\{\mathbf{u}_k - \mathbf{v}_k\}_{k=1}^{\infty} \subset W_0^{1,p}(\Omega;\mathbb{R}^N)$, so it is sufficient to verify that $\lim_{k\to\infty} \|\nabla\mathbf{u}_k - \nabla\mathbf{v}_k\|_q = 0$. We already know that $\lim_{k\to\infty}\|\mathbf{u}_k - \mathbf{v}_k\|_{1,1} = 0$, so $[\nabla\mathbf{u}_k(\mathbf{x}) - \nabla\mathbf{v}_k(\mathbf{x})] \to \mathbf{0}$ for a.e. $\mathbf{x} \in \Omega$. Let $\delta > 0$ be given. Egorov's theorem implies that there is a measurable set $\mathcal{E} \subseteq \Omega$ such that $|\mathcal{E}| < \delta$ and $\nabla\mathbf{u}_k - \nabla\mathbf{v}_k \to \mathbf{0}$ uniformly on $\Omega \setminus \mathcal{E}$. Thus

$$\lim_{k\to\infty} \|\nabla\mathbf{u}_k - \nabla\mathbf{v}_k\|_q^q = \lim_{k\to\infty} \int\limits_{\mathcal{E}} \|\nabla\mathbf{u}_k - \nabla\mathbf{v}_k\|^q\,d\mathbf{x} < CM^q\delta^{(p-q)/p}.$$

Since $\delta > 0$ is arbitrary, we conclude that $\lim_{k\to\infty}\|\mathbf{u}_k - \mathbf{v}_k\|_{1,q} = 0$. ■

Our next result provides Morrey regularity for minimizers of relaxed functionals.

THEOREM 4. *Let $\delta : \Omega \times [0,\infty) \to \mathbb{R}^*$ be a function such that*

(D_1) *$\delta(\cdot,t)$ is measurable for each $t \in [0,\infty)$;*
(D_2) *$\delta(\mathbf{x},\cdot)$ is continuous and non-decreasing for a.e. $\mathbf{x} \in \Omega$;*
(D_3) *$\delta(\mathbf{x},0) = 0$ for a.e. $\mathbf{x} \in \Omega$.*

With $\{\Sigma_\varepsilon\}_{\varepsilon>0} \subset [0,\infty)$, $\alpha \in L^{1,\kappa}(\Omega)$, $\beta \in [0,\infty)$ and $s \in [0,p^*)$, suppose that the function $g : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^*$ satisfies

(i) *$g(\cdot,\mathbf{u},\mathbf{F})$ is measurable for each $(\mathbf{u},\mathbf{F}) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$;*
(ii) *$g(\mathbf{x},\cdot,\cdot)$ is continuous for a.e. $\mathbf{x} \in \Omega$;*
(iii) *for each $(\mathbf{x},\mathbf{F}) \in \Omega \times \mathbb{R}^{N\times n}$ and each $\mathbf{u},\mathbf{v} \in \mathbb{R}^N$,*

$$|g(\mathbf{x},\mathbf{u},\mathbf{F}) - g(\mathbf{x},\mathbf{v},\mathbf{F})| \le \delta(\mathbf{x},\|\mathbf{u}-\mathbf{v}\|)(1+\|\mathbf{F}\|^2)^{p/2};$$

(iv) *for each $(\mathbf{x},\mathbf{u},\mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$,*

$$0 \le g(\mathbf{x},\mathbf{u},\mathbf{F}) \le \alpha(\mathbf{x}) + \beta(\|\mathbf{u}\|^s + \|\mathbf{F}\|^p);$$

(v) *g is $L^{p,\kappa}$-asymptotically convex, with $\tau_\varepsilon(\mathbf{u}) = \Sigma_\varepsilon\|\mathbf{u}\|^{s/p}$ for each $\varepsilon > 0$.*

Let $K : W^{1,1}(\Omega;\mathbb{R}^N) \to \mathbb{R}^*$ be given by (3), *and let $\overline{K} : W^{1,1}(\Omega;\mathbb{R}^N) \to \mathbb{R}^*$ be the relaxed functional for K. Then there is a $\mathbf{u} \in \mathscr{A} \cap W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$ such that*

$$\overline{K}[\mathbf{u}] = \inf_{\mathbf{v}\in\mathscr{A}} \overline{K}[\mathbf{v}] = \inf_{\mathbf{v}\in\mathscr{A}} K[\mathbf{v}].$$

Proof. To prove the result, it is sufficient to argue that $\overline{K}$ has an integral representation with an integrand satisfying the hypotheses of Theorem 1. For convenience, put $S_\varepsilon(\mathbf{x},\mathbf{u}) := \sigma_\varepsilon(\mathbf{x}) + \tau_\varepsilon(\mathbf{u})$ (see Definition 1). Without loss, we assume that $S_\varepsilon > 1$.

From Statement [III.7] in [1], hypotheses ($D_{1,2,3}$) and (i–iv) imply that the relaxed functional $\overline{K}$ has an integral representation as in (4). According to [1], the integrand $\overline{g}$ is a measurable function and for each $(\mathbf{x},\mathbf{u}) \in \Omega\times\mathbb{R}^N$ the function $\mathbf{F} \mapsto \overline{g}(\mathbf{x},\mathbf{u},\mathbf{F})$ is the quasiconvex envelope of $\mathbf{F} \mapsto g(\mathbf{x},\mathbf{u},\mathbf{F})$, i.e. $\mathbf{F} \mapsto \overline{g}(\mathbf{x},\mathbf{u},\mathbf{F})$ is the largest quasiconvex function satisfying $\overline{g}(\mathbf{x},\mathbf{u},\mathbf{F}) \le g(\mathbf{x},\mathbf{u},\mathbf{F})$ for all $\mathbf{F} \in \mathbb{R}^{N\times n}$.

We claim that $\overline{g}$ satisfies the hypotheses for Theorem 1. For this we recall that if a real-valued function is convex, then it is also quasiconvex. Since the 0 function is convex, from hypothesis (iv) we deduce that for each $(\mathbf{x},\mathbf{u},\mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$,

$$0 \le \overline{g}(\mathbf{x},\mathbf{u},\mathbf{F}) \le \alpha(\mathbf{x}) + \beta(\|\mathbf{u}\|^s + \|\mathbf{F}\|^p),$$

which is condition (i) in Theorem 1. To show that $\overline{g}$ also satisfies condition (ii), for each $\varepsilon \in (0, \Lambda_*/4p^2)$ and $(\mathbf{x},\mathbf{u}) \in \Omega\times\mathbb{R}^N$ we produce a convex function $h_\varepsilon(\mathbf{x},\mathbf{u},\cdot) : \mathbb{R}^{N\times n} \to \mathbb{R}$ such that

$$h_\varepsilon(\mathbf{x},\mathbf{u},\mathbf{F}) \le g(\mathbf{x},\mathbf{u},\mathbf{F})$$

for all $\mathbf{F} \in \mathbb{R}^{N\times n}$ and

$$|h_\varepsilon(\mathbf{x},\mathbf{u},\mathbf{F}) - f(\mathbf{F})| < \varepsilon\|\mathbf{F}\|^p$$

whenever $\|\mathbf{F}\| > \pi(\mathbf{x}, \mathbf{u})$. Here g is $L^{p,\kappa}$-asymptotically related to $f \in \mathscr{U}^p$ and

$$\pi_\varepsilon(\mathbf{x}, \mathbf{u}) := \frac{p2^{p+5}}{p-1} \frac{\Lambda^*}{\Lambda_*} S_\varepsilon(\mathbf{x}, \mathbf{u}) \quad \text{for each } (\mathbf{x}, \mathbf{u}) \in \Omega \times \mathbb{R}^N.$$

Recall that, in Definition 2, there is an $\widetilde{f} \in \mathscr{C}^2([0,\infty))$ such that $f(\mathbf{F}) = \widetilde{f}(\|\mathbf{F}\|^2)$. For each $(\mathbf{x}, \mathbf{u}, \mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$, define

$$h_\varepsilon(\mathbf{x}, \mathbf{u}, \mathbf{F}) := \begin{cases} \widetilde{f}(\pi_\varepsilon^2) - \varepsilon\pi_\varepsilon^p \\ \qquad + \{2\widetilde{f}'(\pi_\varepsilon^2)\pi_\varepsilon - \varepsilon p \pi_\varepsilon^{p-1}\}(\|\mathbf{F}\| - \pi_\varepsilon), & \|\mathbf{F}\| < \pi_\varepsilon, \\ f(\mathbf{F}) - \varepsilon\|\mathbf{F}\|^p, & \|\mathbf{F}\| \ge \pi_\varepsilon. \end{cases}$$

We make several observations, for each $(\mathbf{x}, \mathbf{u}) \in \Omega \times \mathbb{R}^N$ and $\varepsilon \in (0, \Lambda_*/4p^2)$:

(a) From (i), (iii) and (v) in Definition 2, for each $t \in [0,\infty)$ we deduce that

$$\frac{\Lambda_*}{2}(1+t^2)^{(p-2)/2} \le \widetilde{f}'(t^2) \le \Lambda^*(1+t^2)^{(p-2)/2} \tag{6}$$

and

$$2\widetilde{f}'(t^2) + 4\widetilde{f}''(t^2)t^2 \ge \Lambda_*(1+t^2)^{(p-2)/2}. \tag{7}$$

It follows that $2\widetilde{f}'(\pi_\varepsilon^2)\pi_\varepsilon - \varepsilon p \pi_\varepsilon^{p-1} > 0$ and that for each $\mathbf{F}, \boldsymbol{\xi} \in \mathbb{R}^{N\times n}$,

$$\begin{aligned} \frac{\partial^2}{\partial \mathbf{F}^2}[f(\mathbf{F}) - \varepsilon\|\mathbf{F}\|^p] :: [\boldsymbol{\xi} \otimes \boldsymbol{\xi}] &\ge (\Lambda_* - \varepsilon p(p-1))(1+\|\mathbf{F}\|^2)^{(p-2)/2}\|\boldsymbol{\xi}\|^2 \\ &> \frac{\Lambda_*}{2}(1+\|\mathbf{F}\|^2)^{(p-2)/2}\|\boldsymbol{\xi}\|^2, \end{aligned} \tag{8}$$

so $\mathbf{F} \mapsto h_\varepsilon(\mathbf{x}, \mathbf{u}, \mathbf{F})$ is convex on $\mathbb{R}^{N\times n}$.

(b) Whenever $\|\mathbf{F}\| > S_\varepsilon(\mathbf{x}, \mathbf{u})$, we see from hypothesis (v) that

$$|g(\mathbf{x}, \mathbf{u}, \mathbf{F}) - f(\mathbf{F})| < \varepsilon\|\mathbf{F}\|^p, \quad \text{so} \quad g(\mathbf{x}, \mathbf{u}, \mathbf{F}) > f(\mathbf{F}) - \varepsilon\|\mathbf{F}\|^p.$$

Now $\mathbf{F} \mapsto f(\mathbf{F}) - \varepsilon\|\mathbf{F}\|^p$ is convex by (8). Given $\mathbf{F} \in \mathbb{R}^{N\times n} \setminus \{\mathbf{0}\}$, put $\mathbf{G} := \pi_\varepsilon \mathbf{F}/\|\mathbf{F}\|$. Then

$$\begin{aligned} f(\mathbf{F}) - \varepsilon\|\mathbf{F}\|^p &\ge f(\mathbf{G}) - \varepsilon\|\mathbf{G}\|^p + \{2\widetilde{f}'(\|\mathbf{G}\|^2) - \varepsilon p\|\mathbf{G}\|^{p-2}\}\mathbf{G} : [\mathbf{F} - \mathbf{G}] \\ &= \widetilde{f}(\pi_\varepsilon^2) - \varepsilon\pi_\varepsilon^p + \{2\widetilde{f}'(\pi_\varepsilon^2)\pi_\varepsilon - \varepsilon p\pi_\varepsilon^{p-1}\}(\|\mathbf{F}\| - \pi_\varepsilon). \end{aligned}$$

Hence $h_\varepsilon(\mathbf{x}, \mathbf{u}, \mathbf{F}) \le f(\mathbf{F}) - \varepsilon\|\mathbf{F}\|^p < g(\mathbf{x}, \mathbf{u}, \mathbf{F})$ if $\|\mathbf{F}\| > S_\varepsilon(\mathbf{x}, \mathbf{u})$.

(c) Finally, we argue that $g(\mathbf{x}, \mathbf{u}, \mathbf{F}) > h_\varepsilon(\mathbf{x}, \mathbf{u}, \mathbf{F})$ whenever $\|\mathbf{F}\| \le S_\varepsilon(\mathbf{x}, \mathbf{u})$. Suppose that $\|\mathbf{F}\| \le S_\varepsilon$. Since $g(\mathbf{x}, \mathbf{u}, \mathbf{F}) \ge 0$ by assumption, it is sufficient to demonstrate that $h_\varepsilon(\mathbf{x}, \mathbf{u}, \mathbf{F}) \le 0$. Using the definition of h_ε,

(ii) in Definition 2 and (6), we may write

$$h_\varepsilon(\mathbf{x},\mathbf{u},\mathbf{F}) < \widetilde{f}(\pi_\varepsilon^2) - 2\widetilde{f}'(\pi_\varepsilon^2)\pi_\varepsilon^2 + \varepsilon(p-1)\pi_\varepsilon^p + 2\widetilde{f}'(\pi_\varepsilon^2)\pi_\varepsilon S_\varepsilon \tag{9}$$
$$\le \int_0^{\pi_\varepsilon} \{2\widetilde{f}'(t^2)t - 2\widetilde{f}'(\pi_\varepsilon^2)\pi_\varepsilon\}\,dt + \Lambda^*$$
$$+ \varepsilon(p-1)\pi_\varepsilon^p + 2\Lambda^*(1+\pi_\varepsilon^2)^{(p-2)/2}\pi_\varepsilon S_\varepsilon.$$

We work to estimate the integral above, which we denote by I. With (7) in mind

$$I \le \int_0^{\pi_\varepsilon}\int_0^1 \{2\widetilde{f}'(\pi_\varepsilon + s(t-\pi_\varepsilon))$$
$$+ 4\widetilde{f}''(\pi_\varepsilon + s(t-\pi_\varepsilon))(\pi_\varepsilon + s(t-\pi_\varepsilon))^2\}(t-\pi_\varepsilon)\,ds\,dt$$
$$\le \Lambda_* \int_0^{\pi_\varepsilon}\int_0^1 (1 + (\pi_\varepsilon + s(t-\pi_\varepsilon))^2)^{(p-2)/2}(t-\pi_\varepsilon)\,ds\,dt,$$

since the integrand is actually negative. Continuing,

$$I \le \Lambda_* \int_0^{\pi_\varepsilon}\int_0^1 (1 + (\pi_\varepsilon + s(t-\pi_\varepsilon))^2)^{(p-3)/2}(\pi_\varepsilon + s(t-\pi_\varepsilon))(t-\pi_\varepsilon)\,ds\,dt$$
$$= \Lambda_* \int_0^{\pi_\varepsilon}\int_0^1 \frac{d}{ds}[(1 + (\pi_\varepsilon + s(t-\pi_\varepsilon))^2)^{(p-1)/2}]\,ds\,dt$$
$$= \Lambda_* \int_0^{\pi_\varepsilon} (1+t^2)^{(p-1)/2}\,dt - \Lambda_*(1+\pi_\varepsilon^2)^{(p-1)/2}\pi_\varepsilon.$$

Integrating the remaining integral by parts yields

$$I \le -\Lambda_*(p-1)\int_0^{\pi_\varepsilon} t^2(1+t^2)^{(p-3)/2}\,dt. \tag{10}$$

If $p \ge 3$, then $(1+t^2)^{(p-3)/2} \ge t^{p-3}$, and we conclude that

$$I \le -\frac{(p-1)\Lambda_*}{p}\,\pi_\varepsilon^p.$$

If instead $1 < p < 3$, then for $t \ge 1$ we have $(1+t^2)^{(p-3)/2} \ge 2^{(p-3)/2}t^{p-3}$. Since the integrand in (10) is non-negative and $\pi_\varepsilon > 1$, it follows that

$$I \le -\Lambda_* 2^{(p-3)/2}(p-1)\int_1^{\pi_\varepsilon} t^{p-1}\,dt \le -\frac{(p-1)\Lambda_*}{2p}\,(\pi_\varepsilon^p - 1).$$

In either case, we see that

$$I \le -\frac{(p-1)\Lambda_*}{2p}\,(\pi_\varepsilon^p - 1).$$

Inserting this into (9) and using the fact that $\Lambda_* \le \Lambda^*$ and $\pi_\varepsilon, S_\varepsilon > 1$, we may write

$$\begin{aligned} &h_\varepsilon(\mathbf{x},\mathbf{u},\mathbf{F}) \\ &\quad < \frac{p-1}{p}\left\{\frac{3p-1}{2(p-1)}\,\Lambda^* + \frac{p2^{p/2}}{p-1}\,\Lambda^*(1+\pi_\varepsilon^2)^{(p-2)/2}\pi_\varepsilon S_\varepsilon + \varepsilon p\pi_\varepsilon^p - \frac{1}{2}\,\Lambda_*\pi_\varepsilon^p\right\} \\ &\quad \le \frac{p-1}{p}\left\{\frac{p2^{p+2}\Lambda^*}{p-1}\,\pi_\varepsilon^{p-1}S_\varepsilon + \varepsilon p\pi_\varepsilon^p - \frac{1}{2}\,\Lambda_*\pi_\varepsilon^p\right\} \\ &\quad \le \frac{p-1}{p}\left\{\frac{p2^{p+2}\Lambda^*}{p-1}\,\frac{S_\varepsilon}{\pi_\varepsilon} + \varepsilon p - \frac{1}{2}\,\Lambda_*\right\}\pi_\varepsilon^p < \frac{1-p}{8p}\,\Lambda_*\pi_\varepsilon^p < 0, \end{aligned}$$

since

$$\pi_\varepsilon = \frac{p2^{p+5}}{p-1}\,\frac{\Lambda^*}{\Lambda_*}\,S_\varepsilon \quad \text{and} \quad \varepsilon < \frac{\Lambda_*}{4p^2}.$$

As was argued before, this implies $g(\mathbf{x},\mathbf{u},\mathbf{F}) > h_\varepsilon(\mathbf{x},\mathbf{u},\mathbf{F})$ whenever $\|\mathbf{F}\| \le S_\varepsilon(\mathbf{x},\mathbf{u})$. Summarizing, for each $\varepsilon \in (0, \Lambda_*/4p^2)$ we have produced a function $h_\varepsilon(\mathbf{x},\mathbf{u},\mathbf{F})$ that is convex with respect to $\mathbf{F}$, is dominated by $g(\mathbf{x},\mathbf{u},\mathbf{F})$, and dominates $f(\mathbf{F}) - \varepsilon\|\mathbf{F}\|^p$ for $\|\mathbf{F}\| > \pi_\varepsilon(\mathbf{x},\mathbf{u})$. Therefore $h_\varepsilon \le \overline{g} \le g$ for each $\varepsilon \in (0, \Lambda_*/4p^2)$. It follows from hypothesis (v) that

$$\begin{aligned} |\overline{g}(\mathbf{x},\mathbf{u},\mathbf{F}) - f(\mathbf{F})| &\le \overline{g}(\mathbf{x},\mathbf{u},\mathbf{F}) - h_{\varepsilon/3}(\mathbf{x},\mathbf{u},\mathbf{F}) + \frac{\varepsilon}{3}\,\|\mathbf{F}\|^p \\ &\le g(\mathbf{x},\mathbf{u},\mathbf{F}) - f(\mathbf{F}) + \frac{2\varepsilon}{3}\,\|\mathbf{F}\|^p < \varepsilon\|\mathbf{F}\|^p \end{aligned}$$

whenever $\varepsilon \in (0, \Lambda_*/4p^2)$ and $\|\mathbf{F}\| > \pi_{\varepsilon/3}(\mathbf{x},\mathbf{u})$. Therefore $\overline{g}$ is $L^{p,\kappa}$-asymptotically convex and satisfies condition (ii) in Theorem 1. ■

4.2. *Regularity for an optimal design problem.* In [16], R. V. Kohn & G. Strang study the problem of designing a composite of two materials, one a conductor and the other a perfect insulator (see also [5]). The objective is to find a design so that the rate of heat production under N specified loads is constrained while minimizing the amount of conducting material used. Suppose that Ω is simply connected and that $n = 2$, so $\Omega \subset \mathbb{R}^2$. Also, suppose that the conductor occupies a measurable set $\mathcal{S} \subseteq \Omega$. Let $\mathbf{l} = (l_1, \ldots, l_N)$ denote the N current loads that the design is required to accommodate. We assume that $\int_{\partial\Omega} \mathbf{l}\, d\boldsymbol{\lambda} = \mathbf{0}$ and, for simplicity, that $\mathbf{l} \in \mathscr{C}^0(\partial\Omega; \mathbb{R}^N)$. For each $i = 1, \ldots, N$, the rate at which energy is dissipated into heat under the load l_i is

$$J_i(\mathcal{S}) = \int_\Omega \|\boldsymbol{\mu}_i\|^2\, d\mathbf{x}$$

where

$$\text{(11)}\qquad \begin{cases} \operatorname{div} \boldsymbol{\mu}_i = 0 & \text{in } \Omega, \\ \boldsymbol{\mu}_i = \chi_{\mathcal{S}} \boldsymbol{\nabla} v_i & \text{in } \Omega, \\ \boldsymbol{\mu}_i \cdot \boldsymbol{\nu} = l_i & \text{on } \partial\Omega. \end{cases}$$

Here $\boldsymbol{\mu}_i$ is the current profile induced by $\mathbf{l}_i$, v_i is the potential and $\boldsymbol{\nu}$ is the outward pointing unit normal. The design problem considered in [16] is to find $\mathcal{S}$ with minimal measure such that $J_i(\mathcal{S}) \le c_i$ for a specified $\mathbf{c} \in \mathbb{R}^N$.

Under certain technical assumptions, Kohn & Strang have shown that this design problem can be recast as a variational problem. Let $g : \mathbb{R}^{N\times 2} \to [0,\infty)$ be given by

$$\text{(12)}\qquad g(\mathbf{F}) := \begin{cases} 0, & \mathbf{F} = \mathbf{0}, \\ 1 + \|\mathbf{F}\|^2, & \mathbf{F} \neq \mathbf{0}. \end{cases}$$

Select $\overline{\mathbf{u}} \in \mathscr{C}^{1,0}(\Omega;\mathbb{R}^N)$ so that $\boldsymbol{\nabla}\overline{\mathbf{u}} = \mathbf{l}$ on $\partial\Omega$, and set

$$\mathscr{A} := \{\mathbf{u} \in W^{1,1}(\Omega;\mathbb{R}^N) : \mathbf{u} - \overline{\mathbf{u}} \in W_0^{1,1}(\Omega;\mathbb{R}^N)\}.$$

If $\mathbf{u} \in \mathscr{A}$ is a minimizer, over $\mathscr{A}$, for the functional $K : W^{1,1}(\Omega;\mathbb{R}^N) \to \mathbb{R}^*$ given by

$$K[\mathbf{u}] := \int_\Omega g(\boldsymbol{\nabla}\mathbf{u})\, d\mathbf{x},$$

then the components of $\mathbf{u}$ are stream functions for current profiles satisfying (11) and $\mathcal{S}$ can be identified with the support of $\mathbf{u}$. There is a level of complexity being ignored here which involves Lagrange multipliers to ensure that each constraint $J_i(\mathcal{S}) \le c_i$ is satisfied.

In general, there is no solution to this design problem. Nearly optimal designs can be obtained, however, by seeking a minimizing sequence for K in $\mathscr{A}$. Clearly g satisfies the hypotheses in Theorem 3. Thus there is a sequence $\{\mathbf{u}_k\}_{k=1}^\infty \subset \mathscr{A} \cap W^{1,(2,\gamma)}(\Omega;\mathbb{R}^N)$, for all $\gamma \in [0,n)$, such that $\lim_{k\to\infty} K[\mathbf{u}_k] = \inf_{\mathbf{u}\in\mathscr{A}} K[\mathbf{u}]$. Of course, the sequence of associated current profiles will belong to $L^{2,\gamma}(\Omega;\mathbb{R}^N)$ for each $\gamma \in [0,n)$. A natural and important question is whether or not each current profile is actually globally bounded in magnitude. Though the results in [10, 13] suggest that such bounds are obtainable, these results are not applicable in the current setting. We point out that a crucial step in proving the global bounds in [10, 13] is to first establish global Morrey regularity, which is now done. It is therefore reasonable to conjecture that there are nearly optimal solutions to the design problem above such that under each current load l_i, the magnitude of the current profile is globally bounded.

We note that the integrand g, defined in (12), does not satisfy the hypothesis of Theorem 4. Nevertheless, in [16] it is shown that the relaxed functional for K has an integral representation as in (4), where the integrand $\overline{g} : \mathbb{R}^{N\times 2} \to \mathbb{R}$ is given by

$$\overline{g}(\mathbf{F}) := \begin{cases} 2R(\mathbf{F}) - 2\|\mathbf{D}(\mathbf{F})\|, & R(\mathbf{F}) < 1, \\ 1 + \|\mathbf{F}\|^2, & R(\mathbf{F}) \geq 1. \end{cases}$$

Here $R(\mathbf{F}) := (\|\mathbf{F}\|^2 + 2\|\mathbf{D}(\mathbf{F})\|)^{1/2}$ and $\mathbf{D}(\mathbf{F})$ is the $N(N-1)/2$-vector of all 2×2 subdeterminants of $\mathbf{F}$. Thus $\overline{g}$ is L^∞-asymptotically convex, and Theorem 1 implies that the minimizers in $\mathscr{A}$ of the relaxed functional for K belong to $W^{1,(p,\gamma)}(\Omega;\mathbb{R}^N)$ for each $\gamma \in [0,n)$. As indicated in [16], a result due to M. Chipot & L. C. Evans [3] shows that these minimizers are also in $W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$.

4.3. *A new characterization for Sobolev–Morrey spaces.* The regularity result given in Theorem 1 is, in a sense, optimal. In this section, we produce an alternative characterization of mappings belonging to a Sobolev–Morrey space.

THEOREM 5. *Suppose $\Omega \subset \mathbb{R}^n$ is an open bounded set with a $\mathscr{C}^{1,0}$-boundary. Define $K : W^{1,1}(\Omega;\mathbb{R}^N) \to \mathbb{R}^*$ by*

$$K[\mathbf{u}] := \int_\Omega \|\nabla\mathbf{u}(\mathbf{x})\|^p \, d\mathbf{x}.$$

A mapping $\mathbf{u} : \Omega \to \mathbb{R}^N$ belongs to $W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$ if and only if there is a $\nu \in L^{1,\kappa}(\Omega)$ such that $\mathbf{u}$ is a $(K,\{0,\nu\})$-minimizer satisfying $\mathbf{u} - \overline{\mathbf{u}} \in W^{1,1}_0(\Omega;\mathbb{R}^N)$ for some $\overline{\mathbf{u}} \in W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$.

Proof. One direction is an immediate application of Theorem 1. For the other direction, suppose that $\mathbf{u} \in W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$. Then we can put $\nu := \|\nabla\mathbf{u}\|^p$ and $\overline{\mathbf{u}} := \mathbf{u}$. Let $\varrho > 0$ and $\mathbf{x}_0 \in \Omega$ be given. For each $\boldsymbol{\varphi} \in W^{1,1}_0(\Omega \cap \mathcal{B}_{\mathbf{x}_0,\varrho};\mathbb{R}^N)$, we clearly have

$$\begin{aligned} K[\mathbf{u}] &= K[\mathbf{u}+\boldsymbol{\varphi}] + \int_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \{\|\nabla\mathbf{u}\|^p - \|\nabla\mathbf{u} + \nabla\boldsymbol{\varphi}\|^p\} \, d\mathbf{x} \\ &\leq K[\mathbf{u}+\boldsymbol{\varphi}] + \int_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \nu(\mathbf{x}) \, d\mathbf{x}. \ \blacksquare \end{aligned}$$

4.4. *Regularity of solutions to systems of PDE's.* We next turn to weak solutions of systems of partial differential equations. In [17], J. Kristensen & A. Taheri showed that if a system of PDE's asymptotically resembles a p-Laplacian, in the same vein as in hypothesis (ii) below, then the solutions belong to $W^{1,q}_{\mathrm{loc}}$ for all $q \in (p,\infty)$. We establish a global Morrey regularity result in a more general setting. For purposes of comparison, we note that under the same hypotheses assumed in [17], our result would state that the solutions belong to $W^{1,(p,\gamma)}$ for all $\gamma \in [0,n)$.

For this section, we define $L^{p,\kappa} \times L^{\infty}_{\mathrm{loc}}$-asymptotically related vector-valued mappings via an obvious extension of Definition 1.

THEOREM 6. *Given* $\{\Sigma_\varepsilon\}_{\varepsilon>0} \subset [0,\infty)$, $\alpha \in L^{p/(p-1),\kappa}(\Omega)$, $\beta \in [0,\infty)$ *and* $s \in [0,p^*)$, *suppose that the functions* $\mathbf{A} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ *and* $\mathbf{h} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^N$ *satisfy*

(i) *for each* $(\mathbf{x},\mathbf{u},\mathbf{F}) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$,

$$\|\mathbf{A}(\mathbf{x},\mathbf{u},\mathbf{F})\| + |\mathbf{h}(\mathbf{x},\mathbf{u},\mathbf{F})| \le \alpha(\mathbf{x}) + \beta(\|\mathbf{u}\|^{s(p-1)/p} + \|\mathbf{F}\|^{p-1});$$

(ii) *there is a* $g \in \mathscr{U}^p$ *such that* $\mathbf{A}$ *and* $\frac{\partial}{\partial \mathbf{F}} g$ *are* $L^{p,\kappa} \times L^{\infty}_{\mathrm{loc}}$*-asymptotically related to the order* $p-1$, *with* $\tau_\varepsilon(\mathbf{u}) = \Sigma_\varepsilon \|\mathbf{u}\|^{s/p}$ *for each* $\varepsilon > 0$.

Suppose that $\mathbf{u} \in \mathscr{A} \cap W^{1,p}(\Omega;\mathbb{R}^N)$ *is a weak solution to the system*

$$\operatorname{div}[\mathbf{A}(\mathbf{x},\mathbf{u}(\mathbf{x}),\nabla\mathbf{u}(\mathbf{x}))] = \mathbf{h}(\mathbf{x},\mathbf{u}(\mathbf{x}),\nabla\mathbf{u}(\mathbf{x})) \quad \text{in } \Omega,$$

i.e. for each $\boldsymbol{\varphi} \in W^{1,p}_0(\Omega;\mathbb{R}^N)$,

$$\int_\Omega \{\mathbf{A}(\mathbf{x},\mathbf{u}(\mathbf{x}),\nabla\mathbf{u}(\mathbf{x})) : \nabla\boldsymbol{\varphi}(\mathbf{x}) + \mathbf{h}(\mathbf{x},\mathbf{u}(\mathbf{x}),\nabla\mathbf{u}(\mathbf{x}))\cdot\boldsymbol{\varphi}(\mathbf{x})\}\,d\mathbf{x} = 0. \tag{13}$$

Then $\mathbf{u} \in W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$.

Proof. Our argument follows one given in [9], where g is allowed to possess a p-q growth structure but $\mathbf{A}$ and $\mathbf{h}$ are assumed to depend on $\mathbf{u}$ through $\nabla\mathbf{u}$ only. Without loss of generality, we assume that $s \in [p,p^*)$. Define the functional $K : W^{1,1}(\Omega;\mathbb{R}^N) \to \mathbb{R}^*$ by

$$K[\mathbf{v}] := \int_\Omega g(\nabla\mathbf{v}(\mathbf{x}))\,d\mathbf{x}.$$

We will show that there is a non-decreasing $\omega \in \mathscr{C}^0([0,\infty))$, satisfying $\omega(0) = 0$, and $\{\nu_\varepsilon\}_{\varepsilon>0} \subset L^{1,\kappa}(\Omega)$ such that $\mathbf{u}$ is a $(K,\omega,\{\nu_\varepsilon\})$-minimizer.

Let $\varrho \in (0,1]$, $\mathbf{x}_0 \in \Omega$ and $\boldsymbol{\varphi} \in W^{1,1}_0(\Omega;\mathbb{R}^N)$ be given. If $\boldsymbol{\varphi} \notin W^{1,p}_0(\Omega;\mathbb{R}^N)$, then condition (ii) in Definition 2 implies that

$$K[\mathbf{u}] < K[\mathbf{u}+\boldsymbol{\varphi}] = +\infty$$

since $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^N)$ by assumption. We may therefore assume that $\boldsymbol{\varphi} \in W^{1,p}_0(\Omega;\mathbb{R}^N)$. The convexity of g implies

$$K[\mathbf{u}+\boldsymbol{\varphi}] - K[\mathbf{u}] \ge -\int_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \frac{\partial}{\partial\mathbf{F}} g(\nabla\mathbf{u}) : \nabla\boldsymbol{\varphi}\,d\mathbf{x}. \tag{14}$$

We now work to estimate the last integral above, which we denote by I. By hypothesis (ii) and Definition 1, there is a $\sigma_\varepsilon \in L^{p,\kappa}(\Omega)$ such that for a.e. $(\mathbf{x},\mathbf{u}) \in \Omega \times \mathbb{R}^N$,

$$\sup_{\|\mathbf{F}\| > \sigma_\varepsilon(\mathbf{x}) + \Sigma_\varepsilon\|\mathbf{u}\|^{s/p}} \|\mathbf{F}\|^{1-p} \left\| \mathbf{A}(\mathbf{x},\mathbf{u},\mathbf{F}) - \frac{\partial}{\partial\mathbf{F}} g(\mathbf{F}) \right\| < \varepsilon.$$

Let $\varepsilon > 0$ be given. Set $\mathcal{E}_\varepsilon := \{\mathbf{x} \in \Omega : \|\nabla \mathbf{u}(\mathbf{x})\| > \sigma_\varepsilon(\mathbf{x}) + \Sigma_\varepsilon \|\mathbf{u}\|^{s/p}\}$. Using (13) and hypotheses (i) and (ii), we may write

$$\begin{aligned}
I \le & \int_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \left\| \mathbf{A}(\mathbf{x},\mathbf{u},\nabla\mathbf{u}) - \frac{\partial}{\partial \mathbf{F}} g(\nabla \mathbf{u}) \right\| \|\nabla\boldsymbol{\varphi}\|\, d\mathbf{x} \\
& - \int_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \mathbf{A}(\mathbf{x},\mathbf{u},\nabla\mathbf{u}) : \nabla\boldsymbol{\varphi}\, d\mathbf{x} \\
\le & \ \varepsilon \int_{\mathcal{E}_\varepsilon\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \|\nabla\mathbf{u}\|^{p-1}\|\nabla\boldsymbol{\varphi}\|\, d\mathbf{x} + \int_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \mathbf{h}(\mathbf{x},\mathbf{u},\nabla\mathbf{u})\cdot\boldsymbol{\varphi}\, d\mathbf{x} \\
& + C \int_{(\Omega\setminus\mathcal{E}_\varepsilon)\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} (\alpha(\mathbf{x}) + \|\mathbf{u}\|^{s(p-1)/p} + \|\nabla\mathbf{u}\|^{p-1})\|\nabla\boldsymbol{\varphi}\|\, d\mathbf{x}.
\end{aligned}$$

Another use of hypothesis (i) yields

$$\begin{aligned}
I \le & \ \varepsilon \int_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \|\nabla\mathbf{u}\|^{p-1}\|\nabla\boldsymbol{\varphi}\|\, d\mathbf{x} \\
& + \varepsilon \int_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} (\alpha(\mathbf{x}) + \|\mathbf{u}\|^{s(p-1)/p} + \|\nabla\mathbf{u}\|^{p-1})\|\boldsymbol{\varphi}\|\, d\mathbf{x} \\
& + \int_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} (\alpha(\mathbf{x}) + \|\mathbf{u}\|^{s(p-1)/p} + \sigma_\varepsilon(\mathbf{x})^{p-1})\|\nabla\boldsymbol{\varphi}\|\, d\mathbf{x}.
\end{aligned}$$

Applying Young's and Sobolev's inequalities yields (in the following, if it happens that $p^* = +\infty$, then we temporarily redefine $p^* = 2s$)

$$\begin{aligned}
I \le & \ C(\varepsilon+\varrho) \int_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \{\|\nabla\mathbf{u}\|^p + \|\nabla\boldsymbol{\varphi}\|^p\}\, d\mathbf{x} + C \int_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \|\mathbf{u}-\overline{\mathbf{u}}\|^s\, d\mathbf{x} \\
& + C_\varepsilon \int_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \{\alpha(\mathbf{x})^{p/(p-1)} + \sigma_\varepsilon(\mathbf{x})^p + \|\overline{\mathbf{u}}\|^s\}\, d\mathbf{x} \\
\le & \ C(\varepsilon+\varrho) \int_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \{\|\nabla\mathbf{u}\|^p + \|\nabla\boldsymbol{\varphi}\|^p\}\, d\mathbf{x} \\
& + C\varrho^{n(p^*-s)/p^*} \Big(\int_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \|\nabla\mathbf{u}\|^p\, d\mathbf{x} \Big)^{s/p} \\
& + C\varrho^{n(p^*-s)/p^*} \Big(\int_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \|\nabla\overline{\mathbf{u}}\|^p\, d\mathbf{x} \Big)^{s/p} \\
& + C_\varepsilon \int_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \{\alpha(\mathbf{x})^{p/(p-1)} + \sigma_\varepsilon(\mathbf{x})^p + \|\overline{\mathbf{u}}\|^s\}\, d\mathbf{x}.
\end{aligned}$$

Now it can be shown that $W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N) \subset L^{s,\kappa}(\Omega;\mathbb{R}^N)$, so $\|\overline{\mathbf{u}}\|^s \in L^{1,\kappa}(\Omega)$.

Putting

$$\omega(\varrho) := C\Big[\varrho + \varrho^{n(p^*-s)/p^*} \sup_{\mathbf{x}_0\in\Omega}\Big(\int\limits_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \|\nabla\mathbf{u}\|^p\,d\mathbf{x}\Big)^{(s-p)/p}\Big]$$

and

$$\nu_\varepsilon(\mathbf{x}) := C_\varepsilon[M\|\nabla\overline{\mathbf{u}}(\mathbf{x})\|^p + \alpha(\mathbf{x})^{p/(p-1)} + \sigma_\varepsilon(\mathbf{x})^p + \|\overline{\mathbf{u}}(\mathbf{x})\|^s]$$

with $M = \operatorname{diam}(\Omega)^{n(p^*-s)/p^*}(\int_\Omega \|\nabla\overline{\mathbf{u}}\|^p\,d\mathbf{x})^{(s-p)/p}$, we may continue with

$$I \le C(\varepsilon+\omega(\varrho)) \int\limits_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \{\|\nabla\mathbf{u}\|^p + \|\nabla\boldsymbol{\varphi}\|^p\}\,d\mathbf{x} + \int\limits_{\Omega\cap\mathcal{B}_{\mathbf{x}_0,\varrho}} \nu_\varepsilon(\mathbf{x})\,d\mathbf{x}.$$

Inserting this last estimate in (14), we see that $\mathbf{u}$ is a $(K,\omega,\{\nu_\varepsilon\})$-minimizer, with ω and $\{\nu_\varepsilon\}_{\varepsilon>0}$ satisfying the hypotheses of Theorem 1. Therefore $\mathbf{u} \in W^{1,(p,\kappa)}(\Omega;\mathbb{R}^N)$. ∎

4.5. *Regularity for obstacle problems.* In this final section, we state a regularity result for obstacle problems. This result is a vectorial analogue of some results provided by M. Eleuteri [8]. For this section, we fix $p = 2$, so that $s \in [0, 2^*)$. Let $\boldsymbol{\psi} \in W^{1,(2,\kappa)}(\Omega;\mathbb{R}^N)$ be given, and set

$$\mathscr{B} := \{\mathbf{u} \in W^{1,2}(\Omega;\mathbb{R}^N) : u_i \ge \psi_i \text{ for } i = 1,\dots,N\}.$$

We assume that $\mathscr{A}\cap\mathscr{B} \ne \emptyset$.

THEOREM 7. *With $\{\Sigma_\varepsilon\}_{\varepsilon>0} \subset [0,\infty)$, $\alpha \in L^{1,\kappa}(\Omega)$, $\beta \in [0,\infty)$ and $s \in [0,p^*)$, suppose that $g : \Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n} \to \mathbb{R}^*$ satisfies*

(i) *for each $(\mathbf{x},\mathbf{u},\mathbf{F}) \in \Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}$,*

$$|g(\mathbf{x},\mathbf{u},\mathbf{F})| \le \alpha(\mathbf{x}) + \beta(\|\mathbf{u}\|^s + \|\mathbf{F}\|^2);$$

(ii) *g is $L^{2,\kappa}$-asymptotically related to $\mathbf{F}\mapsto\|\mathbf{F}\|^2$, with $\tau_\varepsilon(\mathbf{u}) = \Sigma_\varepsilon\|\mathbf{u}\|^{s/2}$ for each $\varepsilon > 0$.*

If $\mathbf{u} \in \mathscr{A}\cap\mathscr{B}$ is a minimizer, over $\mathscr{A}\cap\mathscr{B}$, for $K : W^{1,1}(\Omega;\mathbb{R}^N) \to \mathbb{R}^*$ defined by

$$K[\mathbf{u}] := \int\limits_\Omega g(\mathbf{x},\mathbf{u}(\mathbf{x}),\nabla\mathbf{u}(\mathbf{x}))\,d\mathbf{x},$$

then $\mathbf{u} \in W^{1,(2,\kappa)}(\Omega;\mathbb{R}^N)$.

The proof is an adaptation of an argument used by F. Duzaar, A. Gastel & F. Grotowski [7]. For details, we refer to [12].

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