EXISTENCE OF SOLUTIONS TO THE POISSON EQUATION IN $L_2$-WEIGHTED SPACES

Abstract. We consider the Poisson equation with the Dirichlet and the Neumann boundary conditions in weighted Sobolev spaces. The weight is a positive power of the distance to a distinguished plane. We prove the existence of solutions in a suitably defined weighted space.

1. Introduction. We study the following boundary value problem for the Poisson equation in weighted spaces:

$$
-\Delta u = f \quad \text{in } \Omega,
$$

$$
\frac{\partial u}{\partial x_3}|_{S_*} = 0,
$$

$$
u|_{S_1} = 0,
$$

$$
u|_{S_0} = 0
$$

where $\Omega \subset \mathbb{R}^3$, $\partial \Omega = S_0 \cup S_1 \cup S_* = S$ (see Fig. 1).

Fig. 1. Domain $\Omega$

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Here, \( S_0 \) is parallel and \( S_1 \) and \( S_* \) are perpendicular to the \( x_3 \) axis, and \( S_1 \) meets the \( x_3 \) axis at the point \( x_3 = a, a < \infty \), while \( S_* \) meets the \( x_3 \) axis at the point \( x_3 = 0 \).

The problem arises as an auxiliary system in the analysis of the inflow-outflow motion described by the Navier–Stokes equations and the aim of this paper is to prepare tools to examine the NS equations. We want to use the estimates derived here to remove some restrictions on the boundary inflow for the Navier–Stokes system.

Let us consider the Navier–Stokes equations with slip boundary conditions on \( S = S_0 \cup S_1 \cup S_* \) with inflow on \( S_* \) and outflow on \( S_1 \). On \( S_0 \) the normal component of the velocity vanishes. The inflow-outflow problem is a difficult and unclear problem for NS (see [L, Ch. 5] and [G, Vol. 2, Ch. 8]). Usually, such a result requires strong restrictions, for example in the proof of global existence in [Z2] the inflow flux either must vanish or must be sufficiently small in time. Our goal is to prove long time existence of solutions to the Navier–Stokes equations without restrictions on the magnitude of the inflow flux. The first and most crucial step is to obtain a global estimate (with nonvanishing inflow flux) for weak solutions. To this end, we homogenize the problem in question by solutions of some elliptic systems where the function \( \eta \) (see [L, Ch. 5, Sect. 4]) is used. The derivative of this function implies the weight introduced in this paper. Moreover, to estimate the nonlinear terms which correspond to \( v \cdot \nabla v \), where \( v \) is the velocity of the fluid, we need an \( L^p \) version, with \( p = 3 \), of Theorem 1 (see [RZ]).

To formulate the main result of this paper we introduce

\[
H^k_{\mu}(\Omega) = \left\{ u : \|u\|_{H^k_{\mu}(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_{\Omega} dx' dx_3 |D_x^\alpha u|^2 x_3^{2(\mu + |\alpha| - k)} \right)^{1/2} < \infty \right\},
\]

where \( k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), \( \mu \in \mathbb{R} \), \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) is a multiindex, \( |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \), \( \alpha_i \in \mathbb{N}_0 \), \( i = 1, 2, 3 \), \( D_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \). Moreover, we denote \( L_{2,\mu}(\Omega) = H^0_{\mu}(\Omega) \).

Note, that \( x_3^{p\mu} \), \( p \in (1, \infty) \), is not a Muckenhoupt weight so the results of Coifman–Fefferman [CF] cannot be applied.

**Theorem 1.** Assume that \( f \in L_{2,\mu}(\Omega) \) can be expressed in the form

\[
(1.2) \quad f = \alpha_{\cdot,x_3} \quad \text{with} \quad \alpha = \tilde{\alpha} \eta(x_3), \quad \tilde{\alpha}|_{x_3=0} = \theta,
\]

where

(i) \( \eta(x_3) \) is a smooth non-increasing function with compact support such that \( 0 \leq \eta \leq 1 \) and \( \eta = 1 \) in a neighbourhood of \( x_3 = 0 \),

(ii) \( \theta \in H^1(S_*) \),

(iii) \( \tilde{\alpha} \) is an extension of \( \theta \) to \( x_3 \geq 0 \) such that

\[
\|\tilde{\alpha}\|_{L_\infty(\mathbb{R}_+;H^1(S_*))} \leq c \|\theta\|_{H^1(S_*)}.
\]
Then a solution $u$ of the problem (1.1) exists and satisfies
\begin{equation}
\|u - u(0)\|_{H^2_\mu(\Omega)} \leq c(\|f\|_{L^2_{2,\mu}(\Omega)} + \|\theta\|_{H^1(S_*)}), \quad \mu \in (0, 1),
\end{equation}
where $u(0)$ denotes $u|_{x_3=0}$.

Remark 1.1. The quantity $u(0) \in H^2(S_*)$ must be calculated independently. To this end let us consider problem (1.1) and let $G(x, y)$ be the Green function to (1.1) of the form
\begin{equation}
G(x, y) = \vartheta(x, y, |x-y|) + g(x, y)
\end{equation}
where $\vartheta(x, y)$ is a smooth function such that
\begin{align*}
\vartheta(x, y) &= \begin{cases} 
1 & \text{for } |x-y| \leq 1, \\
0 & \text{for } |x-y| \geq 2.
\end{cases}
\end{align*}
Then $g(x, y)$ is a solution to the problem
\begin{equation}
-\Delta g = 2\nabla \frac{1}{|x-y|} \nabla \vartheta + \frac{1}{|x-y|} \Delta \vartheta,
\end{equation}
where $\vartheta(x, y)$ is a smooth function such that
\begin{align*}
\frac{\partial g}{\partial n} \bigg|_{S_*} &= -\partial_n \frac{\vartheta}{|x-y|}, \\
g|_{S_0 \cup S_1} &= -\frac{\vartheta}{|x-y|},
\end{align*}
where $y \in \Omega$. Then any solution to (1.1) can be expressed as
\begin{equation}
u(x) = \int_{\Omega} G(x, y) f(y) \, dy.
\end{equation}
Since $f = \alpha_{x_3}$ (see (1.2)) we obtain
\begin{equation}u(x) = \int_{S_*} G(x, y') \alpha(y') \, dy' - \int_{\Omega} \partial y_3 G(x, y) \alpha(y) \, dy.
\end{equation}
Using the form of the Green function we have
\begin{equation}||u(0)||_{H^2(S_*)} \leq c||\theta||_{H^1(S_*)}.
\end{equation}

2. Estimates. Although the solution $u$ can be expressed by the Green function, we have to use another approach to show the properties of the solution described in the main theorem. Namely, to prove Theorem 1 we shall use local considerations.

Lemma 2.2. Assume that $f \in L^2_{2,\mu}(\Omega)$, $\mu \in [0, 1]$. Then there exists a solution to problem (1.1) such that
\begin{equation}u' = u - u(0) \in H^1(\Omega) \cap L^2_{2,-\mu}(\Omega)
\end{equation}
and
\begin{equation}||u'||_{H^1(\Omega)} + ||u'||_{L^2_{2,-\mu}(\Omega)} \leq c ||f'||_{L^2_{2,\mu}(\Omega)},
\end{equation}
where $u(0) = u|_{x_3=0}$ and $f' = f + \Delta u(0)$. 

Poisson equation in $L^2$ weighted spaces
Proof. First we obtain an energy type estimate. We reformulate (1.1) as follows:

\[ -\Delta u' = f' \quad \text{in } \Omega, \]

\[ u'_{x_3} |_{S_*} = 0, \]

\[ u' |_{S_1} = 0, \]

\[ u' |_{S_0} = 0. \]

Multiplying \((2.2)_1\) by \(u'\), integrating by parts and using the boundary condition \((2.2)_2\) we have

\[ \int_{\Omega} |\nabla u'|^2 \, dx \leq \int_{\Omega} f'u' \, dx \]

\[ \leq \left( \int_{\Omega} |f'|^2 x_3^{2\mu} \, dx \right)^{1/2} \left( \int_{\Omega} |u'|^2 x_3^{-2\mu} \, dx \right)^{1/2}. \]

We calculate

\[ u(x', x_3) - u(x', s) = \int_s^{x_3} \partial_{s'} u(x', s') \, ds', \]

where \(x' = (x_1, x_2)\). By the Hölder inequality

\[ \frac{|u(x_3) - u(s)|}{|x_3 - s|} \leq \int_s^{x_3} |\partial_{s'} u(x', s')|^2 \, ds'. \]

Setting \(s = 0\) and integrating the inequality over \(\Omega' = \{x \in \Omega : x_3 = \text{const} \in [0, a]\}\) we have

\[ \int_{\Omega'} \frac{|u(x_3) - u(0)|}{|x_3|} \, dx' \leq \int_{\Omega'} dx' \int_0^{x_3} |\partial_{s'} u(x', s')|^2 \, ds'. \]

Now, we consider the second expression on the r.h.s. of (2.3). We obtain

\[ \int_{\Omega} |u'|^2 x_3^{-2\mu} \, dx \leq \int_{\Omega'} dx' \sup_{x_3} \frac{|u(x', x_3) - u(0)|}{x_3}^{\frac{a}{2\mu-1}} \int_0^{x_3} dx_3 \]

\[ \leq c \int_{\Omega'} \int_0^{a} |\partial_{x_3} u(x', x_3)|^2 \, dx_3 \]

where we used \((2.4)\) and \(\mu < 1\).

Summarizing, we have

\[ \int_{\Omega} |u'|^2 x_3^{-2\mu} \, dx + \int_{\Omega} |\nabla u'|^2 \, dx \leq c \int_{\Omega} |f'|^2 x_3^{2\mu} \, dx, \quad \mu \in [0, 1]. \]

The existence of weak solutions to problem (1.1) satisfying (2.5) now follows from the Lax–Milgram theorem.
To increase regularity of the weak solutions we consider problem (1.1) locally. Let \( \zeta = \zeta(x') \) be a smooth function from a partition of unity. Introducing functions \( \bar{u} = u\zeta, \bar{f} = f \zeta \) problem (1.1) takes the form
\[
\Delta \bar{u} = 2 \nabla u \nabla \zeta + u \Delta \zeta + \bar{f} \equiv g,
\]
(2.6)
\[
\bar{u}|_{\partial \text{supp} \zeta \cap \{x_3 = a\}} = 0,
\]
\[
\bar{u},x_3|_{\partial \text{supp} \zeta \cap \{x_3 = 0\}} = 0.
\]
Suppose \( \text{supp} \zeta \cap S_0 \neq \emptyset \). Assume that problem (2.6) is written in local coordinates \( x = (x_1, x_2, x_3) \) with origin in \( \text{supp} \zeta \cap S_0 \). Now by using the local mapping \( y = \Phi(x) \) such that \( y_3 = x_3 \), we flatten the boundary \( \text{supp} \zeta \cap S_0 \). Assume that in the new coordinates \( y \) the flat boundary \( \text{supp} \zeta \cap S_0 \) takes the form \( y_1 = 0 \). Let \( v(y) = \bar{u}(\Phi^{-1}(y)), h(y) = g(\Phi^{-1}(y)) \). Then problem (2.6) takes the form
\[
\nabla^2_y v = \nabla^2_y v - \nabla^2_y \Phi v + h \equiv k,
\]
(2.7)
\[
v|_{y_1=0} = 0,
\]
\[
v|_{y_3=a} = 0,
\]
\[
v,_{x_3}|_{x_3=0} = 0,
\]
where \( \nabla \Phi = \frac{\partial y}{\partial x}|_{x=\Phi^{-1}(y)} \nabla y \) and \( v \) vanishes outside of \( \text{supp} \zeta|_{x=\Phi^{-1}(y)} \).

Let us consider the reflection with respect to \( y_1 \) such that the reflected function \( u \) satisfies
\[
u(y_1, y_2, y_3) = v(y_1, y_2, y_3), \quad y_1 > 0,
\]
\[
u(y_1, y_2, y_3) = v(-y_1, y_2, y_3), \quad y_1 < 0.
\]
After the above reflection problem (2.7) takes the form
\[
\nabla^2_y u = \tilde{h} \quad \text{in } \mathbb{R}^2 \times (0, a),
\]
(2.8)
\[
u|_{y_3=a} = 0 \quad \text{on } \mathbb{R}^2,
\]
\[
u,_{y_3}|_{y_3=0} = 0 \quad \text{on } \mathbb{R}^2,
\]
where \( u \) has a compact support with respect to \( y' \).

If \( \text{supp} \zeta \cap S_0 = \emptyset \) we can extend problem (2.6) by zero with respect to \( x' = (x_1, x_2) \). Then problem (2.6) can also be expressed as (2.8).

Let us consider the Fourier transform
\[
\hat{u}(\xi, x_3) = \int_{\mathbb{R}^2} e^{-i x' \cdot \xi} u(x', x_3) \, dx',
\]
where \( \xi = (\xi_1, \xi_2), x' \cdot \xi = x_1 \xi_1 + x_2 \xi_2 \). Applying this transformation to problem (2.8), where \( \tilde{h} \) is replaced by \( f \) and \( y \) by \( x \), we obtain, in the domain
\[ \Omega = \{(x', x_3) : x' \in \mathbb{R}^2, 0 \leq x_3 \leq a\}, \]
the problem
\[ -\frac{d^2 \hat{u}}{dx_3^2} + \xi^2 \hat{u} = \hat{f}, \]
\[ \hat{u}|_{x_3=a} = 0, \]
\[ \hat{u}_{x_3}|_{x_3=0} = 0. \]
(2.9)

We can solve this problem explicitly:

**Lemma 2.3.** Problem (2.9) has the solution
\[ \hat{u} = \beta_0 \cosh(|\xi|x_3) - \int_0^{x_3} \frac{\cosh(|\xi|\tau) \hat{f}(\xi, \tau)}{|\xi|} d\tau \sinh(|\xi|x_3) \]
\[ + \int_0^{x_3} \frac{\sinh(|\xi|\tau) \hat{f}(\xi, \tau)}{|\xi|} d\tau \cosh(|\xi|x_3) \]
where
\[ \beta_0 = \int_0^{a} \frac{\sinh(|\xi|\tau) \hat{f}(\xi, \tau)}{|\xi|} d\tau \frac{\sinh(|\xi|a)}{\cosh(|\xi|a)} - \int_0^{a} \frac{\sinh(|\xi|\tau) \hat{f}(\xi, \tau)}{|\xi|} d\tau. \]
(2.10)

**Proof.** General solutions of homogeneous equations (2.9) have the form
\[ \hat{u} = \alpha \sinh(|\xi|x_3) + \beta \cosh(|\xi|x_3). \]
(2.12)

We can find solutions to (2.9) by variation of constants. We have the following equations for \(\alpha(x_3), \beta(x_3)\):
\[ \frac{d\alpha}{dx_3} \sinh(|\xi|x_3) + \frac{d\beta}{dx_3} \cosh(|\xi|x_3) = 0, \]
\[ \frac{d\alpha}{dx_3} \cosh(|\xi|x_3) + \frac{d\beta}{dx_3} \sinh(|\xi|x_3) = \frac{-\hat{f}}{|\xi|}. \]

Solving this yields
\[ \frac{d\alpha}{dx_3} = -\frac{\cosh(|\xi|x_3)}{|\xi|} \hat{f}, \quad \frac{d\beta}{dx_3} = \frac{\sinh(|\xi|x_3)}{|\xi|} \hat{f}. \]

Hence, we get
\[ \alpha = -\int_0^{x_3} \frac{\cosh(|\xi|\tau) \hat{f}(\xi, \tau)}{|\xi|} d\tau, \quad \beta = \int_0^{x_3} \frac{\sinh(|\xi|\tau) \hat{f}(\xi, \tau)}{|\xi|} d\tau. \]

Using the formulas for \(\alpha, \beta\) in (2.12) we postulate the general solution of (2.9)
\[ \hat{u} = \alpha_0 \sinh(|\xi|x_3) + \beta_0 \cosh(|\xi|x_3) - \int_0^{x_3} \frac{\cosh(|\xi|\tau) \hat{f}(\xi, \tau)}{|\xi|} d\tau \sinh(|\xi|x_3) \]
\[ + \int_0^{x_3} \frac{\sinh(|\xi|\tau) \hat{f}(\xi, \tau)}{|\xi|} d\tau \cosh(|\xi|x_3). \]
(2.13)
The boundary conditions (2.9)$_{2,3}$ imply the following equations for $\alpha_0$ and $\beta_0$:

$$\alpha_0 \sinh(|\xi|a) + \beta_0 \cosh(|\xi|a) - \int_0^a \frac{\cosh(|\xi|\tau) \hat{f}(\xi, \tau)}{|\xi|} d\tau \sinh(|\xi|a)$$

$$+ \int_0^a \frac{\sinh(|\xi|\tau) \hat{f}(\xi, \tau)}{|\xi|} d\tau \cosh(|\xi|a) = 0,$$

$$-\alpha_0 |\xi| = 0.$$  

(2.14)

Solving (2.14) with respect to $\alpha_0, \beta_0$ implies formula (2.11) with $\alpha_0 = 0$, and inserting the result in the general solution formula (2.13) we obtain (2.10). This concludes the proof. ■

**Corollary 2.1.** The function $\hat{u}$ given by formula (2.10) is the unique solution to problem (2.9).

Now we obtain an estimate for the solution to problem (2.9), given by (2.10). We set $\hat{u}|_{x_3=0} = \hat{u}(0)$ and $\hat{u} - \hat{u}(0) \equiv \hat{u}$.

**Lemma 2.4.** Assume that

$$\int_{\mathbb{R}^2} \xi^4 d\xi \|\hat{u}(0)\|_{L^2_{2,\mu}(0,a)}^2 < \infty,$$

$$\int_{\mathbb{R}^2} \xi^2 d\xi \|\hat{u}(0)\|_{L^2_{2,\mu}(0,a)}^2 < \infty,$$

$$\int_{\mathbb{R}^2} \xi^2 d\xi \|\hat{u}(0)\|_{L^2_{2,\mu}(0,a)}^2 < \infty.$$

Then the solution $\hat{u}$ to the problem (2.9) satisfies

$$\int_{\mathbb{R}^2} \xi^2 d\xi \|\partial_{x_3} \hat{u}\|_{L^2_{2,\mu-2}(0,a)}^2 < \infty.$$

(2.15)

**Proof.** Multiplying (2.9)$_1$ by $\bar{u}x_3^{2\mu}$ and integrating on $(0, a)$ we get

$$\int_{(0,a)} (-\hat{u}, x_3 x_3 \bar{u} + \xi^2 |\bar{u}|^2) x_3^{2\mu} dx_3 = \int_{(0,a)} \hat{f} \bar{u} x_3^{2\mu} dx_3 - \xi^2 \int_{(0,a)} \hat{u}(0) \bar{u} x_3^{2\mu} dx_3.$$
Integrating by parts we obtain
\[
(2.17) \quad \int_{(0,a)} (|\ddot{u},x_3|^2 x_3^{2\mu} + \xi^2 |\dddot{u}|^2 x_3^{2\mu}) \, dx_3
\]
\[
= -2\mu \int_{(0,a)} \dot{u},x_3 \dddot{u} x_3^{2\mu-1} \, dx_3
\]
\[
+ \int_{(0,a)} \hat{f} \dddot{u} x_3^{2\mu} \, dx_3 - \xi^2 \int_{(0,a)} \hat{u}(0) \dddot{u} x_3^{2\mu} \, dx_3.
\]

By the Hölder and Young inequalities we estimate the first term on the r.h.s. of (2.17) by
\[
\frac{\varepsilon_1}{2} \int_{(0,a)} |\ddot{u},x_3|^2 x_3^{2\mu} \, dx_3 + \frac{4\mu^2}{2\varepsilon_1} \int_{(0,a)} |\dddot{u}|^2 x_3^{2\mu-2} \, dx_3,
\]
the second by
\[
\frac{\varepsilon_2}{2} \xi^2 \int_{(0,a)} |\ddot{u}|^2 x_3^{2\mu} \, dx_3 + \frac{1}{2\varepsilon_2} \xi^2 \int_{(0,a)} |\dddot{f}|^2 x_3^{2\mu} \, dx_3
\]
and the third by
\[
\frac{\varepsilon_3}{2} \xi^2 \int_{(0,a)} |\ddot{u}|^2 x_3^{2\mu} \, dx_3 + \frac{1}{2\varepsilon_3} \xi^2 \int_{(0,a)} |\dot{u}(0)|^2 x_3^{2\mu} \, dx_3.
\]

Setting \(\varepsilon_1 = 1, \varepsilon_2 = \varepsilon_3 = 1/2\), we obtain from (2.17) the inequality
\[
\frac{1}{2} \int_{(0,a)} (|\ddot{u},x_3|^2 + \xi^2 |\dddot{u}|^2) x_3^{2\mu} \, dx_3 \leq 2\mu^2 \int_{(0,a)} |\dddot{u}|^2 x_3^{2\mu-2} \, dx_3 + \frac{1}{\xi^2} \int_{(0,a)} |\dddot{f}|^2 x_3^{2\mu} \, dx_3
\]
\[
+ \xi^2 \int_{(0,a)} |\dot{u}(0)|^2 x_3^{2\mu} \, dx_3.
\]

We multiply this by \(2\xi^2\) and integrate with respect to \(\xi\) to get
\[
(2.18) \quad \int_{\mathbb{R}^2} \xi^2 \, d\xi \int_{(0,a)} (|\ddot{u},x_3|^2 + \xi^2 |\dddot{u}|^2) x_3^{2\mu} \, dx_3 \leq 4\mu^2 \int_{\mathbb{R}^2} \xi^2 \, d\xi \int_{(0,a)} |\dddot{u}|^2 x_3^{2\mu-2} \, dx_3
\]
\[
+ 2 \int_{\mathbb{R}^2} d\xi \int_{(0,a)} |\dddot{f}|^2 x_3^{2\mu} \, dx_3 + 2 \int_{\mathbb{R}^2} \xi^4 \, d\xi \int_{(0,a)} |\dot{u}(0)|^2 x_3^{2\mu} \, dx_3.
\]

This yields an estimate for the first integral in (2.16). To deal with the other terms, we slightly reformulate problem (2.9) to the form
\[
\frac{d^2 \ddot{u}}{dx_3^2} = \xi^2 \ddot{u} - \dddot{f},
\]
\[
\ddot{u}|_{x_3=a} = 0,
\]
\[
\ddot{u},x_3|_{x_3=0} = 0.
\]
We derive the bound
\[ ||\partial_{x_3}^2 \hat{u}||^2_{L^2(0,a)} \leq \xi^4 ||\hat{u}||^2_{L^2(0,a)} + \xi^4 ||\hat{u}(0)||^2_{L^2(0,a)} + ||\hat{f}||_{L^2(0,a)}.\]

Consequently, integrating with respect to \( \xi \), we derive
\[ \int_{\mathbb{R}^2} d\xi \ ||\partial_{x_3}^2 \hat{u}||^2_{L^2(0,a)} \leq \int_{\mathbb{R}^2} \xi^4 d\xi \ ||\hat{u}||^2_{L^2(0,a)} + \int_{\mathbb{R}^2} \xi^4 d\xi \ ||\hat{u}(0)||^2_{L^2(0,a)} + \int_{\mathbb{R}^2} d\xi \ ||\hat{f}||_{L^2(0,a)}. \tag{2.20} \]

On the other hand, by the Hardy inequality we have
\[ \int_{\mathbb{R}^2} d\xi \ ||\partial_{x_3}^2 \hat{u}||^2_{L^2(0,a)} \geq c \int_{\mathbb{R}^2} d\xi \ ||\partial_{x_3} \hat{u}||^2_{L^2(0,a)} \geq c \int_{\mathbb{R}^2} d\xi \ ||\hat{u}||^2_{L^2(0,a)} \tag{2.21} \]

and this with (2.20) gives an estimate for the second integral on the l.h.s. of (2.16). Therefore, combining (2.18), (2.20) and (2.21) we derive (2.16), which concludes the proof. \( \blacksquare \)

Next, we need to improve the lemma dealing with the term on the r.h.s. of (2.16) involving \( \hat{u} \). To this end, we introduce the sets:

\[ Q_1 = \{(\xi, x_3) \in \mathbb{R}^2 \times \mathbb{R}_+: |\xi|^{-1} x_3^{-1} \leq a_1\}, \]

\[ Q_2 = \{(\xi, x_3) \in \mathbb{R}^2 \times \mathbb{R}_+: |\xi|^{-1} x_3^{-1} \geq a_2\}, \]

\[ Q_3 = \{(\xi, x_3) \in \mathbb{R}^2 \times \mathbb{R}_+: a_1 \leq |\xi|^{-1} x_3^{-1} \leq a_2\}, \]

and prove the following result:

**Lemma 2.5.** Let assumptions (2.15)\(^{1,2}\) of Lemma 2.3 be satisfied. Then for a solution \( \hat{u} \) to problem (2.9) and \( \hat{u} = \hat{u} - \hat{u}(0) \),

\[ \int_{\mathbb{R}^2} \xi^2 d\xi \int_{(0,a)} |\hat{u}|^2 x_3^{2\mu-2} dx_3 \leq a_1^2 \int_{Q_1} \xi^4 |\hat{u}|^2 x_3^{2\mu} d\xi dx_3 \]

\[ + \frac{1}{a_2^2} \int_{Q_2} |\hat{u}|^2 x_3^{2\mu-4} d\xi dx_3 + c \int_{\mathbb{R}^2 \times (0,a)} \xi^4 |\hat{u}(0)|^2 x_3^{2\mu} d\xi dx_3 + c \int_{\mathbb{R}^2 \times (0,a)} |\hat{f}|^2 x_3^{2\mu} d\xi dx_3. \tag{2.22} \]

**Proof.** Let us consider the expression

\[ \int_{\mathbb{R}^2} \xi^2 d\xi \int_{(0,a)} |\hat{u}|^2 x_3^{2\mu-2} dx_3 = \sum_{i=1}^3 \int_{Q_i} \xi^2 |\hat{u}|^2 x_3^{2\mu-2} d\xi dx_3 = \sum_{i=1}^3 I_i, \]

where
\[
I_1 \leq a_1^2 \int_{Q_1} |\xi|^{4} |\tilde{u}|^{2} x_3^{2\mu} \, d\xi \, dx_3,
\]
\[
I_2 \leq \frac{1}{a_2^2} \int_{Q_2} |\tilde{u}|^{2} x_3^{2\mu-4} \, d\xi \, dx_3,
\]
\[
I_3 \leq a_2^{2-2\mu} \int_{Q_3} |\xi|^{4-2\mu} |\tilde{u}|^{2} \, d\xi \, dx_3.
\]

Let us express problem (2.19) in the form
\[
\partial_{x_3}^2 \tilde{u} = \xi^2 \tilde{u} - \hat{f} + \xi^2 \tilde{u}(0).
\]

Then from [ZS] and [Z1, (4.12)] we extract the inequality
\[
I_3 \leq c R_2 \int_{(0,a)} \xi^{4} |\hat{u}(0)|^{2} x_3^{2\mu} \, d\xi \, dx_3 + c \int_{(0,a)} |\hat{f}|^{2} x_3^{2\mu} \, d\xi \, dx_3.
\]

Collecting the estimates for \(I_i\) we obtain (2.22), and this concludes the proof.

**Corollary 2.2.** Consider the solution \(\hat{u}\) of (2.9). For sufficiently small \(a_1\) and sufficiently large \(a_2\), assuming (2.15), estimates (2.16) and (2.22) imply

\[
\int_{\mathbb{R}^2} d\xi \|\hat{u}\|_{H^\mu_{2,\mu}(0,a)}^2 + \int_{\mathbb{R}^2} \xi^2 \, d\xi \|\hat{u}\|_{H^\mu_{2,\mu}(0,a)}^2 + \int_{\mathbb{R}^2} \xi^4 \, d\xi \|\tilde{u}\|_{L^2,\mu(0,a)}^2
\]
\[
\leq c \int_{\mathbb{R}^2} d\xi \|\hat{f}\|_{L^2,\mu(0,a)}^2 + c \int_{\mathbb{R}^2} \xi^4 \, d\xi \|\hat{u}(0)\|_{L^2,\mu(0,a)}^2.
\]

Next, we want to estimate the last term in (2.23). It takes the form

\[
\int_{\mathbb{R}^2 \times (0,a)} |\partial_{x'}^2 u(0)|^2 |x_3|^{2\mu} \, dx' \, dx_3 \leq c \int_{\mathbb{R}^2} |\partial_{x'}^2 u(0)|^2 \, dx',
\]

where we used the fact that the support of \(u\) with respect to \(x_3\) is compact.

Now we postulate that the function \(f\) in (1.1) can be written as

\[
f = \alpha x_3 \quad \text{where} \quad \alpha = \tilde{a} \eta(x_3), \quad \tilde{a}|_{x_3=0} = \theta
\]

and \(\eta(x_3)\) is a smooth non-increasing function with compact support, \(0 \leq \eta \leq 1\) and \(\eta(x_3) = 1\) in a neighbourhood of \(\{x_3 = 0\}\). We extend the function \(\theta\) onto \(\mathbb{R}^2\) in \(H^1(\mathbb{R}^2)\) norm using the Hestenes–Whitney theorem in such a way that

\[
\tilde{\theta}|_{S_*} = \theta, \quad \|	ilde{\theta}\|_{H^k(\mathbb{R}^2)} \leq c \|	heta\|_{H^k(S_*)}.
\]
Then we solve the problem
\[
\Delta u = \alpha_{x_3}, \quad x_3 > 0, \\
u_{x_3}|_{x_3=0} = 0.
\]

**Lemma 2.6.** Assume that \( u \) is a solution to (2.26), where \( \alpha \) is described by (1.2). Assume that \( \theta \in H^1(S_*). \) Then
\[
\|\partial_x^2 u(0)\|_{L^2(\mathbb{R}^2)} \leq c\|\theta\|_{H^1(S_*)}.
\]

**Proof.** Using the Neumann function, any solution to (2.26) can be expressed by
\[
u(x) = \int_{\mathbb{R}^3_+} \left( \frac{1}{|x - y|} + \frac{1}{|x - \bar{y}|} \right) \alpha_y dy
\]
where \( \bar{y} = (y_1, y_2, -y_3), \mathbb{R}^3_+ = \{ x \in \mathbb{R}^3 : x_3 > 0 \} \).

Integrating by parts we obtain
\[
u(x) = \int_{\mathbb{R}^3_+} \partial_y \left[ \left( \frac{1}{|x - y|} + \frac{1}{|x - \bar{y}|} \right) \alpha \right] dy - \int_{\mathbb{R}^3_+} \partial_y \left( \frac{1}{|x - y|} + \frac{1}{|x - \bar{y}|} \right) \alpha dy
\]
\[
= \int_{\mathbb{R}^2} \left( \frac{1}{|x - y|} + \frac{1}{|x - \bar{y}|} \right) |y_3 = 0\tilde{\theta} dy' - \int_{\mathbb{R}^3_+} \left( \frac{x_3 - y_3}{|x - y|^3} - \frac{x_3 + y_3}{|x - \bar{y}|^3} \right) \tilde{\alpha} dy
\]
\[
= \int_{\mathbb{R}^2} \frac{2}{|x' - y'|^2 + x_3^2} \tilde{\theta} dy' - \int_{\mathbb{R}^3_+} \left( \frac{x_3 - y_3}{|x - y|^3} - \frac{x_3 + y_3}{|x - \bar{y}|^3} \right) \tilde{\alpha} dy.
\]

In view of this formula, the r.h.s. of (2.24) assumes the form
\[
\int_{\mathbb{R}^2} |\partial_x^2 u(0)|^2 dx'
\]
\[
\leq 4 \int_{\mathbb{R}^2} dx' |\partial_x^2 \tilde{\theta} dy'|^2 + 4 \int_{\mathbb{R}^2} dx' \left| \int_{\mathbb{R}^3_+} \frac{y_3}{\sqrt{|x' - y'|^2 + y_3^2}} \tilde{\theta} dy_3 \right|^2
\]
\[
= 4 \int_{\mathbb{R}^2} dx' \left| \int_{\mathbb{R}^2} \partial_x^2 \frac{1}{|x' - y'|} \tilde{\theta} dy' \right|^2 + 4 \int_{\mathbb{R}^2} dx' \left| \int_{\mathbb{R}^3_+} \frac{y_3}{\sqrt{|x' - y'|^2 + y_3^2}} \tilde{\theta} dy_3 \right|^2
\]
\[
\leq c\|\tilde{\theta}\|_{H^1(\mathbb{R}^2)} \leq c\|\theta\|_{H^1(S_*)}.
\]

This concludes the proof. ■

Using (2.23), (2.27) and applying a partition of unity for \( \Omega \) we deduce the following result:

**Lemma 2.7.** For problem (1.1), where \( f \in L_{2, \mu}(\mathbb{R}^2 \times (0, a)), f \) is expressed by (1.2) and \( \theta \in H^1(\mathbb{R}^2), \) the solution \( u \) satisfies
\[
\|u - u(0)\|_{H^2(\mathbb{R}^2 \times (0, a))} \leq c(\|f\|_{L_{2, \mu}(\mathbb{R}^2 \times (0, a))} + \|\theta\|_{H^1(\mathbb{R}^2)}),
\]
where \( u(0) \) denotes \( u|_{x_3=0}. \)
Proof of Theorem 1. We apply the regularizer technique for elliptic equations (see [S], [LSU]). Let us define two collections of open subsets \( \{ \omega^{(k)} \} \) and \( \{ \Omega^{(k)} \} \), \( k \in \mathcal{M} \cup \mathcal{N} \), such that \( \overline{\omega^{(k)}} \subset \Omega^{(k)} \), \( \bigcup_k \omega^{(k)} = \bigcup_k \Omega^{(k)} = \Omega \cap P \), where \( P \) is any plane perpendicular to the \( x_3 \) axis. We assume that \( \omega^{(k)} \cap S = \emptyset \) for \( k \in \mathcal{M} \) and \( \omega^{(k)} \cap S \neq \emptyset \) for \( k \in \mathcal{N} \). Then \( \bigcup_k \omega^{(k)} \times (0, a) = \bigcup_k \Omega^{(k)} \times (0, a) = \Omega \).

Let \( \zeta^{(k)}(x') \) be a smooth function such that \( 0 \leq \zeta^{(k)}(x') \leq 1 \), \( \zeta^{(k)}(x') = 1 \) for \( x' \in \omega^{(k)} \), \( \text{supp} \zeta^{(k)} \subset \Omega^{(k)} \) and \( |D_x \zeta^{(k)}(x')| \leq (c/|\lambda|)^\nu \), where \( \lambda \) is the diameter of \( \Omega \). Then \( 1 \leq \sum_k \zeta^{(k)}(x') \leq N_0 \). Introducing the function \( \eta^{(k)}(x') = \zeta^{(k)}(x')/\sum_l (\zeta^{(l)}(x'))^2 \) we have \( \text{supp} \eta^{(k)}(x) \subset \Omega^{(k)} \), \( \sum_k \eta^{(k)}(x') \zeta^{(k)}(x') = 1 \), \( |D_x \eta^{(k)}(x')| \leq c/|\lambda|^\nu \).

By \( \xi^{(k)} \), we denote a fixed internal point of \( \omega^{(k)} \) and \( \Omega^{(k)} \) for \( k \in \mathcal{M} \) and a point of \( \overline{\omega^{(k)}} \cap S \) and \( \overline{\Omega^{(k)}} \cap S \) for \( k \in \mathcal{N} \). Let us introduce a local coordinate system \( y = (y_1, y_2) \) with centre at \( \xi^{(k)} \), \( k \in \mathcal{N} \). We assume that \( y_2 = F(y_1) \) describes the part \( S^{(k)} = S \cap \overline{\Omega^{(k)}} \) of the boundary. Let us introduce new coordinates by
\[
(2.29) \quad z_1 = y_1, \quad z_2 = y_2 - F(y_1).
\]
Let \( \Psi_k \) denote the transformation
\[
\Omega^{(k)} \ni y' \mapsto \Psi_k(y') = z' \in \hat{\Omega}^{(k)}, \quad \omega^{(k)} \ni y' \mapsto \Phi_k(y') = z' \in \hat{\omega}^{(k)},
\]
where \( \hat{\omega}^{(k)} \) and \( \hat{\Omega}^{(k)} \) are described by the relations
\[
|y_1| < \lambda, \quad 0 < y_2 - F(y_1) < \lambda, \\
|y_1| < 2\lambda, \quad 0 < y_2 - F(y_1) < 2\lambda,
\]
respectively. Let \( y = Y_k(x) \) be a transformation from global coordinates \( x \) to local coordinates with origin at \( \xi^{(k)} \) which is a composition of a translation and a rotation. We denote \( \Phi_k = \Psi_k \cdot Y_k \). We set
\[
(2.30) \quad \hat{u}^{(k)}(z) = u(\Phi_k^{-1}(z)), \quad \tilde{u}^{(k)}(z) = \hat{u}^{(k)}(z)\zeta^{(k)}(z).
\]
For \( k \in \mathcal{M} \) problem (1.1) translates into the equation
\[
(2.31) \quad -\nabla_x^2 \tilde{u}^{(k)} = \tilde{f}^{(k)} \quad \text{in } \mathbb{R}^2 \times (0, a)
\]
and for \( k \in \mathcal{N} \) into the equation
\[
(2.32) \quad -\nabla_x^2 \tilde{u}^{(k)} = \tilde{f}^{(k)} \quad \text{in } \mathbb{R}^2 \times (0, a),
\]
which is appropriately extended onto \( z_2 < 0 \). Let \( R^{(k)} \) be the operator which solves problems (2.31) and (2.32), respectively. Then we define the operator
\[
Rf = \sum_{k \in \mathcal{M} \cup \mathcal{N}} \eta^{(k)} u^{(k)}
\]
where
\[ u^{(k)}(x) = \begin{cases} R^{(k)} \zeta^{(k)}, & k \in \mathcal{M}, \\ \Phi_k^{-1}(\Phi_k^{-1} R^{(k)}(\Phi_k \zeta^{(k)} f)), & k \in \mathcal{N}. \end{cases} \]

The solvability of problems (2.31) and (2.32) is settled by Lemma 2.4.

Let us introduce spaces \( H = L_{2,\mu}, V = H^2_\mu \). Let \( \mathcal{L} = -\Delta \). Then we examine operators \( T, W, v \), where

\[
(2.33) \quad \mathcal{L} R f = (I + T) f, \quad R \mathcal{L} v = (I + W) v.
\]

We calculate
\[
T f = \sum_{k \in \mathcal{M} \cup \mathcal{N}} (\mathcal{L} \eta^{(k)} u^{(k)} - \eta^{(k)} \mathcal{L} u^{(k)})
+ \sum_{k \in \mathcal{N}} \eta^{(k)} \Phi_k^{-1} [\mathcal{L}(\partial_z - \nabla F \partial_{z_2}) - \mathcal{L}(\partial_z)] R^{(k)}(\Phi_k \zeta^{(k)} f)
= T_1 + T_2
\]

and
\[
T_1 f = \sum_{k \in \mathcal{M}} \eta^{(k)} \Phi_k^{-1} [-\partial_{z_1} F \partial_{z_1} F \partial_{z_2}^2 - \partial_{z_1} F \partial_{z_1} \partial_{z_2}
- \partial_{z_1} F \partial_{z_2} \partial_{z_1} - \partial_{z_1} F \partial_{z_1} \partial_{z_2} \partial_{z_2}] R^{(k)}(\Phi_k \zeta^{(k)} f),
\]
\[
T_2 f = \sum_{k \in \mathcal{M} \cup \mathcal{N}} (\mathcal{L} \eta^{(k)} u^{(k)} - \eta^{(k)} \mathcal{L} u^{(k)})
+ \sum_{k \in \mathcal{N}} \eta^{(k)} \Phi_k^{-1} (-\partial_{z_1}^2 F \partial_{z_2} R^{(k)}(\Phi_k \zeta^{(k)} f)).
\]

Since \( |\partial_{z_1} F| \leq c\lambda \), we have
\[
||T_1 f||_H \leq c\lambda ||f||_H
\]
so \( ||T_1||_H \leq 1 \) for sufficiently small \( \lambda \). On the other hand \( T_2 \) is completely continuous. Similarly, \( W = W_1 + W_2 \), where

\[
W_1 u = \sum_{k \in \mathcal{N}} \eta^{(k)} \Phi_k^{-1} R^{(k)} \left[ -\partial_{z_1} F \partial_{z_1} F \partial_{z_2}^2 - 2 \partial_{z_1} F \partial_{z_1} \partial_{z_2} F \partial_{z_2} (\partial_{z_1} F) \partial_{z_2} \right] \Phi_k \zeta^{(k)} u
\]

and

\[
W_2 u = \sum_{k \in \mathcal{M}} \eta^{(k)} R^{(k)} (\zeta^{(k)} \mathcal{L} - \mathcal{L} \zeta^{(k)}) u + \sum_{k \in \mathcal{N}} \eta^{(k)} \Phi_k^{-1} R^{(k)} \left[ \Phi_k (\zeta^{(k)} \mathcal{L} - \mathcal{L} \zeta^{(k)}) u \right]
- \sum_{k \in \mathcal{N}} \eta^{(k)} \Phi_k^{-1} R^{(k)} (F_{z_1 z_1} \partial_{z_2}) \Phi_k \zeta^{(k)} u.
\]

By the same arguments as above we have \( ||W_1||_V < 1 \) and \( W_2 \) is compact and continuous.
We write (2.33) in the form
\[(I + W_1)v = Rf + W_2v.\]
Here \(R\) is a bounded operator from \(H^2_\mu\) in \(L_{2,\mu}\), \(W_2\) is a completely continuous operator with
\[\|W_2\|_{H^2_\mu(\Omega)} \leq \varepsilon \|v\|_{H^2_\mu(\Omega)} + c(1/\varepsilon)\|v\|_{L_2(\Omega)}\]
and \(W_1\) is the operator with norm less than one. Then from (2.34) we obtain
\[(2.35)\]
\[\|v\|_{H^2_\mu(\Omega)} \leq c\|\mathcal{L}v\|_{H^2_\mu(\Omega)} + c\|v\|_{L_2(\Omega)},\]
where by using the Green function and the Hardy inequality we find that
\[\|v\|_{L_2(\Omega)} \leq c\|\alpha\|_{L_2(\Omega)} \leq c\|f\|_{L_{2,\mu}(\Omega)}.\]
We have
\[\|v\|_{H^2_\mu(\Omega)} \leq c\|\mathcal{L}v\|_{L_{2,\mu}(\Omega)}\]
Hence there exists an inverse operator to \(\mathcal{L}\) so we have existence in \(H^2_\mu(\Omega)\).

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References

Poisson equation in $L^2$ weighted spaces

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