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SHOUQUAN MA and GUIXIANG XU (Beijing)

# GLOBAL WELL-POSEDNESS AND BLOW UP FOR THE NONLINEAR FRACTIONAL BEAM EQUATIONS

*Abstract.* We establish the Strichartz estimates for the linear fractional beam equations in Besov spaces. Using these estimates, we obtain global well-posedness for the subcritical and critical defocusing fractional beam equations. Of course, we need to assume small initial data for the critical case. In addition, by the convexity method, we show that blow up occurs for the focusing fractional beam equations with negative energy.

**1. Introduction.** The present paper is concerned with the Cauchy problem for the nolinear fractional beam equations

$$(1.1)\qquad \begin{cases} u_{tt} + (-\triangle)^s u + u = \lambda |u|^{p-1}u, & t\in\mathbb{R},\ x\in\mathbb{R}^n,\\ u(0)=\phi(x)\in H^s, & u_t(0)=\psi(x)\in L^2,\end{cases}$$

where $1\le s\le 2$, $\lambda\in\mathbb{R}\backslash\{0\}$, $(-\triangle)^s u = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u)$, $u$ is a real-valued function with respect to $t$ and $x$, and $\mathcal{F}$ denotes the Fourier transform. The equation (1.1) is said to be *defocusing* when $\lambda<0$, and *focusing* when $\lambda>0$. Let $(\phi,\psi)\in H^s\times L^2$ and $E$ be the energy associated with the nonlinear equation (1.1), i.e.

$$(1.2)\qquad E(u,u_t)=\int_{\mathbb{R}^n}\left(\frac12 u^2+\frac12|D^s u|^2+\frac12 u_t^2-\frac{\lambda}{p+1}|u|^{p+1}\right)dx,$$

where $D^s=(-\triangle)^{s/2}$. When $s=2$, (1.1) is the classical beam equation, when $s=1$, (1.1) is the Klein–Gordon equation. The global well-posedness and scattering for the Klein–Gordon equation and the beam equation have been extensively studied in [3, 4, 7, 9, 12, 13, 15–19]. These two kinds of equations are different from each other on some facets, for instance, the

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Klein–Gordon equation enjoys finite speed propagation while beam equation only enjoys almost finite speed propagation ([18]). In this paper, we prove global well-posedness for the subcritical and critical defocusing fractional beam equations, and blow up for the focusing fractional beam equations with negative energy.

The paper is organized as follows. In Section 2, we give notation and state the main results. In Section 3, we establish the Strichartz estimates for the linear fractional beam equations. In Section 4, we prove the main results.

**2. Notation and statement of the main results.** Throughout this paper, $C$ and $c$ denote positive universal constants, which can vary according to the context. $A \lesssim B$ means that $A \leq CB$, and $A \sim B$ stands for $A \lesssim B$ and $B \lesssim A$. We denote by $p'$ the Hölder dual exponent of $p \in [1, \infty]$, i.e. $1/p + 1/p' = 1$. For convenience, we write for Lebesgue spaces $L^p := L^p(\mathbb{R}^n)$, Bessel potential spaces $H^{s,p} = H^{s,p}(\mathbb{R}^n) = (I - \triangle)^{-s/2} L^p(\mathbb{R}^n)$, $H^s = H^{s,2}$. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing functions, and $\mathcal{S}'(\mathbb{R}^n)$ be the dual space of $\mathcal{S}(\mathbb{R}^n)$. Given $f \in \mathcal{S}(\mathbb{R}^n)$, its Fourier transform $\mathcal{F}f = \widehat{f}$ is defined by

$$\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

Let us choose a nonnegative radially symmetric function $\chi \in C_c^\infty(\mathbb{R}^n)$ supported in the ball $\{\xi \in \mathbb{R}^n;\ |\xi| \leq 4/3\}$ which equals 1 on the ball $\{\xi \in \mathbb{R}^n;\ |\xi| \leq 3/4\}$. Then the function $\eta(\xi) = \chi(\xi/2) - \chi(\xi)$ is supported in $\mathcal{C} = \{\xi \in \mathbb{R}^n;\ 3/4 \leq |\xi| \leq 8/3\}$ and satisfies

$$\chi(\xi) + \sum_{j\geq 0} \eta(2^{-j}\xi) = 1, \qquad \xi \in \mathbb{R}^n,$$
$$\sum_{j\in\mathbb{Z}} \eta(2^{-j}\xi) = 1, \qquad \xi \in \mathbb{R}^n \setminus \{0\}.$$

We define the frequency localization operators $\varDelta_j$ and $S_j$ as follows:

$$\varDelta_j f = \mathcal{F}^{-1}(\eta(2^{-j}\xi)\mathcal{F}f), \qquad S_j f = \sum_{k\leq j-1} \varDelta_k f.$$

Informally, $\varDelta_j = S_{j+1} - S_j$ is a frequency projection to the annulus $\{\xi \in \mathbb{R}^n;\ |\xi| \approx 2^j\}$, while $S_j$ is a frequency projection to the ball $\{\xi \in \mathbb{R}^n;\ |\xi| \leq 2^j\}$. One can check that

$$\varDelta_j \varDelta_k f \equiv 0 \quad \text{if } |j-k| \geq 2. \tag{2.1}$$

Now we can give the definition of Besov spaces.

DEFINITION 2.1. Let $\sigma \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. The *inhomogeneous Besov space* $B^{\sigma}_{p,q}$ is defined by

$$B^{\sigma}_{p,q} = \{f \in \mathcal{S}'(\mathbb{R}^n);\ \|f\|_{B^{\sigma}_{p,q}} < \infty\}. \tag{2.2}$$

Here

$$\|f\|_{B^{\sigma}_{p,q}} = \begin{cases} \Big(\sum_{j\geq 0} 2^{j\sigma q}\|\Delta_j f\|_p^q\Big)^{1/q} + \|S_0(f)\|_p & \text{for } q < \infty, \\ \sup_{j\geq 0} 2^{j\sigma}\|\Delta_j f\|_p + \|S_0(f)\|_p & \text{for } q = \infty. \end{cases}$$

We refer to [1, 14, 22] for more details.

DEFINITION 2.2. A pair $(q, r)$ is said to be *F-admissible*, written $(q, r) \in \tilde{\Lambda}$, if $2 \leq q, r \leq \infty$, $(q, r, n) \neq (2, \infty, 2)$, and

$$\frac{2}{q} \leq \delta(r) := n\left(\frac{1}{2} - \frac{1}{r}\right). \tag{2.3}$$

In particular, if equality holds in (2.3), we say that $(q, r)$ is *sharp F-admissible* and we write $(q, r) \in \Lambda$.

Our main results are the following theorems.

THEOREM 2.1. *Let $\lambda < 0$, $1 \leq s \leq 2$, $1 < p \leq \frac{n+2s}{n-2s}$, and $(\phi, \psi) \in H^s \times L^2$. Then we have:*

(1) *When $1 < p \leq \frac{n}{n-2s}$ and $n > 2s$, there exists a unique global solution $u$ to (1.1) such that $u \in \mathcal{C}(\mathbb{R}, H^s) \cap \mathcal{C}^1(\mathbb{R}, L^2)$ and $u$ depends continuously on the initial data.*
(2) *When $\frac{n}{n-2s} < p < \frac{n+2s}{n-2s}$ and $n > 2s$, there exists a unique global solution $u$ to (1.1) such that $u \in \mathcal{C}(\mathbb{R}, H^s) \cap \mathcal{C}^1(\mathbb{R}, L^2) \cap L^{q_1}_{\mathrm{loc}}(\mathbb{R}, B^{s-\beta(r)}_{r,2})$, where*

$$(q_1, r, \beta(r)) = \left(\frac{2(ps+2)}{(n-2s)p-n}, \frac{2(ps+2)n}{4n+(4s+(s-2)n)p}, \frac{2-s}{2}\delta(r)\right). \tag{2.4}$$

*Moreover, $u$ depends continuously on the initial data.*
(3) *When $n < 2s$ and $p > 1$, there exists a unique global solution $u$ to (1.1) such that $u \in \mathcal{C}(\mathbb{R}, H^s) \cap \mathcal{C}^1(\mathbb{R}, L^2)$ and $u$ depends continuously on the initial data.*
(4) *When $p = \frac{n+2s}{n-2s}$ and $n > 2s$, if $\|(\phi, \psi)\|_{H^s\times L^2}$ is sufficiently small, then there exists a unique global solution $u$ to (1.1) such that $u \in \mathcal{C}(\mathbb{R}, H^s) \cap \mathcal{C}^1(\mathbb{R}, L^2) \cap L^{q_1}(\mathbb{R}, B^{s-\beta(r)}_{r,2})$, where $(q_1, r, \beta(r))$ is as in (2.4). Moreover, $u$ depends continuously on the initial data.*

THEOREM 2.2. *Let $\lambda > 0$ and $1 < p \le \frac{n+2s}{n-2s}$. Assume that $(\phi,\psi) \in H^s \times L^2$ and satisfies*

$$E(\phi,\psi) = \int_{\mathbb{R}^n} \left(\frac{1}{2}\phi^2 + \frac{1}{2}\psi^2 + \frac{1}{2}|D^s\phi|^2 - \frac{\lambda}{p+1}|\phi|^{p+1}\right) dx < 0.$$

*Then the solution to* (1.1) *blows up in finite time.*

**3. The Strichartz estimates.** In this section, we will give the Strichartz estimates for the linear fractional beam equation

$$\begin{cases} u_{tt} + (-\triangle)^s u + u = f(t,x), & (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ u(x,0) = \phi(x), & u_t(x,0) = \psi(x). \end{cases} \tag{3.1}$$

The solution $u(x,t)$ of (3.1) is given by

$$u(x,t) = \dot{K}(t)\phi(x) + K(t)\psi(x) + \int_0^t K(t-\tau)f(x,\tau)\,d\tau,$$

where $K(t) = \omega^{-1}\sin(\omega t)$, $\omega = \sqrt{1+(-\triangle)^s}$. Let us define

$$W(x,t) := \dot{K}(t)\phi(x) + K(t)\psi(x),$$

$$(\mathcal{G}f)(x,t) := \int_0^t K(t-\tau)f(x,\tau)\,d\tau.$$

We denote by $J_m(r)$ the Bessel function

$$J_m(r) = \frac{(r/2)^m}{\Gamma(m+1/2)\pi^{1/2}} \int_{-1}^{1} e^{irt}(1-t^2)^{m-1/2}\,dt, \qquad m > -1/2.$$

Some properties of $J_m(r)$ are listed in the following lemma.

LEMMA 3.1 ([10, 20, 21]). *Let $0 < r < \infty$ and $m > -1/2$. Then:*

(1) $J_m(r) \le Cr^m$;
(2) $\frac{d}{dr}(r^{-m}J_m(r)) = -r^{-m}J_{m+1}(r)$;
(3) $J_m(r) \le Cr^{-1/2}$.

Let $I = \mathbb{R}$ or $I \subset \mathbb{R}$ be an interval with $0 \in I$. Then we have the following *Strichartz estimates.*

PROPOSITION 3.1. *Assume that $\sigma_1, \sigma_2, \mu \in \mathbb{R}$, $2 \le q_1, q_2, r_1, r_2 \le \infty$ and the following conditions are satisfied:*

$$\begin{gathered} 0 \le \frac{2}{q_i} \le \min(\delta(r_i), 1), \qquad (q_i, r_i, n) \ne (2, \infty, 2), \qquad i = 1,2, \\ \sigma_1 + \delta(r_1) - \frac{s}{q_1} = \mu, \qquad \sigma_2 + \delta(r_2) - \frac{s}{q_2} = s - \mu. \end{gathered} \tag{3.2}$$

*Let $Y^\mu = H^\mu \times H^{\mu-s}$. Then*

(1) $(W, W_t) \in (\mathcal{C}(I, Y^\mu) \cap L^{q_1}(I, B^{\sigma_1}_{r_1,2})) \times L^{q_1}(I, B^{\sigma_1-\mu}_{r_1,2})$ *and*

$$\|W\|_{L^{q_1}(I,B^{\sigma_1}_{r_1,2})} + \|W_t\|_{L^{q_1}(I,B^{\sigma_1-s}_{r_1,2})} \le C\|(\phi,\psi)\|_{Y^\mu} \tag{3.3}$$

*for* $(\phi, \psi) \in Y^\mu$,

(2) $\mathcal{G}f \in \mathcal{C}(I, H^\mu) \cap L^{q_1}(I, B^{\sigma_1}_{r_1,2})$ *and*

$$\|\mathcal{G}f\|_{L^{q_1}(I,B^{\sigma_1}_{r_1,2})} \le C\|f\|_{L^{q_2'}(I,B^{-\sigma_2}_{r_2',2})} \tag{3.4}$$

*for* $f \in L^{q_2'}(I, B^{-\sigma_2}_{r_2',2})$,

(3) *we have*

$$\|u\|_{L^{q_1}(I,B^{\sigma_1}_{r_1,2})} + \|u_t\|_{L^{q_1}(I,B^{\sigma_1-s}_{r_1,2})} \le C\|(\phi,\psi)\|_{Y^\mu} + \|f\|_{L^{q_2'}(I,B^{-\sigma_2}_{r_2',2})} \tag{3.5}$$

*for* $(\phi, \psi) \in H^\mu$, $f \in L^{q_2'}(I, B^{-\sigma_2}_{r_2',2})$.

*Proof.* The argument follows from [14, Chapter 5]. Since $(I-\triangle)^{\mu/2}$ is an isomorphism from $B^\sigma_{r,2}$ into $B^{\sigma-\mu}_{r,2}$, it suffices to prove (3.3)–(3.5) for $\mu = 0$. Note that

$$\widehat{K(t)f}(t,\xi) = \frac{e^{it\omega(\xi)} - e^{-it\omega(\xi)}}{2i\omega(\xi)}\widehat{f}(t,\xi),$$
$$\widehat{\dot{K}(t)f}(t,\xi) = \frac{e^{it\omega(\xi)} + e^{-it\omega(\xi)}}{2}\widehat{f}(t,\xi),$$

where $\omega(\xi) = (1+|\xi|^{2s})^{1/2}$. Now we decompose the solution to (3.1) into

$$u = v_+ + v_- + \frac{w_+ - w_-}{2},$$

where

$$v_\pm(t,x) = \int_{\mathbb{R}^n} e^{i(\pm t\sqrt{1+|\xi|^{2s}}+x\cdot\xi)}\hat{\phi}_\pm(\xi)\,d\xi, \qquad \hat{\phi}_\pm(\xi) = \frac{1}{2}\hat{\phi}(\xi) \pm \frac{\hat{\psi}(\xi)}{2i\omega(\xi)},$$
$$w_\pm(t,x) = \int_0^t\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{\pm i(t-\tau)\omega(\xi)}\frac{\hat{f}(\xi,\tau)}{i\omega(\xi)}\,d\xi\,d\tau.$$

Since the Fourier restriction estimates on the hypersurfaces $\{(\tau,\xi) \in \mathbb{R}^{n+1};\ \tau = \omega(\xi)\}$ and $\{(\tau,\xi) \in \mathbb{R}^{n+1};\ \tau = -\omega(\xi)\}$ are the same, it suffices to show that under the conditions

$$\begin{aligned} &0 \le \frac{2}{q_i} \le \min(\delta(r_i), 1), \qquad (q_i, r_i, n) \ne (2, \infty, 2), \qquad i = 1, 2, \\ &\sigma_i + \delta(r_i) - \frac{s}{q_i} = 0, \qquad i = 1, 2, \end{aligned} \tag{3.6}$$

we have

$$\|U(t)\phi\|_{L^{q_1}(I,B^{\sigma_1}_{r_1,2})} \le C\|\phi\|_2, \tag{3.7}$$

$$\Big\|\int_{\mathbb{R}} U(t-\tau)f(x,\tau)\,d\tau\Big\|_{L^{q_1}(I,B^{\sigma_1}_{r_1,2})} \le C\|f\|_{L^{q_2'}(I,B^{-\sigma_2}_{r_2',2})}, \tag{3.8}$$

$$\Big\|\int_0^t U(t-\tau)f(x,\tau)\,d\tau\Big\|_{L^{q_1}(I,B^{\sigma_1}_{r_1,2})} \le C\|f\|_{L^{q_2'}(I,B^{-\sigma_2}_{r_2',2})}, \tag{3.9}$$

where $U(t)\phi = \mathcal{F}^{-1}e^{it\sqrt{1+|\xi|^{2s}}} * \phi$. Our proof consists of five steps:

(1) By Littlewood–Paley decomposition and scaling analysis, we reduce the homogeneous estimate (3.7) to the estimate (3.10) of the localized operators $T_k$.
(2) By Littlewood–Paley decomposition and scaling analysis, we reduce the inhomogeneous estimates (3.8) and (3.9) to the estimates (3.13)–(3.16) of the corresponding localized operators.
(3) By using the $TT^*$ method and the Christ–Kiselev lemma, the estimates (3.10), (3.15) and (3.16) can be deduced from (3.17), which is equivalent to (3.13) and (3.14).
(4) We obtain (3.17) for the diagonal case $q_1 = q_2$, $r_1 = r_2$ from the Bessel function expression of the Fourier transform of a radial function, the method of stationary phase, Young's inequality and Hardy–Littlewood–Sobolev's inequality.
(5) By bilinear interpolation and some special estimates, we obtain the non-diagonal estimate and the endpoint case.

Step 1. Let $\widetilde{\eta} \in \mathcal{C}_c^\infty$ equal 1 on a neighborhood of $\operatorname{supp}\eta$, and $\widetilde{\chi} \in \mathcal{C}_c^\infty$ equal 1 on a neighborhood of $\operatorname{supp}\chi$. We denote by $T_1, T_2$ the truncated operators

$$T_1\phi(t,x) = \int_{\mathbb{R}^n} e^{i(t\sqrt{2^{-2js}+|\xi|^{2s}}+x\cdot\xi)}\widetilde{\eta}(\xi)\hat{\phi}(\xi)\,d\xi,$$

$$T_2\phi(t,x) = \int_{\mathbb{R}^n} e^{i(t\sqrt{1+|\xi|^{2s}}+x\cdot\xi)}\widetilde{\chi}(\xi)\hat{\phi}(\xi)\,d\xi.$$

Therefore, in order to prove (3.7), it suffices to show

$$\|T_k\phi\|_{L^{q_1}(I,L^{r_1})} \le C\|\phi\|_{L^2} \quad (k=1,2). \tag{3.10}$$

In fact, by Littlewood–Paley decomposition, we have

$$\phi(x) = S_0\widetilde{S_0}\phi + \sum_{j\ge 0}\widetilde{\Delta}_j\Delta_j\phi, \qquad \widetilde{\Delta}_j = \Delta_{j-1}+\Delta_j+\Delta_{j+1},$$

where $\widehat{\widetilde{S_0}f}(\xi)=\widetilde{\chi}(\xi)\hat{f}(\xi)$, $\widetilde{\chi}(\xi)=\chi(\xi/2)$ and $\widehat{\widetilde{\Delta_0}f}(\xi)=\widetilde{\eta}(\xi)\hat{f}(\xi)$. We denote

$$U_1(t)\phi := \int_{\mathbb{R}^n} e^{i(t\sqrt{2^{-2js}+|\xi|^{2s}}+x\cdot\xi)}\hat{\phi}(\xi)\,d\xi.$$

By scaling, we have

$$\begin{aligned} U(t)\Delta_j\phi &= \int_{\mathbb{R}^n} e^{i(x\cdot\xi+t\sqrt{1+|\xi|^{2s}})}\widetilde{\eta}(2^{-j}\xi)\widehat{\Delta_j\phi}(\xi)\,d\xi \\ &= 2^{jn}\int_{\mathbb{R}^n} e^{i2^jx\cdot\xi}e^{i2^{js}t\sqrt{2^{-2js}+|\xi|^{2s}}}\widetilde{\eta}(\xi)\widehat{\Delta_j\phi}(2^j\xi)\,d\xi \\ &= [U_1(2^{js}t)\widetilde{\Delta}_0(\Delta_j\phi)(2^{-j}\cdot)](2^jx) \end{aligned}$$

for $j\ge 0$. This together with (3.6) and (3.10) implies that

$$\begin{aligned} \|U(t)\Delta_j\phi\|_{L^{q_1}(I,L^{r_1})} &\lesssim 2^{-(s/q_1+n/r_1)j}\|\Delta_j\phi(2^{-j}\cdot)\|_{L^2} \\ &\lesssim 2^{-\sigma_1 j}\|\Delta_j\phi\|_{L^2}. \end{aligned} \tag{3.11}$$

In addition, by (3.10), we have

$$\big\|U(t)S_0\phi\big\|_{L^{q_1}(I,L^{r_1})} = \big\|U(t)\widetilde{S_0}S_0\phi\big\|_{L^{q_1}(I,L^{r_1})} \lesssim \|S_0\phi\|_{L^2}. \tag{3.12}$$

Therefore, by Minkowski's inequality, (3.6) and (3.11), we get

$$\begin{aligned} &\|U(t)\phi\|_{L^{q_1}(I,B^{\sigma_1}_{r_1,2})} \\ &\quad\le \|U(t)S_0\phi\|_{L^{q_1}(I,L^{r_1})} + \Big(\sum_{j\ge 0}2^{2j\sigma_1}\|U(t)\Delta_j\phi\|^2_{L^{q_1}(I,L^{r_1})}\Big)^{1/2} \\ &\quad\lesssim \|S_0\phi\|_{L^2} + \Big(\sum_{j\ge 0}\|\Delta_j\phi\|^2_{L^2}\Big)^{1/2} \approx \|\phi\|_{L^2}. \end{aligned}$$

STEP 2. In order to prove (3.8), it suffices to show

$$\Big\|\int_{\mathbb{R}} U_1(t-\tau)|\widetilde{\Delta}_0|^2 f\,d\tau\Big\|_{L^{q_1}(I,L^{r_1})} \le C\|f\|_{L^{q_2'}(I,L^{r_2'})} \tag{3.13}$$

and

$$\Big\|\int_{\mathbb{R}} U(t-\tau)\mathcal{F}^{-1}(|\widetilde{\chi}|^2)*f\,d\tau\Big\|_{L^{q_1}(I,L^{r_1})} \le C\|f\|_{L^{q_2'}(I,L^{r_2'})}. \tag{3.14}$$

In fact, for $j \geq 0$, by scaling, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} U(t-\tau)\Delta_j f(x,\tau)\,d\tau \\
&= \int_{\mathbb{R}^{n+1}} e^{i(x\cdot\xi+(t-\tau)\omega(\xi))}|\widetilde{\eta}(2^{-j}\xi)|^2 \widehat{\Delta_j f}(\tau,\xi)\,d\xi\,d\tau \\
&= 2^{jn}\int_{\mathbb{R}^{n+1}} e^{i(2^j x\cdot\xi+(t-\tau)2^{js}\sqrt{2^{-2js}+|\xi|^{2s}})}|\widetilde{\eta}(\xi)|^2 \widehat{\Delta_j f}(\tau,2^j\xi)\,d\xi\,d\tau \\
&= 2^{jn-js}\int_{\mathbb{R}^{n+1}} e^{i(2^j x\cdot\xi+(2^{js}t-\tau)\sqrt{2^{-2js}+|\xi|^{2s}})}|\widetilde{\eta}(\xi)|^2 \widehat{\Delta_j f}(2^{-js}\tau,2^j\xi)\,d\xi\,d\tau \\
&= 2^{-js}\int_{\mathbb{R}^{n+1}} e^{i(2^j x\cdot\xi+(2^{js}t-\tau)\sqrt{2^{-2js}+|\xi|^{2s}})}|\widetilde{\eta}(\xi)|^2 \mathcal{F}(\Delta_j f(\cdot,2^{-j}\cdot))(2^{-js}\tau,\xi)\,d\xi\,d\tau.
\end{aligned}
$$

This together with (3.6) and (3.13) yields

$$
\begin{aligned}
&\Big\|\int_{\mathbb{R}} U(t-\tau)\Delta_j f(x,\tau)\,d\tau\Big\|_{L^{q_1}(I,L^{r_1})} \\
&\qquad = 2^{-(s/q_1+n/r_1)j-js}\bigg\|\int_{\mathbb{R}} U_1(t-\tau)|\widetilde{\Delta}_0|^2\Delta_j f\Big(\frac{x}{2^j},\frac{\tau}{2^{js}}\Big)\,d\tau\bigg\|_{L^{q_1}(2^{js}I,L^{r_1})} \\
&\qquad \lesssim 2^{-(s/q_1+n/r_1)j-js}\bigg\|\Delta_j f\Big(\frac{x}{2^j},\frac{\tau}{2^{js}}\Big)\bigg\|_{L^{q_2'}(2^{js}I,L^{r_2'})} \\
&\qquad \approx 2^{-\sigma_1 j-\sigma_2 j}\|\Delta_j f(x,\tau)\|_{L^{q_2'}(I,L^{r_2'})},
\end{aligned}
$$

which together with Minkowski's inequality and (3.14) implies that

$$
\begin{aligned}
&\Big\|\int_{\mathbb{R}} U(t-\tau)f(x,\tau)\,d\tau\Big\|_{L^{q_1}(I,B^{\sigma_1}_{r_1,2})} \\
&= \bigg\|\Big\|\int_{\mathbb{R}} U(t-\tau)S_0 f\,d\tau\Big\|_{L^{r_1}} + \Big(\sum_{j\geq 0} 2^{2j\sigma_1}\Big\|\int_{\mathbb{R}} U(t-\tau)\Delta_j f\,d\tau\Big\|^2_{L^{r_1}}\Big)^{1/2}\bigg\|_{L^{q_1}(I)} \\
&\lesssim \Big\|\int_{\mathbb{R}} U(t-\tau)S_0 f\,d\tau\Big\|_{L^{q_1}(I,L^{r_1})} + \Big(\sum_{j\geq 0} 2^{2j\sigma_1}\Big\|\int_{\mathbb{R}} U(t-\tau)\Delta_j f\,d\tau\Big\|^2_{L^{q_1}(I,L^{r_1})}\Big)^{1/2} \\
&\lesssim \|S_0 f\|_{L^{q_2'}(I,L^{r_2'})} + \Big(\sum_{j\geq 0} 2^{-2j\sigma_2}\|\Delta_j f\|^2_{L^{q_2'}(I,L^{r_2'})}\Big)^{1/2} \\
&\lesssim \Big(\|S_0 f\|^2_{L^{q_2'}(I,L^{r_2'})} + \sum_{j\geq 0} 2^{-2j\sigma_2}\|\Delta_j f\|^2_{L^{q_2'}(I,L^{r_2'})}\Big)^{1/2} \\
&\lesssim \|f\|_{L^{q_2'}(I,B^{-\sigma_2}_{r_2',2})}.
\end{aligned}
$$

By the same argument, to prove (3.9), it suffices to show that

$$\Big\| \int_0^t U_1(t-\tau)|\widetilde{\Delta}_0|^2 f\, d\tau \Big\|_{L^{q_1}(I,L^{r_1})} \lesssim \|f\|_{L^{q_2'}(I,L^{r_2'})}, \tag{3.15}$$

and

$$\Big\| \int_0^t U(t-\tau)\mathcal{F}^{-1}(|\widetilde{\chi}|^2) * f\, d\tau \Big\|_{L^{q_1}(I,L^{r_1})} \lesssim \|f\|_{L^{q_2'}(I,L^{r_2'})}. \tag{3.16}$$

Step 3. Now we show that (3.10) is equivalent to (3.13) and (3.14) by the $TT^*$ method. We write $T_k\phi$ as

$$T_k\phi(t,x) = \int_{\mathbb{R}^{n+1}} e^{ix\cdot\xi + it\theta_k}\gamma_k(\xi)\hat{\phi}(\xi)\, d\tau\, d\xi, \qquad k=1,2,$$

where

$$\theta_1(\xi) = \sqrt{2^{-2js} + |\xi|^{2s}}, \qquad \gamma_1(\xi) = \widetilde{\eta}(\xi),\ j \geq 0,$$
$$\theta_2(\xi) = \sqrt{1+|\xi|^{2s}}, \qquad \gamma_2(\xi) = \widetilde{\chi}(\xi).$$

By the $TT^*$ method, to prove (3.10), it suffices to show

$$\|T_kT_k^* f\|_{L^{q_1}(I,L^{r_1})} \leq C\|f\|_{L^{q_2'}(I,L^{r_2'})}, \tag{3.17}$$

where $(q_1,r_1),(q_2,r_2) \in \tilde{\Lambda}$. Note that for any $f \in \mathcal{S}(\mathbb{R}^{n+1})$,

$$\begin{aligned}\langle T_k\phi, f\rangle_{t,x} &= \langle e^{it\theta_k}\gamma_k(\xi)\hat{\phi}(\xi), \hat{f}(t,\xi)\rangle_{t,\xi} \\ &= \int_{\mathbb{R}^n} \phi(x)\Big(\iint_{\mathbb{R}^{n+1}} e^{it\theta_k - ix\cdot\xi}\gamma_k(\xi)\overline{\hat{f}}(t,\xi)\, d\xi\, dt\Big)\, dx.\end{aligned}$$

Consequently, we get

$$T_k^* f(x) = \iint_{\mathbb{R}^{n+1}} e^{ix\cdot\xi - is\theta_k}\bar{\gamma}_k(\xi)\widehat{f}(s,\xi)\, ds\, d\xi$$

and

$$\widehat{T_kT_k^* f}(\xi) = \int_{\mathbb{R}} e^{i(t-s)\theta_k}|\gamma_k|^2(\xi)\widehat{f}(s,\xi)\, ds.$$

From the above, we have

$$T_kT_k^* f(t,x) = \int_{\mathbb{R}} W_k(t-s)f(s,x)\, ds, \tag{3.18}$$

where

$$W_k(t)f(x) = \int_{\mathbb{R}^n} e^{i(t\theta_k + x\cdot\xi)}|\gamma_k|^2\hat{f}(\xi)\, d\xi =: K_t^k * f(x).$$

From (3.18), we can see that (3.17) is the same as (3.13) and (3.14). In addition, by the Christ–Kiselev Lemma [6], (3.15) and (3.16) can also be obtained from (3.13) and (3.14).

STEP 4. Now we prove (3.17) for $q_1 = q_2$ and $r_1 = r_2$. It suffices to establish the estimates

$$\|K_t^k * f\|_{L^2} \le \|f\|_{L^2} \tag{3.19}$$

and

$$\|K_t^k * f\|_{L^\infty} \le C(1+t)^{-n/2}\|f\|_{L^1}. \tag{3.20}$$

In fact, by interpolation between (3.19) and (3.20), we have

$$\|K_t^k * f\|_{L^r} \le C(1+t)^{-\delta(r)}\|f\|_{L^{r'}}, \qquad 2 \le r \le \infty. \tag{3.21}$$

Hence by Young's inequality and Hardy–Littlewood–Sobolev's inequality, we obtain (3.17) for $q_1 = q_2$ and $r_1 = r_2$.

Noting that $\widehat{K_t^k}(\xi) = e^{it\theta_k}|\gamma_k|^2$, it is easy to see that (3.19) holds. By Young's inequality, we have

$$\|K_t^k * f\|_{L^\infty} \le C\|K_t^k\|_{L^\infty}\|f\|_{L^1},$$

and it suffices to show

$$\|K_t^k(x)\|_{L^\infty} \le C(1+|t|)^{-n/2}, \qquad k = 1, 2. \tag{3.22}$$

To prove the above decay estimate, we follow the methods of [2, 8, 12].

We first consider $K_t^2(x)$. Let

$$\theta(\xi) = |\widetilde{\chi}(\xi)|^2, \qquad \vartheta(\xi) = \theta(2^{-1}\xi) - \theta(\xi), \qquad \vartheta_j(\xi) := \vartheta(2^{-j}\xi),$$
$$I_j(x,t) = \int_{\mathbb{R}^n} e^{i(t\theta_2 + x\cdot\xi)}\vartheta_j(\xi)\,d\xi.$$

Then

$$\|K_t^2\|_{L^\infty} \le \sum_{j \le -1} \|I_j(\cdot,t)\|_{L^\infty}. \tag{3.23}$$

Note that

$$I_j(x,t) = 2^{jn}\int e^{i2^j x\cdot\xi + it\theta_2(2^j\xi)}\vartheta(\xi)\,d\xi =: M_j(2^j x, t).$$

Hence we have

$$\|I_j(\cdot,t)\|_{L^\infty} = \|M_j(\cdot,t)\|_{L^\infty}. \tag{3.24}$$

In the case $n = 1$, we immediately get

$$\|M_j(\cdot,t)\|_{L^\infty} \lesssim 2^j. \tag{3.25}$$

A simple calculation shows that

$$\left|\frac{d^m}{d\xi^m}\left(\frac{1}{\theta_2'(2^j\xi)}\right)\right| \lesssim 2^{-j(2s-1)} \qquad \text{for } \xi \in \operatorname{supp}\vartheta,\ m \ge 0.$$

Therefore, if $|x| \le 1$, then $|\partial_\xi^m(e^{ix\xi}\vartheta(\xi))| \lesssim 1$. Using integration by parts, we see that for any $q \ge 0$,

$$\|M_j(\cdot,t)\|_{L^\infty} \lesssim |t|^{-q}2^{j(1-2sq)}. \tag{3.26}$$

If $|x|>1$, let $j_0$ be the smallest integer such that $|x|\le|t|2^{2j_0s}$. Then

$$|x|\approx|t|2^{2j_0s}.$$

For $|j-j_0|>C\gg1$, let $w_1(\xi)=x\xi+t\theta_2(2^k\xi)$. Then $|w_1'(\xi)|\ge c|t|2^{2js}$. Using integration by parts, we also find that for any $q\ge0$,

$$\|M_k(\cdot,t)\|_{L^\infty}\lesssim|t|^{-q}2^{k(1-2sq)}. \tag{3.27}$$

For $|j-j_0|\le C$, noting that $|x|>1$ and $s\ge1$, by the van der Corput Lemma and $|w_1''(\xi)|\gtrsim|t|2^{2js}$, we have

$$\|M_j(\cdot,t)\|_{L^\infty}\lesssim|t|^{-1/2}2^{j(1-s)}\lesssim|t|^{-1/2}\left(\frac{|x|}{|t|}\right)^{(1-s)/2s}\lesssim|t|^{-1/2s}. \tag{3.28}$$

Hence, by taking $q$ sufficiently large, from (3.25)–(3.28), we get

$$\begin{aligned}\sum_{j\le-1}\|M_j(\cdot,t)\|_{L^\infty}&\lesssim\sum_{|j-j_0|\le C}\|M_j(\cdot,t)\|_{L^\infty}+\sum_{|j-j_0|>C}\|M_j(\cdot,t)\|_{L^\infty}\\&\lesssim\sum_{|j-j_0|\le C}|t|^{-1/2s}+\sum_{|j-j_0|>C}\min(2^j,|t|^{-q}2^{j(1-2sq)})\\&\lesssim\sum_{|j-j_0|\le C}|t|^{-1/2s}+\sum_{2^j<|t|^{-1/2s}}2^j+\sum_{2^j>|t|^{-1/2s}}|t|^{-q}2^{j(1-2sq)}\\&\lesssim|t|^{-1/2s}.\end{aligned}$$

Therefore, we have

$$\|K_t^2(\cdot)\|_{L^\infty}\lesssim(1+|t|)^{-1/2s}\lesssim(1+|t|)^{-1/2}.$$

Next we consider the case $n\ge2$. Similar to the case $n=1$, it suffices to estimate $\|M_j(\cdot,t)\|_{L^\infty}$. By the radial symmetry of the phase function and polar coordinates transformation, we will reduce the estimate of (3.22) to an oscillatory integral relating to the Bessel function in one dimension. It is well-known that the Fourier transform of a radial function $f$ is still radial [20] and

$$\widehat{f}(\xi)=2\pi\int_0^\infty f(r)r^{n-1}(r|\xi|)^{-(n-2)/2}J_{(n-2)/2}(r|\xi|)\,dr,$$

hence we have

$$\begin{aligned}M_j(x,t)&=2^{jn}\int_{\mathbb{R}^n}e^{ix\cdot\xi+it\theta_2(2^j\xi)}\vartheta(\xi)\,d\xi\\&=2\pi2^{jn}\int_0^\infty e^{it\theta_2(2^jr)}\vartheta(r)r^{n-1}(r|x|)^{-(n-2)/2}J_{(n-2)/2}(r|x|)\,dr,\end{aligned}$$

from which we obtain the trivial estimate

$$\|M_j(\cdot,t)\|_{L^\infty}\lesssim2^{jn}. \tag{3.29}$$

From Lemma 3.1, we easily deduce that for $0 \le |x| \le 2$ and for any $k \ge 0$,

$$\frac{d^k}{dr^k}\big(\vartheta(r) r^{n-1}(r|x|)^{-(n-2)/2} J_{(n-2)/2}(r|x|)\big) \le c_k. \tag{3.30}$$

It is known (see [10, Ch. 1, equation (1.5)]) that

$$r^{-(n-2)/2} J_{(n-2)/2}(r) = c_n \Re(e^{ir} h(r)), \tag{3.31}$$

where $h$ satisfies

$$\left|\frac{d^k}{dr^k} h(r)\right| \le c_k (1+r)^{-(n-1)/2-k},$$

which implies that for $|x| > 2$ and any $k \ge 0$,

$$\left|\frac{d^k}{dr^k}(\vartheta(r) r^{n-1} h(r|x|))\right| \le c_k |x|^{-(n-1)/2}. \tag{3.32}$$

Now we divide the argument into the following two cases.

Case 1: $|x| \le 2$. We denote

$$L_r = \frac{1}{it 2^j \theta_2'(2^j r)} \frac{d}{dr}.$$

We see that

$$L_r(e^{it\theta_2(2^j r)}) = e^{it\theta_2(2^j r)}, \qquad (L_r)^* f = -\frac{1}{it 2^j} \frac{d}{dr}\left(\frac{1}{\theta_2'(2^j r)} f\right).$$

We find that for any $m \ge 0$, and $r \sim 1$,

$$\frac{d^m}{dr^m}\left(\frac{1}{\theta_2'(2^j r)}\right) \le c_m 2^{-j(2s-1)}. \tag{3.33}$$

Let $\widetilde{\vartheta}(r) = \vartheta(r) r^{n-1}$. By integration by parts, we have, for any $q \in \mathbb{N}$,

$$\begin{aligned} M_j(x,t) &= 2^{jn} \int_0^\infty e^{it\theta_2(2^j r)} \widetilde{\vartheta}(r)(r|x|)^{-(n-2)/2} J_{(n-2)/2}(r|x|)\, dr \\ &= 2^{jn} \int_0^\infty (L_r)^q (e^{it\theta_2(2^j r)}) \widetilde{\vartheta}(r)(r|x|)^{-(n-2)/2} J_{(n-2)/2}(r|x|)\, dr \\ &= \frac{2^{jn}}{(-it2^j)^q} \sum_{m=0}^{q} \sum_{(a_1,\dots,a_q)\in X_m^q} C_{q,m} \int_0^\infty e^{it\theta_2(2^j r)} \prod_{l=1}^{q} \partial_r^{a_l}\left(\frac{1}{\theta_2'(2^j r)}\right) \\ &\qquad \times \partial_r^{q-m}(\widetilde{\vartheta}(r)(r|x|)^{-(n-2)/2} J_{(n-2)/2}(r|x|))\, dr, \end{aligned}$$

where $X_m^q = \{(a_1,\dots,a_q) \in (\mathbb{Z}^+)^q : 0 \le a_1 < \dots < a_q \le q, a_1 + \dots + a_q = m\}$. It follows from (3.30) and (3.33) that

$$\|M_j(\cdot,t)\|_{L^\infty} \lesssim |t|^{-q} 2^{j(n-2sq)}. \tag{3.34}$$

CASE 2: $|x| \geq 2$. It follows from (3.31) that

$$\begin{aligned} M_j(x) &= c_n 2^{jn} \int_0^\infty e^{it\theta_2(2^j r)} \widetilde{\vartheta}(r)\big(e^{ir|x|}h(r|x|) + e^{-ir|x|}\bar{h}(r|x|)\big)\,dr \\ &= c_n 2^{jn} \int_0^\infty e^{it\theta_2(2^j r)} \widetilde{\vartheta}(r) e^{ir|x|}h(r|x|)\,dr \\ &\quad + c_n 2^{jn} \int_0^\infty e^{it\theta_2(2^j r)} \widetilde{\vartheta}(r) e^{-ir|x|}\bar{h}(r|x|))\,dr \\ &=: B_1 + B_2. \end{aligned}$$

Without loss of generality, we can assume that $t > 0$. For $B_1$, let

$$\omega_1(r) = t\theta_2(2^j r) + r|x|.$$

Since $\theta_2'(r) > 0$, we have $\omega_1'(r) \geq ct2^{2js}$, and

$$\frac{d^m}{dr^m}\left(\frac{1}{\omega_1'(2^j r)}\right) \leq c_m 2^{-j(2s-1)}. \tag{3.35}$$

By integration by parts, we also get

$$\|B_1\|_{L^\infty} \lesssim |t|^{-q} 2^{j(n-2sq)} \quad \text{for any } q \geq 0.$$

For $B_2$, let

$$\omega_2(r) = t\theta_2(2^j r) - r|x|.$$

Note that if $|x| = t2^j\theta_2'(2^j r)$, then $\omega_2'(r) = 0$, that is, the phase function $\omega_2$ has a stationary point. Now we divide the integration region into two parts. One is far from the stationary point, for which we can utilize integration by parts. The other is near the stationary point, where we can utilize the van der Corput Lemma.

CASE 2a:

$$|x| > 2 \sup_{r\in[3/2,16/3]} t2^j\theta_2'(2^j r) \quad \text{or} \quad |x| < \frac{1}{2} \inf_{r\in[3/2,16/3]} t2^j\theta_2'(2^j r).$$

In this case, we see that

$$|\omega_2'(r)| \geq ct2^{2js}, \qquad r \sim 1.$$

We have

$$\frac{d^m}{dr^m}\left(\frac{1}{\omega_2'(2^j r)}\right) \leq c_m 2^{-j(2s-1)}. \tag{3.36}$$

This together with (3.32) implies that

$$\|B_2\|_{L^\infty} \lesssim |t|^{-q} 2^{j(n-2sq)} \quad \text{for any } q \geq 0.$$

Case 2b: $\frac{1}{2}\inf_{r\in[3/2,16/3]} t2^j\theta_2'(2^jr) \le |x| \le 2\sup_{r\in[3/2,16/3]} t2^j\theta_2'(2^jr)$. Note that

$$|\omega_2''(r)| \ge c|t|2^{2js}.$$

By the van der Corput Lemma and (3.32), we have

$$\begin{aligned}\|B_2\|_{L^\infty} &\lesssim 2^{jn}(t2^{2js})^{-1/2}\int_0^\infty \left|\frac{d}{dr}(\widetilde{\vartheta}(r)e^{ir|x|}h(r|x|))\right| dr\\ &\lesssim 2^{jn}(t2^{2js})^{-1/2}|x|^{-(n-1)/2} \lesssim t^{-n/2}2^{j(n-sn)}.\end{aligned}$$

This together with (3.29) implies that

$$\|B_2\|_{L^\infty} \lesssim |t|^{-q}2^{j(n-2sq)} \quad \text{for } 0 \le q \le n/2.$$

Now we turn to estimating $\|K_t^2(x)\|_{L^\infty}$. Similar to the case $n=1$, if $j_0 \le -1$ and $|x| \sim t2^{2j_0s} \ge 2$, then

$$\|M_{j_0}(\cdot,t)\|_{L^\infty} \lesssim |t|^{-n/2}2^{j_0(n-sn)} \lesssim t^{-n/2}\left(\frac{|x|}{t}\right)^{\frac{n-sn}{2s}} \lesssim t^{-n/2s}.$$

If $|j-j_0| > C \gg 1$, then

$$|M_j(\cdot,t)| \lesssim |t|^{-q}2^{j(n-2sq)} \quad \text{for any } q \ge 0.$$

Hence, choosing $q > n/2s$, we have

(3.37)

$$\begin{aligned}\|K_t^2(\cdot)\|_{L^\infty} &\le \sum_{j\le -1}\|M_j(\cdot,t)\|_{L^\infty}\\ &\lesssim \sum_{|j-j_0|\le C}\|M_j(\cdot,t)\|_{L^\infty} + \sum_{|j-j_0|\ge C}\|M_j(\cdot,t)\|_{L^\infty}\\ &\lesssim \sum_{|j-j_0|\le C}|t|^{-n/2s} + \sum_{|j-j_0|>C}\min(2^{jn},|t|^{-q}2^{j(n-2sq)})\\ &\lesssim \sum_{|j-j_0|\le C}|t|^{-n/2s} + \sum_{2^j<|t|^{-1/2s}}2^{jn} + \sum_{2^j>|t|^{-1/2s}}|t|^{-q}2^{j(n-2sq)}\\ &\lesssim |t|^{-n/2s}.\end{aligned}$$

Therefore, we obtain

$$\|K_t^2(\cdot)\|_{L^\infty} \lesssim (1+|t|)^{-n/2s} \lesssim (1+|t|)^{-n/2}.$$

Now we consider $K_t^1(x)$. Note that

$$\theta_1(\xi) = 2^{-js}\theta_2(2^j|\xi|).$$

If $r \sim 1$, $m\in\mathbb{N}$ and $r=|\xi|$, then

$$\left|\frac{d}{dr}(\theta_1(r))\right| \sim 1, \qquad \left|\frac{d^2}{dr^2}(\theta_1(r))\right| \sim 1, \quad \text{and} \quad \left|\frac{d^{m+2}}{dr^{m+2}}(\theta_1(r))\right| \lesssim 1.$$

Similar to the proof of the decay estimate of $K_t^2(x)$, we get the desired decay estimate (3.22).

STEP 5. Now we show that (3.17) holds for any $(q_1, r_1), (q_2, r_2) \in \widetilde{\Lambda}$. By the Sobolev embedding, it suffices to show that (3.17) holds for $(q_1, r_1)$, $(q_2, r_2) \in \Lambda$. In fact, we suppose $\rho \in \mathcal{C}_c^\infty$ and equals 1 on a neighborhood of $\operatorname{supp} \widetilde{\eta}$. Hence, for any pairs $(q_1, r_1), (q_2, r_2) \in \tilde{\Lambda}$, there exist $(q_1, \tilde{r}_1), (q_2, \tilde{r}_2) \in \Lambda$ such that $\tilde{r}_1 \le r_1$ and $\tilde{r}_2 \le r_2$. By Bernstein's inequality, Young's inequality and (3.17) for $(q_1, \tilde{r}_1), (q_2, \tilde{r}_2) \in \Lambda$, we get

$$\begin{aligned}\|T_k T_k^* f\|_{L^{q_1}(I, L^{r_1})} &\lesssim \|T_k T_k^* f\|_{L^{q_1}(I, L^{\tilde{r}_1})} \lesssim \|\mathcal{F}^{-1}(\rho) * f\|_{L^{q_2'}(I, L^{\tilde{r}_2'})} \\ &\lesssim \|\mathcal{F}^{-1}(\rho) * f\|_{L^{q_2'}(I, L^{r_2'})} \lesssim \|f\|_{L^{q_2'}(I, L^{r_2'})}.\end{aligned}$$

Now we prove (3.17) for $(q_1, r_1), (q_2, r_2) \in \Lambda$, $(2/q_i, \delta(r_i)) \neq (1, 1)$, $i = 1, 2$. We define the bilinear operator as

$$\begin{aligned}B(f, g) &= \langle T_k T_k^* f(t, x), g(t, x)\rangle_{t,x} \\ &= \int_{\mathbb{R}} \Big\langle \int_{\mathbb{R}} W_k(t-\tau) f(\tau, x)\, d\tau, g(t, x)\Big\rangle_x\, dt.\end{aligned} \tag{3.38}$$

So it is enough to show

$$|B(f, g)| \le C\|f\|_{L^{q_1'}(I, L^{r_1'})} \|g\|_{L^{q_2'}(I, L^{r_2'})}, \qquad (q_1, r_1), (q_2, r_2) \in \Lambda. \tag{3.39}$$

CASE 1: $(q_1, r_1) \in \Lambda$, $(q_2, r_2) = (\infty, 2)$. Let $S(t) f := \mathcal{F}^{-1}(e^{it\theta} \bar{\gamma}(\xi) \hat{f}(\xi))$. From Hölder's inequality and Hardy–Littlewood–Sobolev's inequality, we have

$$\begin{aligned}\Big\|\int_{\mathbb{R}} S(-t) f(t, x)\, dt\Big\|_{L^2}^2 &= \int_{\mathbb{R}^2} \langle S(-\tau) f(\tau, x), S(-t) f(t, x)\rangle_x\, d\tau\, dt \\ &= \int_{\mathbb{R}^2} \langle W(t-\tau)(\tau) f(\tau, x), f(t, x)\rangle_x\, d\tau\, dt \\ &= \int_{\mathbb{R}} \Big\langle \int_{\mathbb{R}} W_k(t-\tau) f(\tau, x)\, d\tau, f(t, x)\Big\rangle_x\, dt \\ &\lesssim \Big\|\int_{\mathbb{R}} |t-\tau|^{-\delta(r_1)} \|f(\tau, x)\|_{L^{r_1'}}\, d\tau\Big\|_{L^{q_1}(I)} \|f\|_{L^{q_1'}(I, L^{r_1'})} \\ &\lesssim \|f\|_{L^{q_1'}(I, L^{r_1'})}^2.\end{aligned} \tag{3.40}$$

Hence, we have

$$\Big\|\int_{\mathbb{R}} S(-t) f(t, x)\, dt\Big\|_{L^2} \lesssim \|f\|_{L^{q_1'}(I, L^{r_1'})}. \tag{3.41}$$

It follows that

$$|B(f,g)| = \Big|\int_{\mathbb{R}}\Big\langle\int_{\mathbb{R}} W_k(t-\tau)f(\tau,x)\,d\tau, g(t,x)\Big\rangle_x\,dt\Big| \tag{3.42}$$
$$= \Big|\int_{\mathbb{R}^2}\langle W_k(t-\tau)f(\tau,x), g(t,x)\rangle_x\,d\tau\,dt\Big|$$
$$= \Big|\int_{\mathbb{R}^2}\langle S(-\tau)f(\tau,x), S(-t)g(t,x)\rangle_x\,d\tau\,dt\Big|$$
$$= \Big|\int_{\mathbb{R}}\Big\langle\int_{\mathbb{R}} S(-\tau)f(\tau,x)\,d\tau, S(-t)g(t,x)\Big\rangle_x\,dt\Big|$$
$$\lesssim \|f\|_{L^{q_1'}(I,L^{r_1'})}\|g\|_{L^1(I,L^2)}.$$

Case 2: $(q_1,r_1)=(q_2,r_2)=(q,r)\in\Lambda$. By (3.21) and Hardy–Littlewood–Sobolev's inequality, we have

$$|B(f,g)| = \Big|\int_{\mathbb{R}}\Big\langle\int_{\mathbb{R}} W(t-\tau)f(\tau,x)\,d\tau, g(t,x)\Big\rangle_x\,dt\Big| \tag{3.43}$$
$$\le \Big\|\int_{\mathbb{R}}|t-s|^{-\delta(r)}\|f(s,x)\|_{L^{r'}}\Big\|_{L^q(I)}\|g\|_{L^{q'}(I,L^{r'})}$$
$$\lesssim \|f\|_{L^{q'}(I,L^{r'})}\|g\|_{L^{q'}(I,L^{r'})}.$$

Case 3: $(q_1,r_1)=(\infty,2)$, $(q_2,r_2)\in\Lambda$. Similar to Case 1, by (3.38) and (3.41), we get

$$|B(f,g)| = \Big|\int_{\mathbb{R}}\Big\langle S(-\tau)f(\tau,x), \int_{\mathbb{R}} S(-t)g(t,x)\,dt\Big\rangle_x\,d\tau\Big|$$
$$\lesssim \|f\|_{L^1(I,L^2)}\|g\|_{L^{q_2'}(I,L^{r_2'})}.$$

The estimate (3.39) for the general case follows from the above special cases and the interpolation theorem [5]. For the endpoint case, we refer to [11]. This completes the proof of the proposition. ∎

## 4. The proof of the main results

*Proof of Theorem 2.1.* Let the operator $\mathcal{T}$ be defined by

$$\mathcal{T}(u)(t) = \dot{K}(t)\phi(x) + K(t)\psi(x) + \lambda\int_0^t K(t-\tau)(|u|^{p-1}u)(\tau)\,d\tau. \tag{4.1}$$

Now we divide the proof into several parts.

Case 1: $1<p\le n/(n-2s)$ and $n>2s$. Fix $T,M>0$ to be chosen later. We consider the metric space

$$\mathcal{X} = \{u\in L^\infty((-T,T),H^s);\ \|u\|_{L^\infty((-T,T),H^s)}\le M\} \tag{4.2}$$

equipped with the distance

$$d(u,v) = \|u-v\|_{L^\infty((-T,T),H^s)}.$$

It follows from Plancherel's theorem and Sobolev's embedding that

$$\begin{aligned}\tag{4.3} \|\mathcal{T}u\|_{H^s} &\le \|\phi\|_{H^s} + \|\psi\|_{L^2} + |\lambda| \int_0^t \big\||u|^{p-1}u\big\|_{L^2}\,d\tau \\ &\le \|\phi\|_{H^s} + \|\psi\|_{L^2} + |\lambda| \int_0^t \|u\|_{L^{2p}}^p\,d\tau \\ &\le \|\phi\|_{H^s} + \|\psi\|_{L^2} + C|\lambda|T\|u\|^p_{L^\infty((-T,T),H^s)}.\end{aligned}$$

Here we choose $M = 2(\|\phi\|_{H^s} + \|\psi\|_{L^2})$, $T = T_1 = \frac{1}{2}(C|\lambda|)^{-1}M^{1-p}$. Then $\mathcal{T}$ maps $\mathcal{X}$ to $\mathcal{X}$. On the other hand, we have

$$\begin{aligned}\tag{4.4} d(\mathcal{T}u,\mathcal{T}v) &\le C_1|\lambda| \int_0^t \big\||u-v|(|u|^{p-1}+|v|^{p-1})\big\|_{L^2}\,d\tau \\ &\le C_1|\lambda| \int_0^t \|u-v\|_{L^{2p}}(\|u\|^{p-1}_{L^{2p}} + \|v\|^{p-1}_{L^{2p}})\,d\tau \\ &\le 2C_1|\lambda|TM^{p-1}d(u,v).\end{aligned}$$

Here we choose $M = 2(\|\phi\|_{H^s} + \|\psi\|_{L^2})$, $T = T_1 = \frac{1}{4}(|\lambda|C_1)^{-1}M^{1-p}$. Then $\mathcal{T}$ is contractive. By Banach's fixed point theorem, $\mathcal{T}$ has a unique fixed point $u \in X$, i.e. $u$ satisfies (4.1). By the standard argument, $\mathcal{T}u \in \mathcal{C}([-T,T],H^s)$, and so $u \in \mathcal{C}([-T,T],H^s)$.

Case 2: $\frac{n}{n-2s} < p < \frac{n+2s}{n-2s}$ and $n > 2s$. Fix $T, M > 0$ to be chosen later. We consider the metric space

$$\tag{4.5} \mathcal{X} = \{u \in L^\infty(I,H^s) \cap L^{q_1}(I,B^{s-\beta(r)}_{r,2});\ \|u\|_{L^\infty(I,H^s)} + \|u\|_{X(I)} \le M\}$$

equipped with the distance

$$d(u,v) = \|u-v\|_{L^\infty(I,H^s)} + \|u-v\|_{X(I)}.$$

Here

$$X(I) = L^{q_1}(I, B^{s-\beta(r)}_{r,2}), \qquad I = (-T,T),$$

$$(q_1, r, \beta(r)) = \left(\frac{2(ps+2)}{(n-2s)p-n}, \frac{2(ps+2)n}{4n+(4s+(s-2)n)p}, \frac{2-s}{2}\delta(r)\right).$$

It follows from Strichartz's estimate (3.5), Hölder's inequality, Sobolev embedding $L^q \hookrightarrow B^0_{q,2}$ for $1 < q \le 2$ and the embedding $B^{s-\beta(r)}_{r,2} \hookrightarrow L^\nu$ that

$$
\begin{aligned}
(4.6)\qquad & \|\mathcal{T}u\|_{X(I)} + \|\mathcal{T}u\|_{L^\infty(I,H^s)} \\
&\le C(\|\phi\|_{H^s} + \|\psi\|_{L^2}) + C\big\||u|^{p-1}u\big\|_{L^{q_2'}(I,B^0_{r',2})} \\
&\le C(\|\phi\|_{H^s} + \|\psi\|_{L^2}) + C\big\||u|^{p-1}u\big\|_{L^{q_2'}(I,L^{r'})} \\
&\le C(\|\phi\|_{H^s} + \|\psi\|_{L^2}) + C\|u\|^p_{L^{q_1}(I,L^\nu)}T^\alpha \\
&\le C(\|\phi\|_{H^s} + \|\psi\|_{L^2}) + C\|u\|^p_{X(I)}T^\alpha,
\end{aligned}
$$

where $q_2 = \frac{s(sp+2)}{(n-2s)p-n}$, $s - \beta(r) - n/r = -n/\nu$, $1/r' = p/\nu$, $1/q_2' = p/q_1 + \alpha$, $\alpha = 1 - \frac{(n-2s)p-n}{2s}$, and $C$ does not depend on $u, \phi, \psi$ or $t$. Choosing $M = 2C(\|\phi\|_{H^s} + \|\psi\|_{L^2})$, $T = T_1 = (M^{1-p}/2C)^{1/\alpha}$, we find that $\mathcal{T}$ maps $\mathcal{X}$ to $\mathcal{X}$. On the other hand,

$$
\begin{aligned}
(4.7)\qquad d(\mathcal{T}u,\mathcal{T}v) &\le C_0\big\||u|^{p-1}u - |v|^{p-1}v\big\|_{L^{q_2'}(I,B^0_{r',2})} \\
&\le C_0\big\||u-v|(|u|^{p-1} + |v|^{p-1})\big\|_{L^{q_2'}(I,L^{r'})} \\
&\le C_0(\|u\|^{p-1}_{L^{q_1}(I,L^\nu)} + \|v\|^{p-1}_{L^{q_1}(I,L^\nu)})T^\alpha\|u-v\|_{L^{q_1}(I,L^\nu)} \\
&\le C_0(\|u\|^{p-1}_{X(I)} + \|v\|^{p-1}_{X(I)})T^\alpha\|u-v\|_{X(I)} \\
&\le 2C_0M^{p-1}T^\alpha d(u,v),
\end{aligned}
$$

where we use Strichartz's estimate (3.5), Hölder's inequality, the Sobolev embedding $L^q \hookrightarrow B^0_{q,2}$ for $1 < q \le 2$ and the embedding $B^{s-\beta(r)}_{r,2} \hookrightarrow L^\nu$. Let $T = T_2$ be such that $2C_0M^{p-1}T_2^\alpha = 1/2$. Then $\mathcal{T}$ is a contraction mapping. Following the standard argument, we deduce that $u \in \mathcal{C}(\mathbb{R}, H^s) \cap \mathcal{C}^1(\mathbb{R}, L^2) \cap L^{q_1}_{\mathrm{loc}}(\mathbb{R}, B^{s-\beta(r)}_{r,2})$, and the continuous dependence.

Case 3: $n < 2s$ and $p > 1$. Noting that in this case $H^s \hookrightarrow L^\infty$, we easily obtain

$$\|\mathcal{T}u\|_{H^s} \le \|\phi\|_{H^s} + \|\psi\|_{L^2} + |\lambda|CT\|u\|^p_{L^\infty((-T,T),H^s)}$$

and

$$
\begin{aligned}
&\|\mathcal{T}u - \mathcal{T}v\|_{L^\infty((-T,T),H^s)} \\
&\qquad\le C|\lambda|T(\|u\|_{L^\infty((-T,T),H^s)} + \|v\|_{L^\infty((-T,T),H^s)})^{p-1}\|u-v\|_{L^\infty((-T,T),H^s)}
\end{aligned}
$$

as in Case 1. By the same argument as in Case 1, one can get the uniqueness and existence of a solution to (1.1) such that $u \in \mathcal{C}(\mathbb{R}, H^s) \cap \mathcal{C}^1(\mathbb{R}, L^2)$ and $u$ depends continuously on the initial data.

CASE 4: $p = (n+2s)/(n-2s)$ and $n > 2s$. We modify slightly the proof of CASE 2. By replacing the interval $I$ by $\mathbb{R}$ and utilizing the smallness of $\|(\phi,\psi)\|_{H^s\times L^2}$, we get global existence directly from (4.6), (4.7) and the contraction principle. ∎

*Proof of Theorem 2.2.* From the proof of Theorem 2.1 and the regularity argument, we get local existence and conservation of energy of the solution to (1.1),

$$E := E(u(t), u_t(t)) = \int_{\mathbb{R}^n} \left(\frac{1}{2}u_t^2 + \frac{1}{2}|D^s u|^2 + G(u)\right) dx$$

for $t \in [0,T)$. Here $G(u) = \frac{1}{2}u^2 - \frac{\lambda}{p+1}|u|^{p+1}$ and $[0,T)$ is the maximal lifespan of the solution to (1.1). Multiplying (1.1) by $u$ and integrating, we have

$$\frac{1}{2}\frac{d^2}{dt^2}\int_{\mathbb{R}^n} u^2\,dx = \int_{\mathbb{R}^n} (u_t^2 - |D^s u|^2 - ug(u))\,dx,$$

where $g(u) = u - \lambda|u|^{p-1}u$. From this we subtract $(2+4\alpha)$ times the energy $E$ and choose $\alpha = (p-1)/8$, noting that $ug(u) \le (2+4\alpha)G(u)$, we obtain

$$\begin{aligned}\frac{d^2}{dt^2}I(t) &= \int_{\mathbb{R}^n} \big((2+2\alpha)u_t^2 + 2\alpha|D^s u|^2 + (2+4\alpha)G(u) - ug(u)\big)\,dx \\ &\quad - (2+4\alpha)E \\ &\ge (2+2\alpha)\int_{\mathbb{R}^n} u_t^2\,dx - (2+4\alpha)E,\end{aligned} \tag{4.8}$$

where $I(t) = \frac{1}{2}\int_{\mathbb{R}^n} u^2\,dx$. Multiplying (4.8) by $I$ and using Schwarz's inequality, we get

$$I(I'' + (2+4\alpha)E) > (1+\alpha)\int_{\mathbb{R}^n} u_t^2\,dx \int_{\mathbb{R}^n} u^2\,dx \ge (1+\alpha)(I')^2. \tag{4.9}$$

Let $H(t) = I(t) - (t+\tau)^2 E$. By (4.9), we have

$$\begin{aligned}HH'' - (1+\alpha)(H')^2 &= I(I'' + (2+4\alpha)) - (1+\alpha)(I')^2 - 4(1+\alpha)EI \\ &\quad - (t+\tau)^2 E(I'' + (2+4\alpha)E) + 4(t+\tau)I'E(1+\alpha) \\ &> -(1+\alpha)E[(t+\tau)I' - 2I]^2/I > 0.\end{aligned}$$

If we choose $\tau$ so large that $H'(0) > 0$, then $J = H^{-\alpha}$ satisfies $J''(t) = -\alpha(HH'' - (1+\alpha)(H')^2)/H^{\alpha+2} < 0$ and $J(0) > 0$, $J'(0) < 0$. It follows that $J(t) \le J(0) + tJ'(0)$, so $J(T) = 0$ for some $T > 0$. Thus if the solution exists up to time $T$, we have

$$\int_{\mathbb{R}^n} u^2\,dx \to \infty \quad \text{as } t \to T,$$

which implies that the solution blows up at time $T$. ∎

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Shouquan Ma
Graduate School
China Academy of Engineering Physics
P.O. Box 2101
Beijing, China, 100088
E-mail: mashouquan@126.com

Guixiang Xu
Institute of Applied Physics
and Computational Mathematics
P.O. Box 8009
Beijing, China, 100088
E-mail: xu_guixiang@iapcm.ac.cn

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