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RECURRENCES FOR THE COEFFICIENTS OF SERIES EXPANSIONS WITH RESPECT TO CLASSICAL ORTHOGONAL POLYNOMIALS

Abstract. Let $\{P_k\}$ be any sequence of classical orthogonal polynomials. Further, let f be a function satisfying a linear differential equation with polynomial coefficients. We give an algorithm to construct, in a compact form, a recurrence relation satisfied by the coefficients a_k in $f = \sum_k a_k P_k$. A systematic use of the basic properties (including some nonstandard ones) of the polynomials $\{P_k\}$ results in obtaining a low order of the recurrence.

1. Introduction. Let $\{P_k(x)\}$ be any system of classical orthogonal polynomials, i.e. associated with the names of Jacobi, Laguerre, Hermite or Bessel. Given a function f, a series expansion

$$f = \sum_k a_k[f] P_k \tag{1.1}$$

is a matter of interest in numerical analysis, applied mathematics and mathematical physics. Important special cases are connection and linearization problems, where $f = \overline{P}_n$ and $f = \overline{P}_m \overline{P}_n$ (m, n nonnegative integers), respectively, and $\{\overline{P}_k\}$ is a family of polynomials (orthogonal or not). In particular, positivity of the connection coefficients $c_k = a_k[\overline{P}_n]$, or the linearization coefficients $l_k = a_k[\overline{P}_m \overline{P}_n]$ is of great importance. See [1, 2, 4, 5, 9, 12, 13, 19–28, 33].

Usually, determination of the expansion coefficients $a_k[f]$ requires a deep knowledge of special (hypergeometric) functions. See, e.g., [1, 2, 4, 9, 15, 18, 19, 25, 26]. It is important to note that, even in the case when it is possible

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to compute these coefficients explicitly, it is often useful to have a recurrence relation of the type

$$\mathcal{L}a_k[f] \equiv \sum_{i=0}^{r} A_i(k)a_{k+i}[f] = B(k), \tag{1.2}$$

where $r \in \mathbb{N}$, and A_i and B are known functions of k. See, e.g., [10, 29]. Equation (1.2) may serve as a tool for detecting certain properties of $a_k[f]$, or for numerical evaluation of these quantities, using a judiciously chosen algorithm (cf. [30]).

In the present paper, we give an algorithmic description of the method generalizing ideas of the papers [11]–[13], to construct the recurrence (1.2) provided f is a solution of the differential equation

$$\boldsymbol{P}_n f(x) \equiv \sum_{i=0}^{n} w_{ni}(x)\boldsymbol{D}^i f(x) = g(x), \tag{1.3}$$

where $\boldsymbol{D} := d/dx$, w_{ni} are polynomials in x, and the coefficients $a_k[g]$ are known. The difference operator $\mathcal{L}$ in (1.2) is given in terms of the polynomial coefficients σ and τ of the Pearson differential equation for the orthogonality weight ϱ associated with $\{P_k(x)\}$ (see Section 3).

Notice that an alternative approach is proposed in [5, 6]; it should be stressed that there exists a class of important problems for which our method gives more refined results, i.e. lower-order recurrences of the type (1.2).

As examples we give recurrence relations for (i) the linearization coefficients of the cube $f = P_n^3$ (Section 4.1); (ii) the coefficients in the parameter derivative representation for classical orthogonal polynomials (Section 4.2); (iii) the connection coefficients between Laguerre–Sobolev and Laguerre polynomials (Section 4.3). For further examples see [12] and [13].

A Maple implementation of the proposed algorithm is given in [14].

2. Classical orthogonal polynomials

2.1. *Basic properties of classical orthogonal polynomials.* Let $\{P_k(x)\}$ be any system of classical orthogonal polynomials (i.e. associated with the names of Jacobi, Laguerre, Hermite or Bessel):

$$\int_I \varrho(x)P_k(x)P_l(x)\,dx = \delta_{kl}h_k \qquad (k, l = 0, 1, \ldots),$$

where $h_k \neq 0$ $(k = 0, 1, \ldots)$; the support I of the weight function ϱ is $[-1, 1]$, $[0, \infty)$, $(-\infty, \infty)$ and $\{z \in \mathbb{C} : |z| = 1\}$, respectively. See Appendix, Table 1.

Besides the three-term recurrence relation

$$xP_k(x) = \xi_0(k)P_{k-1}(x) + \xi_1(k)P_k(x) + \xi_2(k)P_{k+1}(x) \qquad (k = 0, 1, \ldots;\ P_{-1}(x) \equiv 0,\ P_0(x) \equiv 1), \tag{2.1}$$

these polynomials enjoy a number of similar properties which in turn provide their characterizations ([3], pp. 150–152; or [8]; or [17], Chapter II). We shall need three of those properties.

First, the weight function ϱ satisfies a differential equation of Pearson type

$$\boldsymbol{D}(\sigma\varrho) = \tau\varrho, \tag{2.2}$$

where $\boldsymbol{D} := d/dx$, σ is a polynomial of degree at most 2, and τ is a first-degree polynomial.

Second, for every $k \in \mathbb{N}$, the polynomial P_k satisfies the second-order differential equation

$$\boldsymbol{L}P_k(x) \equiv \{\sigma\boldsymbol{D}^2 + \tau\boldsymbol{D}\}P_k(x) = -\lambda_k P_k(x), \tag{2.3}$$

where

$$\lambda_k := -\frac{1}{2}k[(k-1)\sigma'' + 2\tau']. \tag{2.4}$$

Third, we have the so-called structure relation

$$\sigma(x)\lambda_k^{-1}\boldsymbol{D}P_k(x) = \delta_0(k)P_{k-1}(x) + \delta_1(k)P_k(x) + \delta_2(k)P_{k+1}(x). \tag{2.5}$$

Recently, Yáñez *et al.* [31] (see also [32], or [8], or [21]) have shown that the coefficients $\xi_i(k)$ and $\delta_i(k)$ of the relations (2.1) and (2.5), respectively, can be expressed in terms of the coefficients σ and τ of the equation (2.2).

Notice that if ζ is any zero of σ, we have the following differential-recurrence identity:

$$\begin{aligned}\frac{\sigma(x)}{x-\zeta}\boldsymbol{D}[\vartheta(k)\lambda_k^{-1}P_k(x) + \omega(\zeta;k)\lambda_{k+1}^{-1}P_{k+1}(x)]\\ = P_k(x) + \pi(\zeta;k)P_{k+1}(x),\end{aligned} \tag{2.6}$$

where

$$\begin{cases} \vartheta(k) := \xi_0(k)/\delta_0(k), \\ \omega(\zeta;k) := \lambda_{k+1}\,\dfrac{\delta_1(k)\vartheta(k) + \zeta - \xi_1(k)}{\xi_0(k+1)\eta(k) - \delta_0(k+1)}, \\ \pi(\zeta;k) := \eta(k)\omega_2(k). \end{cases} \tag{2.7}$$

Here

$$\eta(k) := \frac{\delta_2(k+1)}{\xi_2(k+1)}.$$

2.2. *Fourier coefficients*

2.2.1. *General case.* Let $\{P_k(x)\}$ be any system of classical orthogonal polynomials. Given a function f define

$$a_k[f] := \frac{1}{h_k} b_k[f], \qquad b_k[f] := \int_I \varrho(x) P_k(x) f(x)\, dx \qquad (k = 0, 1, \ldots). \tag{2.8}$$

We have a formal Fourier expansion $f \sim \sum_{k=0}^{\infty} a_k[f] P_k$.

Let $\mathcal{X}$, $\mathcal{D}$, $\mathcal{P}_\zeta$ and $\mathcal{Q}_\zeta$ be the difference operators defined by

$$\mathcal{X} := \xi_0(k)\mathcal{E}^{-1} + \xi_1(k)\mathcal{I} + \xi_2(k)\mathcal{E}, \tag{2.9}$$

$$\mathcal{D} := \delta_0(k)\mathcal{E}^{-1} + \delta_1(k)\mathcal{I} + \delta_2(k)\mathcal{E}, \tag{2.10}$$

$$\mathcal{P}_\zeta := \mathcal{I} + \pi(\zeta; k)\mathcal{E}, \tag{2.11}$$

$$\mathcal{Q}_\zeta := \omega_1(\zeta; k)\mathcal{I} + \omega_2(\zeta; k)\mathcal{E} \tag{2.12}$$

(cf. (2.1), (2.4), (2.5), and (2.6), respectively), where $\mathcal{I}$ is the *identity operator*, $\mathcal{I} b_k[f] = b_k[f]$, and $\mathcal{E}^m$ the mth *shift operator*, $\mathcal{E}^m b_k[f] = b_{k+m}[f]$ ($m \in \mathbb{Z}$). For simplicity, we write $\mathcal{E}$ in place of $\mathcal{E}^1$.

Further, let us introduce the differential operators $\boldsymbol{U}$ and $\boldsymbol{Z}_\zeta$, ζ being any root of σ, by the following formulae:

$$\boldsymbol{U} := \sigma \boldsymbol{D} + \tau \boldsymbol{I}, \tag{2.13}$$

$$\boldsymbol{Z}_\zeta := (x - \zeta)\boldsymbol{D}. \tag{2.14}$$

Here $\boldsymbol{I}$ is the identity operator.

Using (2.1)–(2.5), and the notation of (2.9)–(2.14), the following can be proved.

LEMMA 2.1 ([13]). *Let $\{P_k\}$ be any sequence of classical orthogonal polynomials. The coefficients $b_k[f]$ satisfy the identities*:

$$b_k[qf] = q(\mathcal{X}) b_k[f] \qquad (q \text{ an arbitrary polynomial}), \tag{2.15}$$

$$\mathcal{D} b_k[\boldsymbol{D} f] = b_k[f], \tag{2.16}$$

$$b_k[\boldsymbol{U} f] = -\lambda_k \mathcal{D} b_k[f], \tag{2.17}$$

$$b_k[\boldsymbol{L} f] = -\lambda_k b_k[f], \tag{2.18}$$

$$\mathcal{P}_\zeta b_k[\boldsymbol{Z}_\zeta f] = \mathcal{Q}_\zeta b_k[f]. \tag{2.19}$$

Later we shall need the following result.

LEMMA 2.2. *Given a zero ζ of σ, let $\mathcal{P}_\zeta$ be the operator defined by* (2.11). *Then*

$$\mathcal{D} = \mathcal{R}_\zeta \mathcal{P}_\zeta, \tag{2.20}$$

where $\mathcal{D}$ is defined in (2.10), *and the operator $\mathcal{R}_\zeta$ is given by*

$$\mathcal{R}_\zeta := \delta_0(k)\mathcal{E}^{-1} + \varrho(\zeta; k)\mathcal{I} \tag{2.21}$$

with

$$\varrho(\zeta; k) := \delta_2(k)/\pi(\zeta; k). \tag{2.22}$$

2.2.2. *The Jacobi and Bessel cases.* Let $\{P_k\}$ be the Jacobi polynomials $P_k^{(\alpha,\beta)}$, or Bessel polynomials $Y_n^{\alpha}(x)$. Given a zero ζ of the polynomial σ associated to $\{P_k\}$, we define the following sequences of operators:

$$\begin{aligned} \mathcal{P}_\zeta^{(m)} &:= \mathcal{I} + \pi^{(m)}(\zeta;k)\mathcal{E}, \\ \mathcal{Q}_\zeta^{(m)} &:= \vartheta(k)\mathcal{I} + \frac{\pi^{(m)}(\zeta;k)}{\eta(k+m)}\,\mathcal{E}, \qquad (m=0,1,\ldots) \\ \mathcal{R}_\zeta^{(m)} &:= \delta_0(k)\mathcal{E}^{-1} + \varrho^{(m)}(\zeta;k)\mathcal{I}, \end{aligned} \tag{2.23}$$

where the notation used is that of (2.7), and

$$\begin{aligned} \pi^{(m)}(\zeta;k) &:= \zeta\,\frac{(k+1)(2k+\gamma+1)_2}{(k+\nu_{-\zeta})(2k+\gamma+m+1)_2}, \\ \varrho^{(m)}(\zeta;k) &:= -2\zeta\,\frac{k+\nu_{-\zeta}+m}{(2k+\gamma+m)_2}, \end{aligned} \qquad (\zeta\in\{-1,+1\}) \tag{2.24}$$

with $\gamma := \alpha+\beta+1$ and

$$\nu_\zeta := \frac{\zeta\cdot\tau(\zeta)}{\sigma''} = \begin{cases} \beta+1 & (\zeta=-1), \\ \alpha+1 & (\zeta=+1), \end{cases} \tag{2.25}$$

in the Jacobi case, and

$$\pi^{(m)}(\zeta;k) := -\,\frac{(2k+\alpha+1)_2(2k+\alpha+2)_2}{2(k+\alpha+1)(2k+\alpha+m+2)_2}, \tag{2.26}$$

$$\varrho^{(m)}(\zeta;k) := \frac{2}{(2k+\alpha+m+1)_2}, \tag{2.27}$$

with $\zeta=0$ in the Bessel case (notice that $\zeta=0$ is the double root of σ in this case). The *Pochhammer symbol* $(a)_m$ is defined by

$$(a)_0 := 1, \qquad (a)_m := a(a+1)\ldots(a+m-1) \qquad (m=1,2,\ldots).$$

It can be checked that

$$\mathcal{P}_\zeta^{(0)} = \mathcal{P}_\zeta, \qquad \mathcal{Q}_\zeta^{(0)} = \mathcal{Q}_\zeta, \qquad \mathcal{R}_\zeta^{(0)} = \mathcal{R}_\zeta, \tag{2.28}$$

$$\mathcal{P}_\zeta^{(m)}\mathcal{Q}_\zeta^{(m-1)} = \mathcal{Q}_\zeta^{(m)}\mathcal{P}_\zeta^{(m-1)} \qquad (m\ge 1), \tag{2.29}$$

$$\mathcal{R}_\zeta^{(m)}\mathcal{P}_\zeta^{(m)} = \mathcal{P}_\zeta^{(m-1)}\mathcal{R}_\zeta^{(m-1)} \qquad (m\ge 1). \tag{2.30}$$

Further, for $i,j\ge 0$, define

$$
\begin{aligned}
\mathcal{S}_\zeta^{(i,j)} &:= \begin{cases} \mathcal{I} & (i<j),\\ \mathcal{P}_\zeta^{(i)}\mathcal{P}_\zeta^{(i-1)}\ldots\mathcal{P}_\zeta^{(j)} & (i\ge j),\end{cases}\\
\mathcal{Z}_\zeta^{(i)} &:= \mathcal{S}_\zeta^{(i-1,0)} \qquad (i\ge 0),\\
\mathcal{U}_\zeta^{(i)} &:= \begin{cases} \mathcal{I} & (i=0),\\ \mathcal{Q}_\zeta^{(i-1)}\ldots\mathcal{Q}_\zeta^{(1)}\mathcal{Q}_\zeta^{(0)} & (i\ge 1),\end{cases}\\
\mathcal{M}_\zeta^{(i,j)} &:= \begin{cases} \mathcal{I} & (i>j),\\ \mathcal{R}_\zeta^{(i)}\mathcal{R}_\zeta^{(i+1)}\ldots\mathcal{R}_\zeta^{(j)} & (0\le i\le j),\end{cases}\\
\mathcal{N}_\zeta^{(i)} &:= \mathcal{M}_\zeta^{(0,i-1)}.
\end{aligned}
\tag{2.31}
$$

Lemma 2.3. *We have*

$$\mathcal{Z}_\zeta^{(v)}\mathcal{D}^r = \mathcal{M}_\zeta^{(v,v+r-1)}\mathcal{Z}_\zeta^{(v+r)} \qquad (v,r=0,1,\ldots).$$

Proof. The main tools used in the proof are Lemma 2.2 and (2.30). ∎

Lemma 2.4. *Let* $\mathcal{T}_i := \mathcal{Z}_\zeta^{(v_i)}\mathcal{D}^{r_i}$ $(i=1,\ldots,m)$, *where* v_i, r_i *are non-negative integers. Define* $\mathcal{T} := \mathcal{Z}_\zeta^{(v)}\mathcal{D}^r$, *where* $v := \max_{1\le i\le m}(r_i+v_i)-r$, $r := \max_{1\le i\le m} r_i$. *Then* $\mathcal{T} = \mathcal{C}_i\mathcal{T}_i$ $(i=1,\ldots,m)$, *where*

$$\mathcal{C}_i := \mathcal{M}_\zeta^{(v,v_i+\gamma_i-1)}\mathcal{S}_\zeta^{(v_i+\gamma_i-1,v_i)}$$

and $\gamma_i := v+r-(v_i+r_i)$.

Proof. Making use of (2.31) and Lemma 2.3, we obtain

$$
\begin{aligned}
\mathcal{C}_i\mathcal{T}_i &= \mathcal{M}_\zeta^{(v,v_i+\gamma_i-1)}\mathcal{S}_\zeta^{(v_i+\gamma_i-1,v_i)}\mathcal{Z}_\zeta^{(v_i)}\mathcal{D}^{r_i}\\
&= \mathcal{M}_\zeta^{(v,v_i+\gamma_i-1)}\mathcal{Z}_\zeta^{(v_i+\gamma_i)}\mathcal{D}^{r_i} = \mathcal{U}_\zeta^{(v)}\mathcal{D}^{r-r_i}\mathcal{D}^{r_i} = \mathcal{T}
\end{aligned}
$$

for any $i=1,\ldots,m$. ∎

Lemma 2.5. *We have*

$$\mathcal{Z}_\zeta^{(i)} b_k[\boldsymbol{Z}_\zeta^i f] = \mathcal{U}_\zeta^{(i)} b_k[f] \qquad (i=0,1,\ldots). \tag{2.32}$$

Proof. We use induction on i. For $i=0$, (2.32) is obviously true, and for $i=1$ it takes the form $\mathcal{P}_\zeta^{(0)} b_k[\boldsymbol{Z}_\zeta f] = \mathcal{Q}_\zeta^{(0)} b_k[f]$, which is equivalent to (2.19). Now, assume that (2.32) holds for some i $(i\ge 1)$. We have

$$\mathcal{Z}_\zeta^{(i+1)} b_k[\boldsymbol{Z}_\zeta^{i+1} f] = \mathcal{P}_\zeta^{(i)}\mathcal{Z}_\zeta^{(i)} b_k[\boldsymbol{Z}_\zeta^i \boldsymbol{Z}_\zeta f] = \mathcal{P}_\zeta^{(i)}\mathcal{U}_\zeta^{(i)} b_k[\boldsymbol{Z}_\zeta f].$$

It can be checked that

$$\mathcal{P}_\zeta^{(i)}\mathcal{U}_\zeta^{(i)} = \mathcal{Q}_\zeta^{(i)}\ldots\mathcal{Q}_\zeta^{(2)}\mathcal{Q}_\zeta^{(1)}\mathcal{P}_\zeta^{(0)}.$$

Hence, by the first part of the proof,

$$\mathcal{Z}_\zeta^{(i+1)} b_k[\boldsymbol{Z}_\zeta^{i+1} f] = \mathcal{U}_\zeta^{(i+1)} b_k[f]. \ ∎$$

The Jacobi case. The case where $P_k = P_k^{(\alpha,\beta)}$ are Jacobi polynomials differs significantly from the others. To begin with, it is the only case where the associated polynomial $\sigma(x)$ has two different real zeros, namely -1 and 1. An important role is played by the following special second-order differential operators:

$$\boldsymbol{K}_\zeta := (\boldsymbol{Z}_\zeta + \nu_\zeta \boldsymbol{I})\boldsymbol{D} \qquad (\zeta \in \{-1, +1\}), \tag{2.33}$$

where ν_ζ is given by (2.25). The following two lemmata contain reformulated and slightly improved results of [11].

LEMMA 2.6. *Let*

$$\boldsymbol{Q} := \boldsymbol{K}_\zeta^q \boldsymbol{Z}_\zeta^r, \tag{2.34}$$

where $q \in \mathbb{Z}^+$, $r \in \{0,1\}$, *and* $\zeta \in \{-1,+1\}$. *Then*

$$\mathcal{Z}_\zeta^{(2q+r)} b_k[\boldsymbol{Q}f] = \mu_q(\zeta;k)\mathcal{E}^q \mathcal{U}_\zeta^{(r)} b_k[f], \tag{2.35}$$

where

$$\mu_0(\zeta;k) := 1, \qquad \mu_q(\zeta;k) := \frac{(k+\nu_\zeta)_q}{\prod_{i=1}^q \delta_0(k+i)} \qquad (q \geq 1). \tag{2.36}$$

Proof. First we prove the identity

$$\mathcal{K}_\zeta b_k[\boldsymbol{K}_\zeta f] = b_k[f], \tag{2.37}$$

where

$$\mathcal{K}_\zeta := \mathcal{R}_{-\zeta}\, \frac{1}{k+\nu_\zeta}\, \mathcal{P}_\zeta.$$

To this end, notice that

$$\mathcal{P}_\zeta b_k[\boldsymbol{Z}_\zeta f + \nu_\zeta f] = (\mathcal{Q}_\zeta + \nu_\zeta \mathcal{P}_\zeta) b_k[f] = (k+\nu_\zeta)\mathcal{P}_{-\zeta} b_k[f]$$

(cf. (2.19), (2.11), (2.12), and (2.25)). Using (2.20), (2.16), and the above equality, we obtain

$$\begin{aligned}\mathcal{K}_\zeta b_k[\boldsymbol{K}_\zeta f] &= \mathcal{R}_{-\zeta}\, \frac{1}{k+\nu_\zeta}\, \mathcal{P}_\zeta b_k[(\boldsymbol{Z}_\zeta + \nu_\zeta \boldsymbol{I})\boldsymbol{D} f] \\ &= \mathcal{R}_{-\zeta}\mathcal{P}_{-\zeta} b_k[\boldsymbol{D}f] = \mathcal{D} b_k[\boldsymbol{D}f] = b_k[f].\end{aligned}$$

Now, it can be checked that

$$\mathcal{Z}_\zeta^{(2m)} = \mu_m(\zeta;k)\mathcal{E}^m\mathcal{K}_\zeta^m, \qquad \mathcal{Z}_\zeta^{(2m+1)} = \mu_m(\zeta;k)\mathcal{E}^m\mathcal{Z}_\zeta^{(1)}\mathcal{K}_\zeta^m, \tag{2.38}$$

for $m = 1, 2, \ldots$ The result follows from Lemma 2.5 and (2.37). ■

LEMMA 2.7. *Let* $\mathcal{T}_1 := \mathcal{Z}_\zeta^{(v)}\mathcal{D}^r$, $\mathcal{T}_2 := \mathcal{Z}_{\zeta_*}^{(u)}\mathcal{D}^s$, *where* $\zeta \neq \zeta_*$, *and* v, r, u, s *are nonnegative integers such that* $v + r \geq u + s$. *Define* $\mathcal{T} := \mathcal{Z}_\zeta^{(w)}\mathcal{D}^t$ *with* $t := \max(u+s, r)$, $w := v + r - t$. *Then* $\mathcal{T} = \mathcal{C}_i\mathcal{T}_i$ $(i = 1, 2)$, *where*

$$\mathcal{C}_1 := \mathcal{M}_\zeta^{(w,v-1)}, \qquad \mathcal{C}_2 := \mathcal{Z}_\zeta^{(w)}\mathcal{D}^{t-u-s}\mathcal{N}_{\zeta_*}^{(u)}.$$

Note that the assumption of the above lemma that the polynomial σ has two different zeros is satisfied in the Jacobi case only.

2.2.3. *Back to the general case.* Let us return to the general setting. We have the following

LEMMA 2.8. *Let $\{P_k(x)\}$ be any system of classical orthogonal polynomials, and let ζ be a root of the associated polynomial σ (in the Laguerre or Hermite case, the value of this parameter is inessential). Further, let*

$$\boldsymbol{Q} := \boldsymbol{D}^p \boldsymbol{K}_\zeta^q \boldsymbol{Z}_\zeta^r \boldsymbol{L}^s \boldsymbol{U}^t, \tag{2.39}$$

where $p, s \in \mathbb{Z}^+$, $t \in \{0, 1\}$ and

$$q \in \begin{cases} \mathbb{Z}^+ & (\textit{Jacobi case}), \\ \{0\} & (\textit{other cases}), \end{cases} \tag{2.40}$$

$$r \in \begin{cases} \{0,1\} & (\textit{Jacobi case}), \\ \mathbb{Z}^+ & (\textit{Bessel case}), \\ \{0\} & (\textit{Laguerre and Hermite cases}). \end{cases} \tag{2.41}$$

Then

$$\mathcal{T} b_k[\boldsymbol{Q} f] = \mathcal{A} b_k[f], \tag{2.42}$$

where

$$\mathcal{T} := \begin{cases} \mathcal{Z}_\zeta^{(2q+r)} \mathcal{D}^p & (\textit{Jacobi, Bessel}), \\ \mathcal{D}^p & (\textit{Laguerre, Hermite}), \end{cases} \tag{2.43}$$

$$\mathcal{A} := \begin{cases} \mu_q(\zeta; k) \mathcal{E}^q \mathcal{U}_\zeta^{(r)} (-\lambda_k)^{s+t} \mathcal{D}^t & (\textit{Jacobi, Bessel}), \\ (-\lambda_k)^{s+t} \mathcal{D}^t & (\textit{Laguerre, Hermite}), \end{cases} \tag{2.44}$$

and μ_q is defined by (2.36).

Proof. This readily follows from Lemmata 2.1, 2.5 and 2.6. ∎

3. Main results.

The first step of the algorithm is to convert the LHS of the equation

$$\boldsymbol{P}_n f(x) \equiv \sum_{i=0}^{n} w_{ni}(x) \boldsymbol{D}^i f(x) = g(x) \tag{3.1}$$

to the form

$$\boldsymbol{P}_n f = \sum_{i=0}^{n} \boldsymbol{Q}_i (z_i f), \tag{3.2}$$

where the z_i are polynomials, and the differential operators $\boldsymbol{Q}_i$ have the form

$$\boldsymbol{Q}_i := \boldsymbol{D}^{p_i} \boldsymbol{K}_{\zeta_i}^{q_i} \boldsymbol{Z}_{\zeta_i}^{r_i} \boldsymbol{L}^{s_i} \boldsymbol{U}^{t_i}, \tag{3.3}$$

where ζ_i is a root of the associated polynomial σ (this refers to the Jacobi and Bessel cases only), $p_i, s_i \in \mathbb{Z}^+$, $t_i \in \{0,1\}$ and

$$(3.4)\qquad q_i \in \begin{cases} \mathbb{Z}^+ & \text{(Jacobi case)},\\ \{0\} & \text{(Bessel, Laguerre and Hermite cases)},\end{cases}$$

$$(3.5)\qquad r_i \in \begin{cases} \{0,1\} & \text{(Jacobi)},\\ \mathbb{Z}^+ & \text{(Bessel)},\\ \{0\} & \text{(Laguerre, Hermite)}.\end{cases}$$

To this end, define the differential operators $\boldsymbol{P}_i$ $(i = 0,1,\ldots,n-1)$ and $\boldsymbol{Q}_j$ $(j = 1,\ldots,n)$, and the polynomials $z_0, z_1, \ldots, z_n$ in the following recursive way.

For $i = n, n-1, \ldots, 1$, given the operator $\boldsymbol{P}_i = \sum_{j=0}^{i} w_{ij}\boldsymbol{D}^j$,

- represent the leading coefficient w_{ii} in the form

$$w_{ii}(x) = [\sigma(x)]^{\alpha_i}(x-\zeta_i)^{\beta_i}u_i(x)$$

where the polynomial u_i has no roots in common with σ, and

$$\alpha_i \begin{cases} \in \mathbb{Z}^+ & \text{(Jacobi, Bessel, Laguerre)},\\ = i & \text{(Hermite)},\end{cases}$$

$$\beta_i \begin{cases} \in \mathbb{Z}^+ & \text{(Jacobi, Bessel)},\\ = 0 & \text{(Laguerre, Hermite)},\end{cases}$$

ζ_i being a root of σ (if $\beta_i = 0$, the value of ζ_i is inessential), then find out which of the following cases holds:

— case A: $\alpha_i \ge m$,
— case B: $\alpha_i + \beta_i \ge m > \alpha$,
— case C: $m > \alpha_i + \beta_i$,

where $m := \lfloor (i+1)/2 \rfloor$;

- define

$$z_i(x) := u_i(x)\begin{cases} [\sigma(x)]^{\alpha_i - m}(x-\zeta)^{\beta_i} & \text{(case A)},\\ (x-\zeta)^{\alpha_i+\beta_i-m} & \text{(case B)},\\ 1 & \text{(case C)};\end{cases}$$

- define

$$r_i := \begin{cases} i \bmod 2 & \text{(case B)},\\ 0 & \text{(case A or C)},\end{cases}\qquad t_i := \begin{cases} i \bmod 2 & \text{(case A)},\\ 0 & \text{(case B or C)},\end{cases}$$

$$s_i := \min\{\alpha_i, \lfloor i/2 \rfloor\},\quad q_i := \min\{\beta_i, \lfloor i/2 \rfloor - t_i\},$$

$$p_i := i - 2q_i - 2s_i - r_i - t_i;$$

- define the operator $\boldsymbol{Q}_i$ by

$$\boldsymbol{Q}_i := \boldsymbol{D}^{p_i}\boldsymbol{K}_{\zeta_i}^{q_i}\boldsymbol{Z}_{\zeta_i}^{r_i}\boldsymbol{L}^{s_i}\boldsymbol{U}^{t_i};$$

• define the operator $\boldsymbol{P}_{i-1}$ of degree $i-1$ as

$$\boldsymbol{P}_{i-1}f(x) := \boldsymbol{P}_i f(x) - \boldsymbol{Q}_i(z_i f)(x).$$

For convenience, set

$$\boldsymbol{Q}_0 := \boldsymbol{I}, \qquad z_0 := w_{0,0}, \tag{3.6}$$

where $w_{0,0}$ is the only coefficient of the operator $\boldsymbol{P}_0$.

Now, we prove

THEOREM 3.1. *Let $\{P_k\}$ be any sequence of classical orthogonal polynomials, and let the function f satisfy the differential equation*

$$\boldsymbol{P}_n f(x) \equiv \sum_{i=0}^{n} \boldsymbol{Q}_i(z_i f)(x) = g(x), \tag{3.7}$$

where the z_i are polynomials, and the ith-order differential operator $\boldsymbol{Q}_i$ ($i = 0, 1, \ldots, n$) is of the form

$$\boldsymbol{Q}_i := \boldsymbol{D}^{p_i} \boldsymbol{K}_\zeta^{q_i} \boldsymbol{Z}_\zeta^{r_i} \boldsymbol{L}^{s_i} \boldsymbol{U}^{t_i}, \tag{3.8}$$

where ζ is a fixed root of the associated polynomial σ (this refers to the Jacobi and Bessel cases only), $p_i, s_i \in \mathbb{Z}^+$, $t_i \in \{0,1\}$ and

$$q_i \in \begin{cases} \mathbb{Z}^+ & (\textit{Jacobi}), \\ \{0\} & (\textit{other families}), \end{cases} \qquad r_i \in \begin{cases} \{0,1\} & (\textit{Jacobi}), \\ \mathbb{Z}^+ & (\textit{Bessel}), \\ \{0\} & (\textit{Laguerre, Hermite}). \end{cases}$$

The Fourier coefficients $a_k[f]$ satisfy the recurrence relation

$$\mathcal{L}(h_k a_k[f]) = B(k), \tag{3.9}$$

where

$$\mathcal{L} := \sum_{i=0}^{n} \mathcal{C}_i \mathcal{A}_i z_i(\mathcal{X}), \tag{3.10}$$

$$B(k) := \mathcal{T}(h_k a_k[g]), \tag{3.11}$$

and

$$\mathcal{T} := \mathcal{Z}_\zeta^{(d)} \mathcal{D}^e, \tag{3.12}$$

$$\mathcal{A}_i := \begin{cases} \mu_{q_i}(\zeta;k)\mathcal{E}^{q_i}\mathcal{U}_\zeta^{(r_i)}(-\lambda_k)^{s_i+t_i}\mathcal{D}^{t_i} & (\textit{Jacobi, Bessel}), \\ (-\lambda_k)^{s_i+t_i}\mathcal{D}^{t_i} & (\textit{Laguerre, Hermite}), \end{cases} \tag{3.13}$$

$$\mathcal{C}_i := \begin{cases} \mathcal{M}_\zeta^{(d,d+e-p_i-1)} \mathcal{S}_\zeta^{(d+e-p_i-1,2q_i+r_i)} & (\textit{Jacobi, Bessel}), \\ \mathcal{D}^{e-p_i} & (\textit{Laguerre, Hermite}), \end{cases} \tag{3.14}$$

with

$$e := \max_{0 \le i \le n} p_i, \qquad d := \max_{0 \le i \le n} (p_i + 2q_i + r_i) - e.$$

The order of the recurrence (3.9) *equals*

$$d+\delta\cdot e+\max_{0\le i\le n,\, z_i\not\equiv 0}(2\deg z_i-\delta\cdot[p_i+q_i-t_i]), \tag{3.15}$$

where $\delta:=\deg\sigma$.

Proof. Observe that by Lemma 2.4, the operators (3.12) and (3.14) satisfy

$$\mathcal{T}^+=\mathcal{C}_i\mathcal{T}_i\quad (i=0,1,\dots,n),\quad \text{where}\quad \mathcal{T}_i:=\mathcal{Z}_\zeta^{(2q_i+r_i)}\mathcal{D}^{p_i}.$$

Obviously, in view of (3.7), we have $b_k[\boldsymbol{P}_nf]=b_k[g]$. Apply the operator $\mathcal{T}$ to both sides of the above equation, then use Lemmata 2.8 and 2.1, and (2.8) to transform the left-hand side of the resulting equation:

$$\begin{aligned}\mathcal{T}b_k[\boldsymbol{P}_nf]&=\sum_{i=0}^{n}\mathcal{C}_i\mathcal{T}_ib_k[\boldsymbol{Q}_i(z_if)]=\sum_{i=0}^{n}\mathcal{C}_i\mathcal{A}_iz_i(\mathcal{X})b_k[f]\\&=\mathcal{L}b_k[f]=\mathcal{L}(h_ka_k[f]).\end{aligned}$$

This implies the identity (3.9) with the operator $\mathcal{L}$ and the function $B(k)$ given by (3.10) and (3.11), respectively.

The formula (3.15) follows easily from (3.10), in view of (3.12)–(3.14). (Notice that the orders of the operators $\mathcal{D}$ and $\mathcal{X}$ are $\deg\sigma$ and 2, respectively; see Tables 3 and 2 in the Appendix). ■

Let us return to the differential operator (3.2). In the Jacobi case, it is possible that not all ζ_i's in (3.3) are equal, so that Theorem 3.1 is not applicable. The next theorem is a reformulated and corrected version of a result in [11].

THEOREM 3.2. *Let* $\{P_k\}$ *be the sequence of Jacobi polynomials, and let* f *satisfy the equation*

$$\boldsymbol{P}_nf(x)\equiv\sum_{i=0}^{n}\boldsymbol{Q}_i(z_if)(x)=g(x), \tag{3.16}$$

where the z_i *are polynomials, and the differential operators* $\boldsymbol{Q}_i$ *have the form*

$$\boldsymbol{Q}_i:=\boldsymbol{D}^{p_i}\boldsymbol{K}_{\zeta_i}^{q_i}\boldsymbol{Z}_{\zeta_i}^{r_i}\boldsymbol{L}^{s_i}\boldsymbol{U}^{t_i}, \tag{3.17}$$

where $\zeta_i\in\{-1,+1\}$, $p_i,q_i,s_i\in\mathbb{Z}^+$, $r_i,t_i\in\{0,1\}$, $p_i+2q_i+r_i+2s_i+t_i=i$, *and the expansion of the function* g *in* P_k *is known. Set*

$$\Omega:=\{1,\dots,n\},\qquad \Omega_\eta:=\{i\in\Omega:\zeta_i=\eta\}\qquad (\eta\in\{-1,+1\}), \tag{3.18}$$

$$v_i:=2q_i+r_i\qquad (i\in\Omega), \tag{3.19}$$

$$e_\eta:=\max_{i\in\Omega_\eta}p_i,\qquad d_\eta:=\max_{i\in\Omega_\eta}(p_i+v_i)-e_\eta\qquad (\eta\in\{-1,\,+1\}), \tag{3.20}$$

$$\omega:=\begin{cases}-1 & \textit{if } d_{-1}+e_{-1}\ge d_1+e_1,\\ +1 & \textit{if } d_{-1}+e_{-1}< d_1+e_1,\end{cases} \tag{3.21}$$

$$e := \max(e_\omega, d_{-\omega} + e_{-\omega}), \qquad d := d_\omega + e_\omega - e. \tag{3.22}$$

Further, define difference operators $\mathcal{T}$, $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$, $\mathcal{J}_\eta$ *by*

$$\mathcal{T} := \mathcal{Z}_\omega^{(d)}\mathcal{D}^e, \tag{3.23}$$

$$\mathcal{A}_i := \mu_{q_i}(\omega;k)\mathcal{E}^{q_i}\mathcal{U}_\omega^{(r_i)}(-\lambda_k)^{s_i+t_i}\mathcal{D}^{t_i} \qquad (i = 0,1,\dots,n), \tag{3.24}$$

$$\mathcal{B}_i := \mathcal{M}_\eta^{(d_\eta, e_\eta+d_\eta-p_i-1)}\mathcal{S}_\eta^{(e_\eta+d_\eta-p_i-1, v_i)} \qquad (i \in \Omega_\eta;\ \eta = -1,+1), \tag{3.25}$$

$$\mathcal{J}_\omega := \mathcal{M}_\omega^{(d,d_\omega-1)}, \qquad \mathcal{J}_{-\omega} := \mathcal{Z}_\omega^{(d)}\mathcal{N}_{-\omega}^{(e-e_{-\omega})}\mathcal{S}_{-\omega}^{(e-e_{-\omega}-1, d_{-\omega})}, \tag{3.26}$$

$$\mathcal{C}_0 := \mathcal{T}, \qquad \mathcal{C}_i := \begin{cases} \mathcal{J}_\omega\mathcal{B}_i & (i \in \Omega_\omega), \\ \mathcal{J}_{-\omega}\mathcal{B}_i & (i \in \Omega_{-\omega}). \end{cases} \tag{3.27}$$

The Fourier coefficients $a_k[f]$ *satisfy the recurrence relation*

$$\mathcal{L}(h_k a_k[f]) = B(k), \tag{3.28}$$

where

$$\mathcal{L} := \sum_{i=0}^{n} \mathcal{C}_i\mathcal{A}_i z_i(\mathcal{X}), \tag{3.29}$$

$$B(k) := \mathcal{T}(h_k a_k[g]). \tag{3.30}$$

The order of the recurrence (3.28) *equals*

$$d + 2e + 2 \max_{0\le i\le n,\ z_i\not\equiv 0}(\deg z_i - p_i - q_i + t_i). \tag{3.31}$$

Proof. Let the operator $\mathcal{W}_\eta$ be given by

$$\mathcal{W}_\eta := \mathcal{Z}_\eta^{(d_\eta)}\mathcal{D}^{e_\eta} \qquad (\eta = \pm 1),$$

where d_η, e_η are the numbers defined in (3.20). By Lemma 2.4,

$$\mathcal{W}_\eta = \mathcal{B}_i\mathcal{T}_i \qquad (i \in \Omega_\eta;\ \eta = \pm 1), \tag{3.32}$$

where $\mathcal{B}_i$ is given by (3.25), and

$$\mathcal{T}_i := \mathcal{Z}_{\zeta_i}^{(v_i)}\mathcal{D}^{p_i} \qquad (i = 0,1,\dots,n). \tag{3.33}$$

Obviously, $\mathcal{T}_i b_k[\boldsymbol{Q}_i f] = \mathcal{A}_i b_k[f]$, where $\mathcal{A}_i$ is the operator (3.24). Now, using Lemma 2.7, we see that the operator (3.23) satisfies $\mathcal{T} = \mathcal{J}_\eta\mathcal{W}_\eta$ $(\eta = \pm 1)$, $\mathcal{J}_\eta$ being given by (3.26). Hence, in view of (3.32) and (3.27), $\mathcal{T} = \mathcal{C}_i\mathcal{T}_i$ $(i = 0,1,\dots,n)$, and

$$\begin{aligned} \mathcal{T}b_k[\boldsymbol{P}_n f] &= \sum_{i=0}^{n} \mathcal{C}_i\mathcal{T}_i b_k[\boldsymbol{Q}_i(z_i f)] = \sum_{i=0}^{n} \mathcal{C}_i\mathcal{A}_i b_k[z_i f] \\ &= \Big\{\sum_{i=0}^{n} \mathcal{C}_i\mathcal{A}_i z_i(\mathcal{X})\Big\} b_k[f] = \mathcal{L}b_k[f], \end{aligned}$$

where $\mathcal{L}$ is given by (3.29). The formula (3.31) follows readily from (3.29), in view of (3.23)–(3.27). ∎

4. Examples

4.1. *Linearization of cubes of classical orthogonal polynomials.* Given a system $\{P_k\}$ of classical orthogonal polynomials, let us construct a recurrence relation satisfied by the coefficients $c_{n,k}$ in

$$P_n^3 = \sum_{k=0}^{3n} c_{n,k} P_k \quad (n \in \mathbb{N}). \tag{4.1}$$

Let us recall the following recent result of Hounkonnou *et al.* [7] (see also [21]).

LEMMA 4.1 ([7]). *Let $\{P_k(x)\}$ be any system of classical orthogonal polynomials. For any $n \in \mathbb{N}$, the cube $w := P_n^3$ satisfies the fourth-order differential equation*

$$\boldsymbol{P}_4 w \equiv \begin{vmatrix} \boldsymbol{R}_2 w & \sigma & 0 \\ \boldsymbol{R}_3 w & \eta & 1 \\ \boldsymbol{R}_4 w & \sigma(\eta' - 3\lambda_n) - 2\tau\eta & 2\eta - \tau \end{vmatrix} = 0, \tag{4.2}$$

where $\eta := \sigma' - 2\tau$, and

$$\boldsymbol{R}_2 := \sigma \boldsymbol{D}^2 + \tau \boldsymbol{D} + 3\lambda_n \boldsymbol{I}, \qquad \boldsymbol{R}_3 := \boldsymbol{D}(\boldsymbol{R}_2 + 4\lambda_n \boldsymbol{I}),$$
$$\boldsymbol{R}_4 := \sigma \boldsymbol{D} \boldsymbol{R}_3 + 4\lambda_n \eta \boldsymbol{D}.$$

Now, the LHS of (4.2) can be written in the form

$$\boldsymbol{P}_4 f \equiv \boldsymbol{L}^2(z_4 f) + \boldsymbol{L}\boldsymbol{U}(z_3 f) + \boldsymbol{L}(z_2 f) + \boldsymbol{U}(z_1 f) + z_0 f, \tag{4.3}$$

where

$$\begin{aligned}
&z_4 := \sigma, \qquad z_3 := 4\tau - 6\sigma', \\
&z_2 := (11\sigma'' - 10\tau' + 10\lambda_n)\sigma + (2\tau - 3\sigma')(\tau - 2\sigma'), \\
&z_1 := (4\tau - 6\sigma')(4\sigma'' - 4\tau' + 5\lambda_n), \\
&z_0 := [(6\sigma'' - 10\tau' + 17\lambda_n)\sigma'' + (2\tau' - 3\lambda_n)^2 - 2\lambda_n\tau']\sigma \\
&\qquad - (\sigma'' - \tau')(4\tau - 6\sigma')\tau + \lambda_n(3\sigma' - 2\tau)(2\sigma' + \tau).
\end{aligned}$$

On applying Theorem 3.1, we obtain the following.

THEOREM 4.2. *Let $\{P_k(x)\}$ be any system of classical orthogonal polynomials. For any $n \in \mathbb{N}$, the linearization coefficients $c_{n,k}$ in* (4.1) *satisfy the fourth-order recurrence relation*

$$\{\lambda_k^2 z_4(\mathcal{X}) + \lambda_k^2 \mathcal{D} z_3(\mathcal{X}) - \lambda_k z_2(\mathcal{X}) - \lambda_k \mathcal{D} z_1(\mathcal{X}) + z_0(\mathcal{X})\}(h_k c_{n,k}) = 0. \tag{4.4}$$

Let C_k^ν be the ultraspherical (Gegenbauer) polynomials,

$$C_k^\nu(x) := \frac{(2\nu)_k}{(\nu + 1/2)_k}\, P_k^{(\nu-1/2,\nu-1/2)}(x) \quad (\nu > -1/2,\ \nu \neq 0).$$

Theorem 4.2 implies that the linearization coefficients $c_{n,k}$ in

$$(C_n^\nu)^3 = \sum_{k=0}^{3n} c_{n,k} C_k^\nu$$

satisfy the three-term recurrence relation

$$A_0(k)c_{n,k-2} + A_1(k)c_{n,k} + A_2(k)c_{n,k+2} = 0 \quad (2 \le k \le 3n+1)$$

where $c_{n,3n} = 1$, $c_{n,m} = 0$ for $m > 3n$, and

$$\begin{aligned} A_0(k) &:= 16(k+\nu+1)(k+\nu-1)_4(k-3n-2) \\ &\qquad \times (k+n+4\nu-2)(k-n+2\nu-2)(k+3n+6\nu-2), \\ A_1(k) &:= 8(k+\nu)_3\{4KN[K+(2\nu-1)(3\nu+1)] \\ &\qquad + (3N-K)(8\nu^2[K+(\nu-1)(6\nu-1)] \\ &\qquad + [8\nu(1-3\nu)-3N+K][K+(\nu-1)(2\nu+1)])\}, \\ A_2(k) &:= (k+1)_2(k+\nu-1)(k+2\nu)_2(k+n+2) \\ &\qquad \times (k-n-2\nu+2)(k+3n+2\nu+2)(k-3n-4\nu+2). \end{aligned}$$

Here $K := k(k+2\nu)$, $N := n(n+2\nu)$.

The linearization coefficients $c_{n,k}$ in

$$H_n^3 = \sum_{k=0}^{3n} c_{n,k} H_k,$$

H_k being the Hermite polynomials, satisfy the recurrence relation

$$D_0(k)c_{n,k-2} + D_1(k)c_{n,k} + D_2(k)c_{n,k+2} = 0 \quad (2 \le k \le 3n+1),$$

where $c_{n,3n} = 1$, $c_{n,m} = 0$ for $m > 3n$, and

$$D_0(k) := 12(3n-k+2), \qquad D_1(k) := 3n(3n+2k+4) - k(7k+4),$$
$$D_2(k) := -(k+1)_2(n+k+2).$$

4.2. *Parameter derivative representation for classical orthogonal polynomials.* Let $P_k(x) = P_k(x; \boldsymbol{c})$ $(k \ge 0)$, be any sequence of classical orthogonal polynomials, depending on parameters, i.e., Jacobi, Laguerre or Bessel polynomials. Here $\boldsymbol{c} = [c_1, \dots, c_p]$ is a parameter vector. In what follows, c is a generic symbol for any member of $\boldsymbol{c}$. Given a function $f(x) = f(x; \boldsymbol{c})$, we use the notation

$$f^{[r]}(x; \boldsymbol{c}) := \frac{\partial^r}{\partial c^r} f(x; \boldsymbol{c}) \quad (r \ge 0).$$

Let us look for recurrences for the coefficients $C_k^{[r]}$ in the expansion

$$P_n^{[r]}(x;\boldsymbol{c}) = \sum_{k=0}^{n} C_k^{[r]} P_k(x;\boldsymbol{c}) \quad (r \ge 1). \tag{4.5}$$

A partial solution to the problem might be a recurrence linking $C_k^{[r]}$ and $C_k^{[r-1]}$ $(r \ge 1)$.

We start from the equality [23]

$$\boldsymbol{L}_n P_n^{[r]} = -r\boldsymbol{M}_n^{(c)} P_n^{[r-1]}, \tag{4.6}$$

where $\boldsymbol{L}_n = \boldsymbol{L} + \lambda_n \boldsymbol{I}$, and

$$\boldsymbol{M}_n^{(c)} := \frac{\partial \sigma}{\partial c}\,\boldsymbol{D}^2 + \frac{\partial \tau}{\partial c}\,\boldsymbol{D} + \frac{\partial \lambda_n}{\partial c}\,\boldsymbol{I}.$$

Let us apply the theory of §3 to

$$f(x) := P_n^{[r-1]}(x), \qquad \boldsymbol{P}_2 := -r\boldsymbol{M}_n^{(c)}, \qquad g(x) := \boldsymbol{L}_n P_n^{[r]}(x).$$

Using Lemma 2.1 and (2.8), we obtain

$$b_k[g] = (\lambda_n - \lambda_k) b_k[P_n^{[r]}] = (\lambda_n - \lambda_k) h_k C_k^{[r]}.$$

Now, observe that for the Jacobi and Bessel polynomials we have

$$\boldsymbol{M}_n^{(c)} = A_{\boldsymbol{c}} \boldsymbol{Z}_\zeta + \frac{\partial \lambda_n}{\partial c}\,\boldsymbol{I},$$

where ζ is a root of σ, and $A_{\boldsymbol{c}} = \text{const}$. Thus, by Theorem 3.1,

$$\mathcal{P}_\zeta(h_k a_k[g]) = -r\{A_{\boldsymbol{c}} \mathcal{Q}_\zeta - n\mathcal{P}_\zeta\}(h_k a_k[P_n^{[r-1]}]),$$

hence

$$\mathcal{P}_\zeta\{(\lambda_n - \lambda_k) h_k C_k^{[r]}\} = -r(A_{\boldsymbol{c}} \mathcal{Q}_\zeta - n\mathcal{P}_\zeta)\{h_k C_k^{[r-1]}\}. \tag{4.7}$$

For instance, in the monic Jacobi case,

$$\frac{\partial^r}{\partial \alpha^r} P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n-1} C_k^{[r]} P_k^{(\alpha,\beta)}(x),$$

formula (4.7) takes the form

$$C_k^{[r]} - (n-k-1)A(k) C_{k+1}^{[r]} = -\frac{r}{n+k+\omega}\, C_k^{[r-1]} + rA(k) C_{k+1}^{[r-1]}, \tag{4.8}$$

where $r \ge 1$, $\omega := \alpha + \beta + 1$, and

$$A(k) := 2\,\frac{(k+1)(k+\beta+1)(n+k+\omega+1)}{(n-k)(n+k+\omega)(2k+\omega+1)_2}.$$

Observing that $C_k^{[0]} = \delta_{nk}$, we deduce the formula

$$C_k^{[1]} = \frac{2^{n-k}}{n-k} \cdot \frac{(k+1)_{n-k}(k+\beta+1)_{n-k}}{(n+k+\omega)(2k+\omega+1)_{2n-2k-1}},$$

equivalent to the one given in [4] (see also [9]). For general r, the formula

$$C_k^{[r]} = u_k^{[r]} C_k^{[1]}$$

can be obtained, where the auxiliary sequence $u_k^{[r]}$ satisfies the recurrence

$$u_{k+1}^{[r]} - u_k^{[r]} = \frac{r}{n-k-1} u_{k+1}^{[r-1]} - \frac{r}{n+k+\omega} u_k^{[r-1]}$$

with the initial condition $u_{n-1}^{[r]} = \frac{r}{2n+\omega-1} u_{n-1}^{[r-1]}$. For instance,

$$u_k^{[2]} = 2\{\psi(2n+\omega) - \psi(n+k+\omega) - \psi(n-k) - \gamma\},$$

where γ is Euler's constant, and $\psi(z) = \Gamma'(z)/\Gamma(z)$. Notice that in [23, §3.2], a nonhomogeneous second-order recurrence relation for $C_k^{[2]}$ is obtained.

4.3. *Connection between Laguerre–Sobolev and Laguerre polynomials.* The monic Laguerre–Sobolev polynomials $\{Q_m^{(\alpha)}\}$ are orthogonal with respect to the inner product

$$(f,g)_S := \int_0^\infty \varrho^{(\alpha)}(x) f(x) g(x)\,dx + \lambda \int_0^\infty \varrho^{(\alpha)}(x) f'(x) g'(x)\,dx,$$

where $\alpha > -1$, $\lambda \geq 0$, and $\varrho^{(\alpha)}(x) := x^\alpha e^{-x}$ is the classical Laguerre weight, associated with the monic Laguerre polynomials $\{L_k^{(\alpha)}\}$. See [16]. For convenience, let us add the superscript α on the related symbols, thus, $\sigma^{(\alpha)}(x) \equiv \sigma(x)$, $\tau^{(\alpha)}(x) \equiv \tau(x)$, $h_k^{(\alpha)} \equiv h_k$, $\mathcal{D}^{(\alpha)} \equiv \mathcal{D}$, $\mathcal{X}^{(\alpha)} \equiv \mathcal{X}$, $\boldsymbol{L}^{(\alpha)} \equiv \boldsymbol{L}$ etc.

We show that the method of Section 3 provides a third-order recurrence relation for the connection coefficients $S_{n,k}$ in

$$Q_n^{(\alpha)} = \sum_{k=0}^{n} S_{n,k} L_k^{(\alpha)} \qquad (\alpha > -1),$$

as well as a second-order recurrence for the coefficients $S_{n,k}^*$ in

$$Q_n^{(\alpha)} = \sum_{k=0}^{n} S_{n,k}^* L_k^{(\alpha-1)} \qquad (\alpha > 0).$$

In view of the well-known identity

$$L_k^{(\alpha-1)} = L_k^{(\alpha)} + k L_{k-1}^{(\alpha)}, \tag{4.9}$$

we have

$$S_{n,k} = S_{n,k}^* + (k+1) S_{n,k+1}^* \qquad (k = 0,1,\ldots,n;\ S_{n,n+1}^* := 0),$$

provided all the quantities are well-defined.

In [16], the following differential equation satisfied by $Q_n^{(\alpha)}$ was given:

$$\begin{aligned}\boldsymbol{F}Q_n^{(\alpha)}(x) &\equiv \{\sigma^{(\alpha)}\boldsymbol{I} - \lambda(\tau^{(\alpha)} - \sigma^{(\alpha)\prime})\boldsymbol{D} - \lambda\sigma^{(\alpha)}\boldsymbol{D}^2\}Q_n^{(\alpha)}(x)\\ &= \sum_{i=-1}^{1} b_{n+i}(n)L_{n+i}^{(\alpha)}(x),\end{aligned}$$

where $b_{n-1}(n) := nc_n$, $b_n(n) := n+1+c_n$, $b_{n+1}(n) := 1$, c_n being a constant. Writing the differential operator $\boldsymbol{F}$ in the form

$$\boldsymbol{F} = \sigma^{(\alpha)}\boldsymbol{I} + \lambda\sigma^{(\alpha)\prime}\boldsymbol{D} - \lambda\boldsymbol{L}^{(\alpha)},$$

applying the method of Section 3 to $f := Q_n^{(\alpha)}$ and $P_k := L_k^{(\alpha)}$, and using the data of Table 2, we conclude that the $S_{n,k}$'s satisfy the recurrence relation

$$\{\mathcal{D}^{(\alpha)}(\mathcal{X}^{(\alpha)} + \lambda\lambda_k^{(\alpha)}\mathcal{I}) + \lambda\mathcal{I}\}(h_k^{(\alpha)}S_{n,k}) = \mathcal{D}^{(\alpha)}(h_k^{(\alpha)}b_k(n)),$$

where $b_k(n) := 0$ for $k < n-1$. The scalar form of the recurrence is

$$\begin{aligned}S_{n,k-2} + [(\lambda+3)k + \alpha - \lambda - 1]S_{n,k-1} + k[(\lambda+3)k + 2\alpha + \lambda + 1]S_{n,k}&\\ + (k)_2(k+\alpha+1)S_{n,k+1} = b_{k-1}(n) + kb_k(n)&\\ (2 \le k \le n+1;\ b_m(n) := 0 \text{ for } m < n-1).&\end{aligned}$$

The starting values are $S_{n,n} := 1$, $S_{n,n+1} := S_{n,n+2} := 0$.

If $\alpha > 0$, we can write the right-hand side of (4.10) as (cf. (4.9))

$$c_nL_n^{(\alpha-1)}(x) + L_{n+1}^{(\alpha-1)}(x).$$

Now, writing the operator $\boldsymbol{F}$ in the form

$$\boldsymbol{F} = \sigma^{(\alpha-1)}\boldsymbol{I} - \lambda\boldsymbol{L}^{(\alpha-1)},$$

and applying Theorem 3.1 to $f := Q_n^{(\alpha)}$ and $P_k := L_k^{(\alpha-1)}$, we obtain the following recurrence relation for the $S^*_{n,k}$'s:

$$\{\mathcal{X}^{(\alpha-1)} + \lambda\,\lambda_k^{(\alpha-1)}\mathcal{I}\}(h_k^{(\alpha-1)}S^*_{n,k}) = h_k^{(\alpha-1)}\delta_{kn}c_n,$$

or, in scalar form,

$$S^*_{n,k-1} + [(\lambda+2)k + \alpha]S^*_{n,k} + (k+1)(k+\alpha)S^*_{n,k+1} = 0 \quad (1 \le k \le n-1).$$

The starting values are $S^*_{n,n} = 1$, $S^*_{n,n-1} = c_n - (\lambda+2)n - \alpha$.

Appendix. In the tables below, we collect some relevant data for the classical orthogonal polynomials.

Table 1. Hypergeometric series representations of the classical monic orthogonal polynomials

Family	Hypergeometric series
Jacobi	$P_k^{(\alpha,\beta)}(x) = (-1)^k \binom{k+\beta}{k}\, {}_2F_1\left(\begin{matrix}-k,\ k+\alpha+\beta+1\\ \beta+1\end{matrix}\,\middle|\,\frac{1+x}{2}\right)$
Laguerre	$L_k^{\alpha}(x) = (\alpha+1)_k(-1)^k\, {}_1F_1\left(\begin{matrix}-k\\ \alpha+1\end{matrix}\,\middle|\,x\right)$
Bessel	$Y_k^{\alpha}(x) = \dfrac{2^k}{(k+\alpha+1)_k}\, {}_2F_0\left(\begin{matrix}-k,\ k+\alpha+1\\ —\end{matrix}\,\middle|\,-\frac{x}{2}\right)$
Hermite	$H_k(x) = x^k\, {}_2F_0\left(\begin{matrix}-k/2,\ -k/2+1/2\\ —\end{matrix}\,\middle|\,-\frac{1}{x^2}\right)$

Table 2. Data for the monic Laguerre and Hermite polynomials

	Laguerre	Hermite
$\sigma(x)$	x	1
$\tau(x)$	$1+\alpha-x$	$-2x$
λ_k	k	$2k$
h_k	$k!\Gamma(k+\alpha+1)$	$\sqrt{\pi}\,2^{-k}k!$
$\xi_0(k)$	$k(k+\alpha)$	$k/2$
$\xi_1(k)$	$2k+\alpha+1$	0
$\xi_2(k)$	1	1
$\delta_0(k)$	$k+\alpha$	$1/2$
$\delta_1(k)$	1	0
$\delta_2(k)$	0	0

Table 3. Data for the monic Jacobi and Bessel polynomials

	Jacobi	Bessel
$\sigma(x)$	x^2-1	x^2
$\tau(x)$	$(\gamma+1)x+\delta$	$(\alpha+2)x+2$
λ_k	$-k(k+\gamma)$	$-k(k+\alpha+1)$
h_k	$2^{2k+\gamma}\dfrac{k!\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\Gamma(2k+\gamma+1)(k+\gamma)_k}$	$\dfrac{(-1)^{k+1}2^{2k}k!}{(\alpha+1)_{2k+1}(k+\alpha+1)_k}$
$\xi_0(k)$	$\dfrac{4k(k+\alpha)(k+\beta)(k+\gamma-1)}{(2k+\gamma-2)_3(2k+\gamma-1)}$	$\dfrac{-4k(k+\alpha)}{(2k+\alpha)(2k+\alpha-1)_3}$
$\xi_1(k)$	$-\dfrac{(\alpha-\beta)(\gamma-1)}{(2k+\gamma-1)(2k+\gamma+1)}$	$-\dfrac{2\alpha}{(2k+\alpha)(2k+\alpha+2)}$
$\xi_2(k)$	1	1
$\delta_0(k)$	$\dfrac{4(k+\alpha)(k+\beta)(k+\gamma-1)}{(2k+\gamma-2)_3(2k+\gamma-1)}$	$-\dfrac{4(k+\alpha)}{(2k+\alpha)(2k+\alpha-1)_3}$
$\delta_1(k)$	$\dfrac{2(\alpha-\beta)}{(2k+\gamma-1)(2k+\gamma+1)}$	$\dfrac{4}{(2k+\alpha)(2k+\alpha+2)}$
$\delta_2(k)$	$-\dfrac{1}{k+\gamma}$	$-\dfrac{1}{k+\alpha+1}$

Note: $\gamma := \alpha+\beta+1$.

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