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LIMIT THEOREMS FOR BIVARIATE EXTREMES OF NON-IDENTICALLY DISTRIBUTED RANDOM VARIABLES

Abstract. The limit behaviour of the extreme order statistics arising from n two-dimensional independent and non-identically distributed random vectors is investigated. Necessary and sufficient conditions for the weak convergence of the distribution function (d.f.) of the vector of extremes, as well as the form of the limit d.f.'s, are obtained. Moreover, conditions for the components of the vector of extremes to be asymptotically independent are studied.

1. Introduction. Only in the last few years are general results on order statistics from non-identically but independently distributed random variables (r.v.'s) coming out. However, most of these results concern recurrence relations for the d.f. (see, e.g., Cao and West, 1997) or for the single and product moments of order statistics (see, e.g., Balakrishnan, 1994, and references therein). The work on the asymptotic behaviour of order statistics from non-identical r.v.'s is still limited. An earlier result and perhaps the most important one for this problem is due to Mejlzer (1949–1956). Under a natural uniformity assumption (that the initial suitably normalized r.v.'s should, in some sense, be individually negligible in the limit), Mejlzer proved that a nondegenerate d.f. $\Phi_1(x)$ is the limiting distribution of a suitably normalized maximum for some sequence of independent r.v.'s if, and only if, either (i) $\log \Phi_1(x)$ is concave, or (ii) $\omega = \sup\{x : \Phi_1(x) < 1\} < \infty$ and $\log \Phi_1(\omega - e^{-x})$, $x > 0$, is concave, or finally (iii) $\alpha = \inf\{x : \Phi_1(x) > 0\}$ is finite and $\log \Phi_1(\alpha + e^x)$, $x > 0$, is concave. Mejlzer and Weissman (1969) extended this result to k th extremes. They proved that the d.f. of the k th extremes (suitably normalized), under the uniformity assumption, converges

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weakly to a nondegenerate limit d.f. $\Phi_k(x)$ if, and only if, the d.f. of the maximum converges weakly to the d.f. $\Phi_1(x)$. Balkema *et al.* (1993) (Theorem A.1) proved Meijler's result on limit distributions for maxima of a sequence of independent r.v.'s, replacing the uniformity assumption by less restrictive conditions (imposed on the normalizing constants). Additional interesting results on independent sequences were given in Weissman (1975a, b), Tiago de Oliveira (1976), Mucci (1977) and de Hann and Verkade (1985). A survey of recent developments can be found in Galambos (1987).

The main aim of this paper is to study the limit behaviour of the order statistics of bivariate independent non-identically distributed r.v.'s. Namely, consider n two-dimensional independent random vectors $\underline{X}_j = (X_{1j}, X_{2j})$, $j = 1, \dots, n$, with the respective d.f.'s $F_j(\underline{x}) = F_j(x_1, x_2) = P(X_{1j} \leq x_1, X_{2j} \leq x_2)$, $j = 1, \dots, n$. The order statistics of the t th components are

$$X_{t,1:n} \leq X_{t,2:n} \leq \dots \leq X_{t,n:n}, \quad t = 1, 2.$$

The main object of this paper is to investigate the limiting distribution of the random vector $\underline{Z}_{k,k':n} = (X_{1,n-k+1:n}, X_{2,n-k'+1:n})$, where k and k' are constants. Necessary and sufficient conditions under which the d.f. of $\underline{Z}_{k,k':n}$ converges weakly to a nondegenerate limit d.f., as well as the form of this limit, will be obtained in Theorem 2.1. Necessary and sufficient conditions (in terms of the original sequence of d.f.'s $\{F_j(\underline{x})\}$) for the components of $\underline{Z}_{k,k':n}$ to be asymptotically independent are derived in Theorem 2.2. The problem can also be stated in terms of $\underline{W}_{k,k':n} = (X_{1,k:n}, X_{2,k':n})$ and $\underline{V}_{k,k':n} = (X_{1,k:n}, X_{2,n-k'+1:n})$ by turning respectively to $(-X_{1j}, -X_{2j})$ and $(-X_{1j}, X_{2j})$, $j = 1, \dots, n$.

The results of this paper can be applied to many natural problems, e.g., project scheduling by PERT technique. In this practical problem we assume a large number n of different activities, each of them has a random duration and a random cost (which usually depends on the duration). In many cases, the durations of different activities are non-identically distributed r.v.'s, and so are the costs of different activities. On the other hand the duration and the cost of each activity are in general dependent r.v.'s. Therefore, we get a sequence of n two-dimensional non-identical r.v.'s $\{\underline{X}_j\} = \{(X_{1j}, X_{2j})\}$, $j = 1, \dots, n$, where X_{1j} and X_{2j} denote respectively the duration and the cost of the j th activity. It is well known that the vectors $\underline{Z}_{1,1:n}$, $\underline{W}_{1,1:n}$ and $\underline{V}_{1,1:n}$ for this sequence play a major role in the investigation of the above stated problem.

Throughout this paper the following conventions and notations will be adopted. For numerical vectors $\underline{x} = (x_1, x_2)$, the components are signified by a subscript. Basic arithmetical operations are always meant component-wise. Thus $\underline{x} \leq \underline{y}$ means $x_t \leq y_t$, $t = 1, 2$. Further, $\underline{x} \pm \underline{y} = (x_1 \pm y_1, x_2 \pm y_2)$,

$\underline{x}y = (x_1y_1, x_2y_2)$ and $\underline{x}/\underline{y} = (x_1/y_1, x_2/y_2)$. The special vectors $\underline{0} = (0, 0)$ and $\underline{\infty} = (\infty, \infty)$ will be used. Let $G_j(\underline{x}) = P(\underline{X}_j > \underline{x})$ be the survival function of $F_j(\underline{x})$, $j = 1, 2, \dots$, and let $\Phi_{\cdot, k':n}(x_2)$, $\Phi_{k, \cdot:n}(x_1)$, $F_{1j}(x_1)$, $F_{2j}(x_2)$, $G_{1j}(x_1) = 1 - F_{1j}(x_1)$ and $G_{2j}(x_2) = 1 - F_{2j}(x_2)$ be the marginal d.f.'s and the marginal survival functions of $\Phi_{k, k':n}(\underline{x}) = P(\underline{Z}_{k, k':n} \leq \underline{x})$, $F_j(\underline{x})$ and $G_j(\underline{x})$, $j = 1, 2, \dots$, respectively. Finally, for any sequence of suitable normalizing constants $\underline{a}_n = (a_{1n}, a_{2n}) > \underline{0}$ and $\underline{b}_n = (b_{1n}, b_{2n})$, write $\underline{Z}_{k, k':n}^* = (\underline{Z}_{k, k':n} - \underline{b}_n)/\underline{a}_n$, $\Phi_{k, k':n}^*(\underline{x}) = \Phi_{k, k':n}(\underline{a}_n \underline{x} + \underline{b}_n)$, $F_{j:n}^*(\underline{x}) = F_j(\underline{a}_n \underline{x} + \underline{b}_n)$, $G_{j:n}^*(\underline{x}) = G_j(\underline{a}_n \underline{x} + \underline{b}_n)$, $F_{tj:n}^*(x_t) = F_{tj}(a_{tn}x_t + b_{tn})$, $G_{tj:n}^*(x_t) = G_{tj}(a_{tn}x_t + b_{tn})$, $t = 1, 2$, $\Phi_{\cdot, k':n}^*(x_2) = \Phi_{\cdot, k':n}(a_{2n}x_2 + b_{2n})$ and $\Phi_{k, \cdot:n}^*(x_1) = \Phi_{k, \cdot:n}(a_{1n}x_1 + b_{1n})$.

We conclude this section with a theorem which is a combination of the results of Juncosa (1949), Sec. 3 (see also Galambos, 1987) and Meizler and Weissman (1969), Theorem 4.1. This theorem will be needed in what follows.

THEOREM 1.1. *Assume that, for suitable normalizing constants $a_{1n} > 0$, b_{1n} (resp. $a_{2n} > 0$, b_{2n}),*

$$(1.1) \quad \begin{aligned} \delta_{1:n} &= \max_{1 \leq j \leq n} G_{1j:n}^*(x_1) \rightarrow 0 \text{ as } n \rightarrow \infty \\ (\text{resp. } \delta_{2:n} &= \max_{1 \leq j \leq n} G_{2j:n}^*(x_2) \rightarrow 0 \text{ as } n \rightarrow \infty). \end{aligned}$$

Then $\Phi_{k, \cdot:n}^(x_1)$ (resp. $\Phi_{\cdot, k':n}^*(x_2)$) converges weakly to a nondegenerate d.f. $\Phi_{k, \cdot}(x_1)$ (resp. $\Phi_{\cdot, k'}(x_2)$) if, and only if, for all x_1 (resp. x_2) for which $\Phi_{k, \cdot}(x_1) > 0$ (resp. $\Phi_{\cdot, k'}(x_2) > 0$), the limit*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n G_{1i:n}^*(x_1) = h_1(x_1) \quad \left(\text{resp. } \lim_{n \rightarrow \infty} \sum_{i=1}^n G_{2i:n}^*(x_2) = h_2(x_2) \right)$$

is finite, and the function

$$(1.2) \quad \Phi_{k, \cdot}(x_1) = \sum_{i=0}^{k-1} \frac{h_1^i(x_1)}{i!} e^{-h_1(x_1)} \quad \left(\text{resp. } \Phi_{\cdot, k'}(x_2) = \sum_{i=0}^{k'-1} \frac{h_2^i(x_2)}{i!} e^{-h_2(x_2)} \right)$$

is a nondegenerate d.f. The actual limit d.f. of

$$Z_{k, \cdot:n}^* = \frac{X_{1, n-k+1:n} - b_{1n}}{a_{1n}} \quad \left(\text{resp. } Z_{\cdot, k':n}^* = \frac{X_{2, n-k'+1:n} - b_{2n}}{a_{2n}} \right)$$

is the one given in (1.2).

As a direct result of this theorem we see that the convergence of $\Phi_{k, \cdot:n}^*(x_1)$ (resp. $\Phi_{\cdot, k':n}^*(x_2)$) for at least one fixed value of k (resp. k') implies its convergence for all fixed values of k (resp. k'). Moreover, the possible types of the function $h_1(x_1)$ (resp. $h_2(x_2)$) may be determined from the above mentioned results of Meizler (1949–1956) (since $\Phi_{1, \cdot}(x_1) = e^{-h_1(x_1)}$ and $\Phi_{\cdot, 1}(x_2) = e^{-h_2(x_2)}$).

2. Main results. It is easy to show that the d.f. of the vector $\underline{Z}_{k,k':n}$ is given by $\Phi_{k,k':n}(\underline{x}) = P(\text{at most } k - 1 \text{ and } k' - 1 \text{ of events } \{X_{1i} > x_1\} \text{ and } \{X_{2j} > x_2\}, i, j = 1, \dots, n, \text{ occur respectively})$. Consequently, by taking into account the fact that the vectors $(X_{1j}, X_{2j}), j = 1, \dots, n$, are independent and by collecting terms according as $\{X_{1j} > x_1, X_{2j} \leq x_2\}$, or $\{X_{1j} > x_1, X_{2j} > x_2\}$, or $\{X_{1j} \leq x_1, X_{2j} > x_2\}$, or $\{X_{1j} \leq x_1, X_{2j} \leq x_2\}$, $j = 1, \dots, n$, we get

$$\begin{aligned}
 (2.1) \quad \Phi_{k,k':n}(\underline{x}) &= \sum_{t=0}^{k-1} \sum_{s=0}^{k'-1} \sum_{r=0 \vee (t+s-n)}^{t \wedge s} \sum_{\mathcal{P}_{r,s,t}} \prod_{j=1}^{t-r} P(X_{1i_j} > x_1, X_{2i_j} \leq x_2) \\
 &\quad \times \prod_{j=t-r+1}^t P(X_{1i_j} > x_1, X_{2i_j} > x_2) \\
 &\quad \times \prod_{j=t+1}^{t+s-r} P(X_{1i_j} \leq x_1, X_{2i_j} > x_2) \\
 &\quad \times \prod_{j=t+s-r+1}^n P(X_{1i_j} \leq x_1, X_{2i_j} \leq x_2) \\
 &= \sum_{t=0}^{k-1} \sum_{s=0}^{k'-1} \sum_{r=0 \vee (t+s-n)}^{t \wedge s} \sum_{\mathcal{P}_{r,s,t}} \prod_{j=1}^{t-r} (G_{1i_j}(x_1) - G_{i_j}(\underline{x})) \\
 &\quad \times \prod_{j=t-r+1}^t G_{i_j}(\underline{x}) \prod_{j=t+1}^{t+s-r} (G_{2i_j}(x_2) - G_{i_j}(\underline{x})) \\
 &\quad \times \prod_{j=t+s-r+1}^n (1 - G_{1i_j}(x_1) - G_{2i_j}(x_2) + G_{i_j}(\underline{x})),
 \end{aligned}$$

where $\min(a, b) = a \wedge b$, $\max(a, b) = a \vee b$, and for $0 \leq s < k$, $0 \leq t < k'$ and $0 \leq r \leq t \wedge s$, $\mathcal{P}_{r,s,t}$ denotes the set of permutations (i_1, \dots, i_n) of $(1, \dots, n)$ such that $i_1 < \dots < i_{t-r}, i_{t-r+1} < \dots < i_t, i_{t+1} < \dots < i_{t+s-r}$ and $i_{t+s-r+1} < \dots < i_n$.

REMARK 2.1. In (2.1), and in what follows, we adopt the convention that the product of the sort $\prod_{i=a}^b A_i$, for any integers a, b and any sequence $\{A_i\}$, is $A_a A_{a+1} \dots A_b$ if $a \leq b$, and 1 if $b < a$. Therefore, if $x_1 \rightarrow \infty$ (say) we get $G_{1j}(x_1), G_j(\underline{x}) \rightarrow 0, j = 1, \dots, n$. We thus have $t \equiv r \equiv 0$, implying

$$\begin{aligned}
 \Phi_{.,k':n}(x_2) &= \sum_{s=0}^{k'-1} \sum_{\mathcal{P}_s} \prod_{j=1}^s (G_{2i_j}(x_2) - G_{i_j}(\underline{x})) \\
 &\quad \times \prod_{j=s+1}^n (1 - G_{1i_j}(x_1) - G_{2i_j}(x_2) + G_{i_j}(\underline{x})),
 \end{aligned}$$

where \mathcal{P}_s is the set of permutations (i_1, \dots, i_n) of $(1, \dots, n)$ such that $i_1 < \dots < i_s$ and $i_{s+1} < \dots < i_n$. Moreover, when $k = k' = 1$, we get

$$\Phi_{1,1:n}(\underline{x}) = \prod_{j=1}^n (1 - G_{1j}(x_1) - G_{2j}(x_2) + G_j(\underline{x})) = \prod_{j=1}^n F_j(\underline{x}).$$

Relation (2.1) may be written in term of permanents as follows:

$$(2.2) \quad \Phi_{k,k':n}(\underline{x}) = \sum_{t=0}^{k-1} \sum_{s=0}^{k'-1} \sum_{r=0 \vee (t+s-n)}^{t \wedge s} \frac{1}{(t-r)!r!(s-r)!(n-t-s+r)!} \\ \times \text{Per} \left[\begin{array}{ccc} \underline{Y}_{1.} & -\underline{Y}_{11} & \underline{Y}_{11} \\ t-r & & r \end{array} \quad \begin{array}{ccc} \underline{Y}_{.1} - \underline{Y}_{11} & \underline{1} - \underline{Y}_{1.} & -\underline{Y}_{.1} + \underline{Y}_{11} \\ s-r & n-t-s+r & \end{array} \right],$$

where $\underline{Y}_{1.} = (G_{11}(x_1) \dots G_{1n}(x_1))'$, $\underline{Y}_{.1} = (G_{21}(x_2) \dots G_{2n}(x_2))'$, $\underline{Y}_{11} = (G_1(\underline{x}) \dots G_n(\underline{x}))'$ and $\underline{1}$ is the $n \times 1$ column vector of ones. Moreover, if $\underline{a}_1, \underline{a}_2, \dots$ are column vectors, then

$$\begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \dots \\ i_1 & i_2 & \dots \end{bmatrix}$$

will denote the matrix obtained by taking i_1 copies of \underline{a}_1 , i_2 copies of \underline{a}_2 and so on. Finally, in (2.2), $\text{Per}(A)$ denotes the permanent of a square matrix A , which is defined similarly to the determinant except that all terms in the expansion have a positive sign (see Minc's book, 1978, and the survey papers of Minc, 1983, 1987). It is worth mentioning that the permanent in (2.2) is a stochastic one.

REMARK 2.2. In view of Remark 2.1, we adopt the convention that

$$\text{Per} \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \dots & \underline{a}_j & \dots \\ i_1 & i_2 & \dots & i_{j=0} & \dots \end{bmatrix}$$

is understood as

$$\text{Per} \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \dots & \underline{a}_{j-1} & \underline{a}_{j+1} & \dots \\ i_1 & i_2 & \dots & i_{j-1} & i_{j+1} & \dots \end{bmatrix}.$$

THEOREM 2.1. Assume that the uniformity assumption

$$(2.3) \quad \delta_{t:n} = \max_{1 \leq i \leq n} G_{ti:n}^*(x_t) \rightarrow 0, \quad t = 1, 2, \text{ as } n \rightarrow \infty,$$

is satisfied. Then the d.f. $\Phi_{k,k':n}^*(\underline{x})$ converges weakly to a nondegenerate d.f. $\Phi_{k,k'}(\underline{x})$ if, and only if, for all \underline{x} for which $\Phi_{k,\cdot}(x_1)$ and $\Phi_{\cdot,k'}(x_2)$ are positive, the limits

$$(2.4) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n G_{ti:n}^*(x_t) = h_t(x_t), \quad t = 1, 2,$$

and

$$(2.5) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n G_{i:n}^*(\underline{x}) = h(\underline{x})$$

are finite, and the function

$$(2.6) \quad \Phi_{k,k'}(\underline{x}) = \sum_{t=0}^{k-1} \sum_{s=0}^{k'-1} \sum_{r=0}^{t \wedge s} \frac{1}{(t-r)!r!(s-r)!} (h_1(x_1) - h(\underline{x}))^{t-r} h^r(\underline{x}) \times (h_2(x_2) - h(\underline{x}))^{s-r} \exp(-h_1(x_1) - h_2(x_2) + h(\underline{x}))$$

is a nondegenerate d.f. The actual limit d.f. of $\underline{Z}_{k,k':n}^*$ is the one given in (2.6).

REMARK 2.3. The conditions of Theorem 2.1 show that the d.f. $\Phi_{k,k':n}^*(\underline{x})$ converges weakly to the nondegenerate d.f. $\Phi_{k,k'}(\underline{x})$ (which is defined by (2.6)), for all fixed values k and k' , if there exist at least two fixed integers k_0 and k'_0 for which the d.f. $\Phi_{k_0,k'_0:n}^*(\underline{x})$ converges weakly to the nondegenerate d.f. $\Phi_{k_0,k'_0}(\underline{x})$. Moreover, the different types of the functions $h_1(x_1)$ and $h_2(x_2)$ can be determined from Theorem 1.1. For the function $h(\underline{x})$, it is easy to show that for all \underline{x} for which $\Phi_{k, \cdot}(x_1) > 0$ and $\Phi_{\cdot, k'}(x_2) > 0$, we have either $0 \leq h(\underline{x}) < h_1(x_1) \wedge h_2(x_2) < \infty$ or $h(\underline{x}) \equiv h_1(x_1) \wedge h_2(x_2) \equiv 0$.

Proof of Theorem 2.1. Since the proof is somewhat lengthy, we split it up into several steps, some of which are of independent interest.

STEP 1. For any $0 \leq y_i < 1/2, i = 1, 2, \dots,$

$$(2.7) \quad \exp\left(-\left(1 + \max_{1 \leq j \leq n} y_j\right) \sum_{i=1}^n y_i\right) \leq \prod_{i=1}^n (1 - y_i) \leq \exp\left(-\sum_{i=1}^n y_i\right).$$

The right-hand inequality remains to hold for all $0 \leq y_i \leq 1$.

Proof. The right-hand inequality follows immediately from the trivial inequality $1 - y \leq e^{-y}$, for all y . The left-hand inequality can be proved by using the inequality $\ln(1 - y) \geq -y - y^2$ whenever $0 \leq y < 1/2$ (which is a consequence of the Taylor expansion). Therefore, for all $0 \leq y_i < 1/2$, we have

$$\begin{aligned} \ln \prod_{i=1}^n (1 - y_i) &= \sum_{i=1}^n \ln(1 - y_i) \geq -\sum_{i=1}^n y_i - \sum_{i=1}^n y_i^2 \\ &\geq -\sum_{i=1}^n y_i - \sum_{i=1}^n y_i \max_{1 \leq j \leq n} y_j \geq -\left(1 + \max_{1 \leq j \leq n} y_j\right) \sum_{i=1}^n y_i. \quad \blacksquare \end{aligned}$$

STEP 2. For each fixed s (independent of n) and for any subset $\mathcal{S} = \{i_1, \dots, i_s : 1 < i_1 < \dots < i_s < n\} \subseteq \mathcal{N} = \{1, \dots, n\}$, (2.4) and (2.5), in view of the uniformity assumptions in (2.3), yield

$$\sum_{j \in \mathcal{N} - \mathcal{S}} G_{tj:n}^*(x_t) \rightarrow h_t(x_t), \quad t = 1, 2, \text{ as } n \rightarrow \infty.$$

Moreover, since

$$\delta_n = \max_{1 \leq j \leq n} G_{j:n}^*(\underline{x}) \leq \delta_{1:n} \wedge \delta_{2:n} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

we get

$$\sum_{j \in \mathcal{N} - \mathcal{S}} G_{j:n}^*(\underline{x}) \rightarrow h(\underline{x}) \quad \text{as } n \rightarrow \infty,$$

for every fixed s and any subset $\mathcal{S} \subseteq \mathcal{N}$.

STEP 3. Let $\{a_{tj:n}\}_{j=1}^{j=n}$, $t = 1, 2, 3$, be sequences such that

$$(2.8) \quad \varepsilon_{t:n} = \max_{1 \leq j \leq n} a_{tj:n} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and

$$\sum_{i=1}^n a_{ti:n} \rightarrow a_t, \quad t = 1, 2, 3, \text{ as } n \rightarrow \infty.$$

Then, for any fixed s_1, s_2 and s_3 , we get

$$\sum_{\mathcal{Q}} \prod_{t=1}^{s_1} a_{1i_t:n} \prod_{t=s_1+1}^{s_1+s_2} a_{2i_t:n} \prod_{t=s_1+s_2+1}^{s_1+s_2+s_3} a_{3i_t:n} \rightarrow \frac{a_1^{s_1}}{s_1!} \frac{a_2^{s_2}}{s_2!} \frac{a_3^{s_3}}{s_3!} \quad \text{as } n \rightarrow \infty,$$

where $\underline{a}_{t:n} = (a_{t1:n} \dots a_{tn:n})'$, $t = 1, 2, 3$, and \mathcal{Q} is the set of permutations $(i_1, \dots, i_{s_1+s_2+s_3})$ of $(1, \dots, s_1 + s_2 + s_3)$ such that $i_1 < \dots < i_{s_1}, i_{s_1+1} < \dots < i_{s_1+s_2}$ and $i_{s_1+s_2+1} < \dots < i_{s_1+s_2+s_3}$.

Proof. In view of (2.8),

$$\left(\sum_{j=1}^n a_{1j:n} \right)^{s_1} \left(\sum_{j=1}^n a_{2j:n} \right)^{s_2} \left(\sum_{j=1}^n a_{3j:n} \right)^{s_3} \rightarrow a_1^{s_1} a_2^{s_2} a_3^{s_3},$$

which is equivalent to

$$\sum_{\substack{1 \leq i_1, i_2, \dots, i_{s_1}, i_{s_1+1}, \dots, \\ i_{s_1+s_2}, \dots, i_{s_1+s_2+s_3} \leq n}} \prod_{t \in \mathcal{C}_1} a_{1i_t:n} \prod_{t \in \mathcal{C}_2} a_{2i_t:n} \prod_{t \in \mathcal{C}_3} a_{3i_t:n} \rightarrow a_1^{s_1} a_2^{s_2} a_3^{s_3} \quad \text{as } n \rightarrow \infty,$$

where $\mathcal{C}_1 = \{1, \dots, s_1\}$, $\mathcal{C}_2 = \{s_1 + 1, \dots, s_1 + s_2\}$ and $\mathcal{C}_3 = \{s_1 + s_2 + 1, \dots, s_1 + s_2 + s_3\}$, or is equivalent to

$$\left(\sum_{\mathcal{C}} + \sum_{\mathcal{C}'} \right) \prod_{t \in \mathcal{C}_1} a_{1i_t:n} \prod_{t \in \mathcal{C}_2} a_{2i_t:n} \prod_{t \in \mathcal{C}_3} a_{3i_t:n} \rightarrow a_1^{s_1} a_2^{s_2} a_3^{s_3} \quad \text{as } n \rightarrow \infty,$$

where $\sum_{\mathcal{C}}$ signifies summation over those subscripts $(i_1, \dots, i_{s_1+s_2+s_3})$ for which all the subscripts related to each \mathcal{C}_j , $j = 1, 2, 3$, are different, i.e., $\mathcal{C} = \{1 \leq i_1 \neq \dots \neq i_{s_1} \leq n; 1 \leq i_{s_1+1} \neq \dots \neq i_{s_1+s_2} \leq n; 1 \leq i_{s_1+s_2+1} \neq \dots \neq i_{s_1+s_2+s_3} \leq n\}$ and \mathcal{C}' is the complement of \mathcal{C} . Clearly,

$$\sum_{\mathcal{C}'} \prod_{t \in \mathcal{C}_1} a_{1i_t:n} \prod_{t \in \mathcal{C}_2} a_{2i_t:n} \prod_{t \in \mathcal{C}_3} a_{3i_t:n} \leq \left(\sum_{\mathcal{C}'_1} + \sum_{\mathcal{C}'_2} + \sum_{\mathcal{C}'_3} \right) \prod_{t \in \mathcal{C}_1} a_{1i_t:n} \prod_{t \in \mathcal{C}_2} a_{2i_t:n} \prod_{t \in \mathcal{C}_3} a_{3i_t:n},$$

where $\sum_{\mathcal{C}'_j}$ signifies summation over those subscripts $(i_1, \dots, i_{s_1+s_2+s_3})$ for which at least two of the subscripts related to \mathcal{C}_j are equal, $j = 1, 2, 3$ (e.g., if $j = 2$ (say) then $\sum_{\mathcal{C}'_2}$ signifies summation over those $(i_1, \dots, i_{s_1+s_2+s_3})$ for which at least two of $1 \leq i_{s_1+1}, \dots, i_{s_1+s_2} \leq n$ are equal). On the other hand, for each $j = 1, 2, 3$, we can write

$$\sum_{\mathcal{C}'_j} \prod_{t \in \mathcal{C}_1} a_{1i_t:n} \prod_{t \in \mathcal{C}_2} a_{2i_t:n} \prod_{t \in \mathcal{C}_3} a_{3i_t:n} = \sum_{l=2}^{s_j} \sum_{\mathcal{C}'_{jl}} \prod_{t \in \mathcal{C}_1} a_{1i_t:n} \prod_{t \in \mathcal{C}_2} a_{2i_t:n} \prod_{t \in \mathcal{C}_3} a_{3i_t:n},$$

where $\sum_{\mathcal{C}'_{jl}}$, $l = 2, \dots, s_j$, signifies summation over those subscripts $(i_1, \dots, i_{s_1+s_2+s_3})$ for which l subscripts related to \mathcal{C}_j are equal, $l = 2, \dots, s_j$, $j = 1, 2, 3$, (e.g., if $j = 2$ (say) then $\sum_{\mathcal{C}'_{2l}}$ signifies summation over those $(i_1, \dots, i_{s_1+s_2+s_3})$ for which l of $1 \leq i_{s_1+1}, \dots, i_{s_1+s_2} \leq n$ are equal). However, in view of (2.8), for each $l = 2, \dots, s_j$ and $j = 1, 2, 3$, it is easy to verify that, as $n \rightarrow \infty$,

$$\begin{aligned} & \sum_{\mathcal{C}'_{1l}} \prod_{t \in \mathcal{C}_1} a_{1i_t:n} \prod_{t \in \mathcal{C}_2} a_{2i_t:n} \prod_{t \in \mathcal{C}_3} a_{3i_t:n} \\ & \leq \varepsilon_{1:n}^{l-1} \sum_{1 \leq i_1, \dots, i_{s_1+s_2+s_3-l+1} \leq n} \prod_{t \in \mathcal{C}_{11}} a_{1i_t:n} \prod_{t \in \mathcal{C}_{21}} a_{2i_t:n} \prod_{t \in \mathcal{C}_{31}} a_{3i_t:n} \rightarrow 0, \\ & \sum_{\mathcal{C}'_{2l}} \prod_{t \in \mathcal{C}_1} a_{1i_t:n} \prod_{t \in \mathcal{C}_2} a_{2i_t:n} \prod_{t \in \mathcal{C}_3} a_{3i_t:n} \\ & \leq \varepsilon_{2:n}^{l-1} \sum_{1 \leq i_1, \dots, i_{s_1+s_2+s_3-l+1} \leq n} \prod_{t \in \mathcal{C}_1} a_{1i_t:n} \prod_{t \in \mathcal{C}_{22}} a_{2i_t:n} \prod_{t \in \mathcal{C}_{32}} a_{3i_t:n} \rightarrow 0, \end{aligned}$$

and

$$\sum_{\mathcal{C}'_{3l}} \prod_{t \in \mathcal{C}_1} a_{1i_t:n} \prod_{t \in \mathcal{C}_2} a_{2i_t:n} \prod_{t \in \mathcal{C}_3} a_{3i_t:n} \leq \varepsilon_{3:n}^{l-1} \sum_{1 \leq i_1, \dots, i_{s_1+s_2+s_3-l+1} \leq n} \prod_{t \in \mathcal{C}_1} a_{1i_t:n} \prod_{t \in \mathcal{C}_2} a_{2i_t:n} \prod_{t \in \mathcal{C}_{33}} a_{3i_t:n} \rightarrow 0,$$

where $\mathcal{C}_{11} = \{1, \dots, s_1 - l + 1\}$, $\mathcal{C}_{21} = \{s_1 - l + 2, \dots, s_1 + s_2 - l + 1\}$, $\mathcal{C}_{31} = \{s_1 + s_2 - l + 2, \dots, s_1 + s_2 + s_3 - l + 1\}$, $\mathcal{C}_{22} = \{s_1 + 1, \dots, s_1 + s_2 - l + 1\}$, $\mathcal{C}_{32} = \{s_1 + s_2 - l + 2, \dots, s_1 + s_2 + s_3 - l + 1\}$ and $\mathcal{C}_{33} = \{s_1 + s_2 + 1, \dots$

$\dots, s_1 + s_2 + s_3 - l + 1\}$, from which we deduce that

$$\sum_{C'} \prod_{t \in C_1} a_{1i_t:n} \prod_{t \in C_2} a_{2i_t:n} \prod_{t \in C_3} a_{3i_t:n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

On the other hand, we have

$$\begin{aligned} \sum_C \prod_{t \in C_1} a_{1i_t:n} \prod_{t \in C_2} a_{2i_t:n} \prod_{t \in C_3} a_{3i_t:n} \\ = s_1!s_2!s_3! \sum_Q \prod_{t \in C_1} a_{1i_t:n} \prod_{t \in C_2} a_{2i_t:n} \prod_{t \in C_3} a_{3i_t:n}. \end{aligned}$$

Hence the result. ■

STEP 4. Let n be sufficiently large such that $G_{ti:n}^*(x_t) < 1/2$, $t = 1, 2$ (consequently, $G_{i:n}^*(\underline{x}) < 1/2$). Then

$$\begin{aligned} (2.9) \quad & \sum_{t=0}^{k-1} \sum_{s=0}^{k'-1} \sum_{r=0 \vee (t+s-n)}^{t \wedge s} \sum_{\mathcal{P}_{r,s,t}} \prod_{j=1}^{t-r} (G_{1i_j:n}^*(x_1) - G_{i_j:n}^*(\underline{x})) \\ & \times \prod_{j=t-r+1}^t G_{i_j:n}^*(\underline{x}) \prod_{j=t+1}^{t+s-r} (G_{2i_j:n}^*(x_2) - G_{i_j:n}^*(\underline{x})) \\ & \times e^{-(1+\delta_{1:n}+\delta_{2:n}) \sum_{j=t+s-r+1}^n (G_{1i_j:n}^*(x_1) + G_{2i_j:n}^*(x_2) - G_{i_j:n}^*(\underline{x}))} \\ & \leq \Phi_{k,k':n}^*(\underline{x}) \\ & \leq \sum_{t=0}^{k-1} \sum_{s=0}^{k'-1} \sum_{r=0 \vee (t+s-n)}^{t \wedge s} \sum_{\mathcal{P}_{r,s,t}} \prod_{j=1}^{t-r} (G_{1i_j:n}^*(x_1) - G_{i_j:n}^*(\underline{x})) \prod_{j=t-r+1}^t G_{i_j:n}^*(\underline{x}) \\ & \times \prod_{j=t+1}^{t+s-r} (G_{2i_j:n}^*(x_2) - G_{i_j:n}^*(\underline{x})) e^{-\sum_{j=t+s-r+1}^n (G_{1i_j:n}^*(x_1) + G_{2i_j:n}^*(x_2) - G_{i_j:n}^*(\underline{x}))}. \end{aligned}$$

Proof. Upon replacing the functions $\Phi_{k,k':n}(\underline{x})$, $G_{1j}(x_1)$, $G_{2j}(x_2)$ and $G_j(\underline{x})$, $j = 1, \dots, n$, in (2.1) respectively by $\Phi_{k,k':n}^*(\underline{x})$, $G_{1j:n}^*(x_1)$, $G_{2j:n}^*(x_2)$ and $G_{j:n}^*(\underline{x})$, $j = 1, \dots, n$, the proof follows immediately by applying inequality (2.7) to (2.1) (note that $\max_{t+s-r+1 \leq j \leq n} (G_{1i_j:n}^*(x_1) + G_{2i_j:n}^*(x_2) - G_{i_j:n}^*(\underline{x})) \leq \delta_{1:n} + \delta_{2:n}$). ■

We now conclude the first part of Theorem 2.1 by taking the limit of (2.9), as $n \rightarrow \infty$, as follows. Apply Step 3 with $\underline{a}_{1:n} = (G_{11:n}^*(x_1) - G_{1:n}^*(\underline{x}), G_{12:n}^*(x_1) - G_{2:n}^*(\underline{x}), \dots, G_{1n:n}^*(x_1) - G_{n:n}^*(\underline{x}))'$, $\underline{a}_{2:n} = (G_{1:n}^*(\underline{x}), \dots, G_{n:n}^*(\underline{x}))'$, $\underline{a}_{3:n} = (G_{21:n}^*(x_2) - G_{1:n}^*(\underline{x}), G_{22:n}^*(x_2) - G_{2:n}^*(\underline{x}), \dots, G_{2n:n}^*(x_2) - G_{n:n}^*(\underline{x}))'$, $s_1 = t - r$, $s_2 = r$ and $s_3 = s - r$. On the other hand, by applying Step 2 with $\mathcal{S} = \{i_1, \dots, i_{s_1+s_2+s_3}\}$, the sum in the exponent of e in (2.9) tends to $h_1(x_1) + h_2(x_2) - h(\underline{x})$. This completes the proof. ■

We now turn to the proof of the converse of the theorem. The idea of the proof is based heavily on the proof of Theorem 5.3.1 of Galambos (1987) (the converse part of Theorem 5.3.1). Namely, we assume that the uniformity assumptions in (2.3) are satisfied. Furthermore, assume the d.f. of $Z_{k,k':n}^*$ converges weakly to a limit d.f. $\Phi_{k,k'}(\underline{x})$, given by (2.6). Let \underline{x} be such that $\Phi_{k,\cdot}(x_1) > 0$ and $\Phi_{\cdot,k'}(x_2) > 0$. We shall prove the validity of (2.4), for $t = 1, 2$, and (2.5). The proof will be accomplished in the following two steps.

STEP 5. For any fixed h_1, h_2 and all values of h for which $0 < h < h_1 \wedge h_2 < \infty$, the function

$$\begin{aligned} &\tilde{\Phi}_{k,k'}(h_1, h_2, h) \\ &= \sum_{t=0}^{k-1} \sum_{s=0}^{k'-1} \sum_{r=0}^{t \wedge s} \frac{1}{(t-r)!r!(s-r)!} (h_1-h)^{t-r} h^r (h_2-h)^{s-r} \exp(-h_1-h_2+h) \end{aligned}$$

is uniquely determined by h .

Proof. The proof immediately follows if we observe that

$$\begin{aligned} \exp(h_1+h_2-h) \frac{\partial \tilde{\Phi}_{k,k'}(h_1, h_2, h)}{\partial h} &= - \sum_{j=0}^{k'-1} \sum_{i=1}^{k-1} \sum_{r=0}^{(i-1) \wedge j} \frac{(h_1-h)^{i-r-1} h^r (h_2-h)^{j-r}}{(i-r-1)!r!(j-r)!} \\ &\quad - \sum_{j=1}^{k'-1} \sum_{i=0}^{k-1} \sum_{r=0}^{i \wedge (j-1)} \frac{(h_1-h)^{i-r} h^r (h_2-h)^{j-r-1}}{(i-r)!r!(j-r-1)!} \\ &\quad + \sum_{j=1}^{k'-1} \sum_{i=1}^{k-1} \sum_{r=1}^{i \wedge j} \frac{(h_1-h)^{i-r} h^{r-1} (h_2-h)^{j-r}}{(i-r)!(r-1)!(j-r)!} \\ &\quad + \sum_{j=0}^{k'-1} \sum_{i=0}^{k-1} \sum_{r=0}^{i \wedge j} \frac{(h_1-h)^{i-r} h^r (h_2-h)^{j-r}}{(i-r)!r!(j-r)!}, \end{aligned}$$

which after routine calculations yields

$$\begin{aligned} &\frac{\partial \tilde{\Phi}_{k,k'}(h_1, h_2, h)}{\partial h} \\ &= \sum_{r=0}^{(k-1) \wedge (k'-1)} \frac{(h_1-h)^{k-r-1} h^r (h_2-h)^{k'-r-1}}{(k-r-1)!r!(k'-r-1)!} \exp(-h_1-h_2+h) > 0, \end{aligned}$$

when h_1 and h_2 are fixed and $0 < h < h_1 \wedge h_2 < \infty$, i.e., the function $\tilde{\Phi}_{k,k'}(h_1, h_2, h)$ is strictly increasing in h when h_1 and h_2 are fixed (see Lemma 2.2 of Barakat, 1999, and Barakat, 1997). Hence the result. ■

STEP 6. Since (2.6) implies (2.4) for $t = 1, 2$ (by letting respectively $x_1 \rightarrow \infty, x_2 \rightarrow \infty$ and by applying Theorem 1.1), the elementary inequalities $G_{j:n}^*(\underline{x}) \leq G_{1j:n}^*(x_1) \wedge G_{2j:n}^*(x_2), j = 1, \dots, n$, yield that $\{\sum_{j=1}^n G_{j:n}^*(\underline{x})\}$ is bounded. Therefore, we can select a subsequence $\{n'\}$ of $\{n\}$ for which (2.5) holds. Let us repeat the first part of the proof (Steps 1–4) for this subsequence. We find that the limit $\Phi_{k,k'}(\underline{x})$ of $\Phi_{k,k':n'}^*(\underline{x})$ satisfies (2.6) where the limit in (2.5) may depend on the actual subsequence $\{n'\}$. However, in view of the result of Step 5, we conclude from the representation (2.6) that (2.5) cannot depend on n' . This completes the proof. ■

COROLLARY 2.1. *Let (2.3)–(2.5) be satisfied with $a_{1n} = a_{2n}$ and $b_{1n} = b_{2n}$. Then $\omega_1 \equiv \omega_2 \leq \infty$, where $\omega_1 = \sup\{x_1 : \Phi_{k,\cdot}(x_1) < 1\}$ and $\omega_2 = \sup\{x_2 : \Phi_{\cdot,k'}(x_2) < 1\}$.*

Proof. We first note that Theorem A.3 of Balkema *et al.* (1993) is closely related to the special case $k = k' = 1$ of Theorem 2.1. Therefore, together with Remark 2.3 it implies the conclusion of the present corollary. ■

The following result is an immediate consequence of Theorem 2.1.

COROLLARY 2.2. *Assume that $\underline{Z}_{k,k':n}^*$ has a nondegenerate asymptotic distribution $\Phi_{k,k'}(\underline{x})$. Then the components of $\underline{Z}_{k,k':n}^*$ are asymptotically independent if, and only if, the limit in (2.5) is identically zero i.e., $h(\underline{x}) = 0$.*

We are now in a position to obtain a general condition under which the components of the vector $\underline{Z}_{k,k':n}^*$ will be asymptotically independent. The following remark will be needed.

REMARK 2.4. Let $\bar{F}_{t:n}(x_t) = n^{-1} \sum_{j=1}^n F_{tj:n}(x_t), t = 1, 2$, and $\bar{F}_n(\underline{x}) = n^{-1} \sum_{j=1}^n F_j(\underline{x})$. Then, keeping the notations of Section 1, (2.4) and (2.5) may be written respectively as $n\bar{G}_{t:n}^*(x_t) = n(1 - \bar{F}_{t:n}^*(x_t)) \rightarrow h_t(x_t), t = 1, 2$, as $n \rightarrow \infty$ and $n\bar{G}_n^*(\underline{x}) \rightarrow h(\underline{x})$ as $n \rightarrow \infty$, where $\bar{G}_{t:n}^*(x_t), t = 1, 2$, and $\bar{G}_n^*(\underline{x})$ denote respectively the survival functions of $\bar{F}_{t:n}^*(x_t), t = 1, 2$, and $\bar{F}_n^*(\underline{x})$.

THEOREM 2.2. *Let $\hat{x}_{t:n} = \sup\{x_t : \bar{F}_{t:n}(x_t) < 1\}, t = 1, 2$. Let further $\lim_{n \rightarrow \infty} \hat{x}_{t:n} = \hat{x}_t \leq \infty, t = 1, 2$. Then, the components of the vector $\underline{Z}_{k,k':n}^*$ are asymptotically independent if, and only if*

$$(2.10) \quad \lim_{\substack{n \rightarrow \infty \\ \underline{x} \rightarrow \hat{\underline{x}}}} \frac{\bar{G}_n(\underline{x})}{1 - \bar{F}_n(\underline{x})} = 0,$$

where $\hat{\underline{x}} = (\hat{x}_1, \hat{x}_2)$.

Proof. We first observe that, in view of Corollary 2.2, the components of the vector $\underline{Z}_{k,k':n}^*$ are asymptotically independent if, and only if, $h(\underline{x}) = 0$. Thus, in order to prove the theorem, we have to prove $h(\underline{x}) = 0$ if, and only

if, (2.10) holds. On the other hand, in view of the elementary relation

$$(2.11) \quad \overline{G}_n(\underline{x}) = \overline{G}_{1:n}(x_1) + \overline{G}_{2:n}(x_2) - (1 - \overline{F}_n(\underline{x})),$$

(2.10) is equivalent to

$$(2.12) \quad \lim_{\substack{n \rightarrow \infty \\ \underline{x} \rightarrow \hat{\underline{x}}}} \frac{\overline{G}_{1:n}(x_1) + \overline{G}_{2:n}(x_2)}{\overline{G}_{1:n}(x_1) + \overline{G}_{2:n}(x_2) - \overline{G}_n(\underline{x})} = 1.$$

Suppose now that (2.12) holds. Appealing to (2.4), (2.5) and Remark 2.4, we can easily get

$$\frac{h_1(x_1) + h_2(x_2)}{h_1(x_1) + h_2(x_2) - h(\underline{x})} = 1$$

for all \underline{x} such that $h_1(x_1), h_2(x_2) < \infty$, i.e., for all \underline{x} such that $\Phi_{k,\cdot}(x_1), \Phi_{\cdot,k'}(x_2) > 0$ (note that, in view of Remark 2.4, $\underline{a}_n \underline{x} + \underline{b}_n \rightarrow \hat{\underline{x}}$ when $\Phi_{k,\cdot}(x_1), \Phi_{\cdot,k'}(x_2) > 0$), which yields $h_1(x_1) + h_2(x_2) - h(\underline{x}) = h_1(x_1) + h_2(x_2)$, i.e., $h(\underline{x}) = 0$. Conversely, suppose that $h(\underline{x}) = 0$ and (2.10) does not hold. Then there exist $\varepsilon > 0$ and sequences $\{\underline{v}_m\} = \{(v_{1m}, v_{2m})\}$, $\{n_m\}$ such that $\underline{v}_m \rightarrow \hat{\underline{x}}$ and $n_m \rightarrow \infty$ as $m \rightarrow \infty$, and

$$(2.13) \quad \overline{G}_{1:n_m}(v_{1m}) + \overline{G}_{2:n_m}(v_{2m}) \geq \overline{G}_{n_m}(\underline{v}_m) \geq \varepsilon(1 - \overline{F}_{n_m}(\underline{v}_m)).$$

Now choose a subsequence $\{n_l^*\} = \{n_{m_l}\}$ of $\{n_m\}$ so that

$$(2.14) \quad \begin{aligned} \frac{h_1(x_1) + h_2(x_2)}{n_l^* + 1} &\leq \overline{G}_{1:n_l^*}(v_{1l}^*) + \overline{G}_{2:n_l^*}(v_{2l}^*) - 2\overline{G}_{n_l^*}(\underline{v}_l^*) \\ &\leq \overline{G}_{1:n_l^*}(v_{1l}^*) + \overline{G}_{2:n_l^*}(v_{2l}^*) \leq \frac{h_1(x_1) + h_2(x_2)}{n_l^*}. \end{aligned}$$

(Note that $n\overline{G}_n^*(\underline{x}) \rightarrow 0$. Therefore, $n_l^*\overline{G}_{n_l^*}^*(\underline{v}_l^*) \rightarrow 0$, where $\underline{v}_{m_l} = \underline{v}_l^* = (v_{1l}^*, v_{2l}^*)$ and $\{\underline{v}_{m_l}\}$ is a subsequence of $\{\underline{v}_m\}$.) Clearly, we have either

- (i) $\overline{F}_{n_l^*}^*(\underline{x}) \geq \overline{F}_{n_l^*}^*(\underline{v}_l^*)$ for infinitely many l , or
- (ii) $\overline{F}_{n_l^*}^*(\underline{x}) < \overline{F}_{n_l^*}^*(\underline{v}_l^*)$ for infinitely many l .

If (i) holds, then for such l ,

$$(2.15) \quad n_l^*(1 - \overline{F}_{n_l^*}^*(\underline{x})) \leq n_l^*(1 - \overline{F}_{n_l^*}^*(\underline{v}_l^*)).$$

Clearly, we have

$$\begin{aligned} n_l^*(1 - \overline{F}_{n_l^*}^*(\underline{v}_l^*)) &= h_1(x_1) + h_2(x_2) \\ &\quad + n_l^* \left(-\frac{h_1(x_1) + h_2(x_2)}{n_l^*} + \overline{G}_{1:n_l^*}(v_{1l}^*) + \overline{G}_{2:n_l^*}(v_{2l}^*) - \overline{G}_{n_l^*}(v_l^*) \right) \\ &\leq h_1(x_1) + h_2(x_2) \\ &\quad + n_l^* \left(-\frac{h_1(x_1) + h_2(x_2)}{n_l^*} + \frac{h_1(x_1) + h_2(x_2)}{n_l^*} - \varepsilon(1 - \overline{F}_{n_l^*}^*(\underline{v}_l^*)) \right), \end{aligned}$$

which yields

$$(1 + \varepsilon)n_l^*(1 - \bar{F}_{n_l^*}^*(y_l^*)) \leq h_1(x_1) + h_2(x_2).$$

It follows that

$$\liminf_{l \rightarrow \infty} n_l^*(1 - \bar{F}_{n_l^*}^*(y_l^*)) < h_1(x_1) + h_2(x_2),$$

and hence by (2.15),

$$\liminf_{l \rightarrow \infty} n_l^*(1 - \bar{F}_{n_l^*}^*(\underline{x})) < h_1(x_1) + h_2(x_2),$$

which contradicts the fact that $n(1 - \bar{F}_n^*(\underline{x})) = n\bar{G}_{1:n}^*(x_1) + n\bar{G}_{2:n}^*(x_2) - n\bar{G}_n^*(\underline{x}) \rightarrow h_1(x_1) + h_2(x_2)$ as $n \rightarrow \infty$ (note that $n\bar{G}_n^*(\underline{x}) \rightarrow 0$).

If (ii) holds, then for such ℓ ,

$$(2.16) \quad n_\ell^*(1 - \bar{F}_{n_\ell^*}^*(\underline{x})) > n_\ell^*(1 - \bar{F}_{n_\ell^*}^*(y_\ell^*)).$$

Clearly, we have (in view of (2.13) and (2.14))

$$\begin{aligned} n_l^*(1 - \bar{F}_{n_l^*}^*(y_l^*)) &= h_1(x_1) + h_2(x_2) \\ &+ n_l^* \left(-\frac{h_1(x_1) + h_2(x_2)}{n_l^*} + (\bar{G}_{1:n_l^*}(v_{1l}^*) + \bar{G}_{2:n_l^*}(v_{2l}^*) - 2\bar{G}_{n_l^*}(y_l^*)) + \bar{G}_{n_l^*}(y_l^*) \right) \\ &\geq h_1(x_1) + h_2(x_2) \\ &+ n_l^* \left(-\frac{h_1(x_1) + h_2(x_2)}{n_l^*} + \frac{h_1(x_1) + h_2(x_2)}{n_l^* + 1} + \varepsilon(1 - \bar{F}_{n_l^*}^*(y_l^*)) \right) \\ &= h_1(x_1) + h_2(x_2) + \varepsilon n_l^*(1 - \bar{F}_{n_l^*}^*(y_l^*)) - \frac{h_1(x_1) + h_2(x_2)}{n_l^* + 1}, \end{aligned}$$

which yields

$$(1 - \varepsilon)n_l^*(1 - \bar{F}_{n_l^*}^*(y_l^*)) \geq h_1(x_1) + h_2(x_2) - \frac{h_1(x_1) + h_2(x_2)}{n_l^* + 1}.$$

Since clearly $n_l^* \rightarrow \infty$, it follows that (since $0 < h_1(x_1) + h_2(x_2) < \infty$ by assumption)

$$\limsup_{l \rightarrow \infty} n_l^*(1 - \bar{F}_{n_l^*}^*(y_l^*)) > h_1(x_1) + h_2(x_2),$$

and hence by (2.16),

$$\limsup_{l \rightarrow \infty} n_l^*(1 - \bar{F}_{n_l^*}^*(\underline{x})) > h_1(x_1) + h_2(x_2),$$

which contradicts again the fact that $n(1 - \bar{F}_n^*(\underline{x})) = n\bar{G}_{1:n}^*(x_1) + n\bar{G}_{2:n}^*(x_2) - n\bar{G}_n^*(\underline{x}) \rightarrow h_1(x_1) + h_2(x_2)$, as $n \rightarrow \infty$. Hence the result. ■

REMARK 2.5. Theorem 2.2 can be considered as a substantial generalization of Theorem 2.3 of Barakat (1999). On the other hand, Theorem 2.3

proves the necessity of the Geffroy condition (see Sarhan and Greenberg, 1962) in the case of independent and identically distributed r.v.'s.

We now briefly go over parallel results for $\underline{V}_{k,k':n}$. Keeping the notation of Remark 2.1 and using the elementary relation $\min(X_{11}, X_{12}, \dots, X_{1n}) = -\max(-X_{11}, -X_{12}, \dots, -X_{1n})$, it is easy to write explicitly the d.f. of $\underline{V}_{k,k':n}$:

$$\Psi_{k,k':n}(\underline{x}) = \Phi_{.,k':n}(x_2) - M_{k,k':n}(\underline{x}),$$

where

$$\begin{aligned} M_{k,k':n}(\underline{x}) &= P(X_{1,k:n} \geq x_1, X_{2,n-k'+1:n} < x_2) \\ &= \sum_{t=0}^{k-1} \sum_{s=0}^{k'-1} \sum_{r=0 \vee (t+s-n)}^{t \wedge s} \sum_{\mathcal{P}_{r,s,t}} \prod_{j=1}^{t-r} F_{i_j}(\underline{x}) \\ &\quad \times \prod_{j=t-r+1}^t (F_{1i_j}(x_1) - F_{i_j}(\underline{x})) \prod_{j=t+1}^{t+s-r} G_{i_j}(\underline{x}) \\ &\quad \times \prod_{j=t+s-r+1}^n (1 - G_{2i_j}(x_2) + F_{i_j}(\underline{x})). \end{aligned}$$

Moreover, for any sequence of suitable normalizing constants $\underline{a}'_n = (\alpha_{1n}, a_{2n}) > \underline{0}$ and $\underline{b}'_n = (\beta_{1n}, b_{2n})$, in view of the identity $\sum_{i=1}^n G_{i:n}^*(\underline{x}) = \sum_{i=1}^n G_{2i:n}^*(x_2) - \sum_{i=1}^n F_{1i:n}^*(x_1) + \sum_{i=1}^n F_{i:n}^*(\underline{x})$ we have the following theorem.

THEOREM 2.3. *Assume that the uniformity assumptions*

$$\delta'_{1:n} = \max_{1 \leq i \leq n} F_{1i:n}^*(x_1) \rightarrow 0 \quad \text{and} \quad \delta_{2:n} = \max_{1 \leq i \leq n} G_{2i:n}^*(x_2) \rightarrow 0$$

as $n \rightarrow \infty$ are satisfied. Then the d.f. $\Psi_{k,k':n}^*(\underline{x}) = \Psi_{k,k':n}(\underline{a}'_n \underline{x} + \underline{b}'_n)$ converges weakly to a nondegenerate d.f. $\Psi_{k,k'}(\underline{x})$ if, and only if, for all \underline{x} for which the two marginals $\Psi_{k, \cdot}(x_1)$ and $\Psi_{\cdot, k'}(x_2)$ of $\Psi_{k,k'}(\underline{x})$ are positive, the limits

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n F_{1i:n}^*(x_1) = \ell_1(x_1), \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n G_{2i:n}^*(x_2) = h_2(x_2)$$

are finite, and moreover at least one of the limits

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n G_{i:n}^*(\underline{x}) = h'(\underline{x}), \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n F_{i:n}^*(\underline{x}) = \ell'(\underline{x})$$

is finite. The function $\Psi_{k,k'}(\underline{x}) = \Phi_{.,k'}(x_2) - M_{k,k'}(\underline{x})$, where

$$\begin{aligned} (2.17) \quad M_{k,k'}(\underline{x}) &= \sum_{t=0}^{k-1} \sum_{s=0}^{k'-1} \sum_{r=0}^{t \wedge s} \frac{1}{(t-r)!r!(s-r)!} \\ &\quad \times \ell'^{t-r}(\underline{x})(\ell_1(x_1) - \ell'(\underline{x}))^r h'^{s-r}(\underline{x}) e^{-h_2(x_2) - \ell'(\underline{x})}, \end{aligned}$$

is a nondegenerate d.f. The actual limit d.f. of $\underline{V}_{k,k':n}^* = (\underline{V}_{k,k':n}^* - \underline{b}'_n) / \underline{a}'_n$ is the one given in (2.17). The components of $\underline{V}_{k,k':n}^*$ are asymptotically independent if and only if either $h'(\underline{x}) = h_2(x_2)$ or $\ell'(\underline{x}) = \ell_1(x_1)$.

Finally, we have the following theorem.

THEOREM 2.3. *Let $\tilde{x}_{1:n} = \inf\{x_1 : \bar{F}_{1:n}(x_1) > 0\}$. Let further $\lim_{n \rightarrow \infty} \tilde{x}_{1:n} = \tilde{x}_1 \geq -\infty$. Then the components of the vector $\underline{V}_{k,k':n}^*$ are asymptotically independent if, and only if,*

$$\lim_{\substack{n \rightarrow \infty \\ \underline{x} \rightarrow (\tilde{x}_1, \tilde{x}_2)}} \frac{\bar{G}_{2:n}(x_2) - \bar{G}_n(\underline{x})}{\bar{G}_{2:n}(x_2) + \bar{F}_n(\underline{x})} = 0 \quad \left(\text{or} \quad \lim_{\substack{n \rightarrow \infty \\ \underline{x} \rightarrow (\tilde{x}_1, \tilde{x}_2)}} \frac{\bar{F}_{1:n}(x_1) - \bar{F}_n(\underline{x})}{\bar{G}_{2:n}(x_2) + \bar{F}_n(\underline{x})} = 0 \right).$$

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