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Ioannis K. Argyros (Lawton, OK)

LOCAL CONVERGENCE THEOREMS FOR NEWTON'S METHOD FROM DATA AT ONE POINT

Abstract. We provide local convergence theorems for the convergence of Newton's method to a solution of an equation in a Banach space utilizing only information at one point. It turns out that for analytic operators the convergence radius for Newton's method is enlarged compared with earlier results. A numerical example is also provided that compares our results favorably with earlier ones.

1. Introduction. In this study, we are concerned with the problem of approximating a solution x^* of an equation

$$F(x) = 0, \tag{1}$$

where F is sufficiently many times Fréchet-differentiable on an open, convex subset D of a Banach space X, with values in a Banach space Y.

Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0,\ x_0 \in D) \tag{2}$$

has been used to generate a sequence converging to x^*. There is an extensive literature on local and semilocal convergence theorems for Newton's method. We refer the reader to [1]–[9] and the references there for such results.

Here we introduce some local results for Newton's method, which enable us to obtain a convergence radius larger than in earlier results [5], [7]–[10]. That is, we obtain a wider range of initial choices x_0 than it was possible before. This information is important and also finds applications in step length

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selection in predictor–corrector continuation procedures [4], [5], [7]–[10]. See also Remark 5 for other applications.

At the end of the study we provide a numerical example to show that indeed our results can provide a larger convergence radius than before.

2. Convergence analysis. We state the following local convergence theorem for Newton's method:

THEOREM 1. *Let $F : D \subseteq X \to Y$ be twice Fréchet-differentiable. Assume*:

(a) *there exists a simple zero $x^* \in D$ of F*;

(b) *there exists a constant $\ell \geq 0$ such that*

$$\|F'(x^*)^{-1}F''(x)\| \leq \ell \quad (x \in D); \tag{3}$$

(c)

$$\overline{U}\left(x^*, r_1 = \frac{2}{3\ell}\right) = \{x \in X \mid \|x - x^*\| \leq r_1\} \subseteq D. \tag{4}$$

Then Newton's method $\{x_n\}$ $(n \geq 0)$ generated by (2) *is well defined, remains in $\overline{U}(x^*, r_1)$ for all $n \geq 0$, and converges to x^* provided that $x_0 \in \overline{U}(x^*, r_1)$. Moreover, the following error bounds hold for all $n \geq 0$*:

$$\|x_{n+1} - x^*\| \leq \frac{\ell}{2[1 - \ell\|x_n - x^*\|]}\,\|x_n - x^*\|^2. \tag{5}$$

Proof. It follows from (3) that there exists $D_0 \subseteq D$ such that F' is ℓ-Lipschitz on D_0, i.e.,

$$\|F'(x^*)^{-1}[F'(x) - F'(y)]\| \leq \ell\|x - y\| \quad (x \in D_0). \tag{6}$$

Without loss of generality we can assume $D_0 = D$. The rest of the theorem follows exactly as in [9]. ∎

REMARK 1. In order for us to replace ℓ in Lipschitz conditions or as a bound on Fréchet derivatives in convergence theorems for Newton's method, assume F is analytic on D, set

$$\gamma(x) = \sup_{k>1}\left\|\frac{1}{k!}\,F'(x)^{-1}F^{(k)}(x)\right\|^{1/(k-1)} \quad (x \in D), \tag{7}$$

and

$$\gamma = \gamma(x^*). \tag{8}$$

Moreover, assume that

$$\overline{U}(x^*, r/\gamma) \subseteq D, \quad r \in [0, 1/\gamma).$$

Then, for all $x \in U(x^*, r)$, $i = 1, 2, \ldots,$ we get

$$\text{(9)}\quad \|F'(x^*)^{-1}F^{(i+1)}(x)\| = \left\| \sum_{k=0}^{\infty} \frac{1}{k!}\, F'(x^*)^{-1}F^{(i+k+1)}(x^*)(x-x^*)^k \right\|$$
$$\le \sum_{k=0}^{\infty}(k+i+1)(k+i)\gamma^{k+i}\|x-x^*\|^k$$
$$= \gamma^i \sum_{k=0}^{\infty}(k+i+1)(k+i)(\gamma\|x-x^*\|)^k$$
$$\le \delta_{i+1} \equiv \frac{(i+1)!\gamma^i}{(1-\gamma r)^{i+2}}.$$

It follows from (6) that ℓ can be replaced by δ_2 ($\gamma \neq 0$). In this case the convergence condition is

$$\text{(10)}\qquad r \le \frac{(1-\gamma r)^3}{3\gamma},$$

which becomes

$$\text{(11)}\qquad z^3 - 3z^2 + 6z - 1 \le 0, \qquad z = r\gamma.$$

Solving (11) we finally deduce that Newton's method converges, provided that

$$x_0 \in \overline{U}(x^*, r_2) \subseteq D,$$

where

$$\text{(12)}\qquad r_2 = \frac{.182269}{\gamma} \qquad (\gamma \neq 0).$$

Hence, we showed:

THEOREM 2. *Let* $F : D \subseteq X \to Y$ *be analytic*, x^* *be as in Theorem* 1, γ *and* r_2 *as defined by* (8) *and* (12) *respectively. Moreover*, *assume*:

$$\text{(13)}\qquad r_2 \in (0, 1/\gamma),$$
$$\text{(14)}\qquad x_0 \in \overline{U}(x_0, r_2),$$
$$\text{(15)}\qquad \overline{U}(x_0, r_2) \subseteq D.$$

Then the conclusions of Theorem 1 *for Newton's method hold with* δ_2 *and* r_2 *replacing* ℓ *and* r_1 *respectively.*

The following local convergence theorem was essentially proved, e.g., [2] or [3].

THEOREM 3. *Let* $F : D \subseteq X \to Y$ *be an* $(m+1)$*-times* ($m \ge 2$, *an integer*) *Fréchet-differentiable operator and* x^* *be as in Theorem* 1. *Assume that there exist nonnegative constants* α_j, $j = 2, \ldots, m+1$, *such that*:

$$\|F'(x^*)^{-1}F^{(j)}(x^*)\| \le \alpha_j, \qquad j = 2, \dots, m, \tag{16}$$

$$\|F'(x^*)^{-1}F^{(m+1)}(x)\| \le \alpha_{m+1} \qquad (x \in D). \tag{17}$$

Denote by r_3 *the minimum positive zero, guaranteed to exist by Descartes' rule of signs, of the function* g *given by*

$$g(r) = \beta_m r^m + \beta_{m-1} r^{m-1} + \dots + \beta_1 r + \beta_0, \tag{18}$$

where

$$\beta_m = \frac{2m+1}{(m+1)!}\,\alpha_{m+1}, \tag{19}$$

$$\beta_i = \frac{i + (i+2)!(i+1)}{(m+1)!m!}\,\alpha_{i+1}, \qquad i = 2, \dots, m-1, \tag{20}$$

$$\beta_1 = \frac{3}{2}\alpha_2, \tag{21}$$

$$\beta_0 = -1. \tag{22}$$

Then Newton's method $\{x_n\}$ $(n \ge 0)$ *generated by* (2) *is well defined, remains in* $\overline{U}(x^*, r_3)$ *for all* $n \ge 0$ *and converges to* x^* *provided that* $x_0 \in \overline{U}(x^*, r_3)$. *Moreover, the following error bounds hold for all* $n \ge 0$:

$$\|x_{n+1} - x^*\| \le a_n \|x_n - x^*\|^2, \tag{23}$$

where

$$b_n = \frac{m}{(m+1)!}\,\alpha_{m+1}\|x_n - x^*\|^{m-1} + \frac{(m-1)\alpha_m}{m!}\,\|x_n - x^*\|^{m-2} + \dots + \frac{\alpha_2}{2!}, \tag{24}$$

$$c_n = 1 - \alpha_2\|x_n - x^*\| - \dots - \frac{\alpha_m}{(m-1)!}\,\|x_n - x^*\|^{m-1} - \frac{\alpha_{m+1}}{m!}\,\|x_n - x^*\|^m, \tag{25}$$

$$a_n = \frac{b_n}{c_n}. \tag{26}$$

Remark 2. Note that condition (17) implies the weaker α_{m+1}-Lipschitz condition used in the proof of Theorem 2 in [2].

Remark 3. We can now argue as we did after Theorem 1. Replace α_{m+1} in Theorem 3 by δ_{m+1} and denote by r_4 the minimum positive zero of the function h defined as g with δ_{m+1} replacing α_{m+1}.

We proved:

Theorem 4. *Let $F : D \subseteq X \to Y$ be analytic, let x^*, α_j, $j = 2, \dots, m$, be as in Theorem 3, and let γ and r_4 be defined above. Moreover, assume:*

$$r_4 \in (0, 1/\gamma), \tag{27}$$

$$x_0 \in \overline{U}(x^*, r_4), \tag{28}$$

$$\overline{U}(x^*, r_4) \subseteq D. \tag{29}$$

Then the conclusions of Theorem 3 for Newton's method hold with δ_{m+1} and r_4 replacing α_{m+1} and r_3 respectively.

Remark 4. Note that condition (15) in Theorem 2 or condition (29) in Theorem 4 are automatically satisfied when $D = X$.

3. Applications

Remark 5. As noted in [1]–[5], [8]–[10] our results can be used for projection methods such as Arnold's, the generalized minimum residual method (GMRES), the generalized conjugate residual method (GCR), and for combined Newton-like/finite-difference projection methods.

Remark 6. The results obtained here can also be used to solve equations of the form $F(x) = 0$, where F' satisfies the autonomous differential equation ([4], [7])

$$F'(x) = T(F(x)), \tag{30}$$

where $T : Y \to X$ is a known continuously sufficiently many times Fréchet-differentiable operator. Since $F'(x^*) = T(F(x^*)) = T(0)$,

$$F''(x^*) = F'(x^*)T'(F(x^*)) = T(0)T'(0)$$

etc., we can apply the results obtained here without actually knowing the solution x^* of equation (1).

We complete our study with such an example.

Example. Let $X = Y = \mathbb{R}$, $D = U(0, 1)$, and define a function F on D by

$$F(x) = e^x - 1. \tag{31}$$

Then it can easily be seen that we can set $T(x) = x + 1$ in (30).

Using (4), (6), (8), (12), (16)–(18), and (31) we obtain, for $m = 2$,

$$\alpha_2 = 1, \quad \alpha_3 = e, \quad \gamma = .5,$$

$$r_1 = .245253, \quad r_2 = .364538, \quad r_3 = .411254048, \quad r_4 = .3822432.$$

Hence, our results provide a wider choice for x_0 than the corresponding ones in [9], [10, Theorem 3.1, p. 585]. This observation is important and also finds applications in step length selection in predictor–corrector continuation procedures [5], [8], [10].

References

[1] I. K. Argyros, *Improved error bounds for Newton-like iterations under Chen–Yamamoto assumptions*, Appl. Math. Lett. 10 (1997), no. 4, 97–100.

[2] —, *Local convergence theorems for Newton's method using outer or generalized inverses and m-Fréchet differentiable operators*, Math. Sci. Res. Hot-Line 4 (2000), no. 4, 47–56.

[3] —, *A Newton–Kantorovich theorem for equations involving m-Fréchet differentiable operators and applications in radiative transfer*, J. Comput. Appl. Math. 131 (2001), 149–159.

[4] I. K. Argyros and F. Szidarovszky, *The Theory and Applications of Iteration Methods*, CRC Press, Boca Raton, FL, 1993.

[5] P. N. Brown, *A local convergence theory for combined inexact-Newton/finite-difference projection methods*, SIAM J. Numer. Anal. 24 (1987), 407–434.

[6] J. M. Gutiérrez, *A new semilocal convergence theorem for Newton's method*, J. Comput. Appl. Math. 79 (1997), 131–145.

[7] L. V. Kantorovich and G. R. Akilov, *Functional Analysis*, Pergamon Press, New York, 1982.

[8] F. A. Potra, *On Q-order and R-order of convergence*, J. Optim. Theory Appl. 63 (1989), 415–431.

[9] W. C. Rheinboldt, *An adaptive continuation process for solving systems of nonlinear equations*, in: Banach Center Publ. 3, PWN, Warszawa, 1977, 129–142.

[10] T. J. Ypma, *Local convergence of inexact Newton methods*, SIAM J. Numer. Anal. 21 (1984), 583–590.

Department of Mathematics
Cameron University
Lawton, OK 73505, U.S.A.
E-mail: ioannisa@cameron.edu

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