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WOJCIECH NIEMIRO (Toruń)
RYSZARD ZIELIŃSKI (Warszawa)

UNIFORM ASYMPTOTIC NORMALITY FOR THE BERNOULLI SCHEME

Abstract. It is easy to notice that no sequence of estimators of the probability of success θ in a Bernoulli scheme can converge (when standardized) to $N(0,1)$ *uniformly in* $\theta \in\ ]0,1[$. We show that the uniform asymptotic normality can be achieved if we allow the sample size, that is, the number of Bernoulli trials, to be chosen sequentially.

1. Introduction. Zieliński (2004) pointed out that for the Bernoulli scheme with the probability of success θ, the central limit theorem (CLT) does *not* hold uniformly in $\theta \in\ ]0,1[$. For any fixed n (the number of trials), the normal approximation deteriorates and its error exceeds $1/4$ if θ is close to 0 or close to 1. In our paper we consider the following question: does there exist a sequence of estimators of θ which is uniformly asymptotically normal? The answer is *yes* provided that we take into consideration *sequential* estimators (which use a random number of observations, depending on the outcomes of previous trials).

2. Main results. Let $Z_1, \ldots, Z_n, \ldots$ be a sequence of real-valued statistics defined on a statistical space $(\Omega, \{P_\theta : \theta \in \Theta\}, \mathcal{F})$.

DEFINITION 2.1. The sequence Z_n is *uniformly asymptotically normal* (UAN) if for some functions $\mu(\theta)$ and $\sigma(\theta) \neq 0$,

$$(1)\qquad \sup_{\theta} \sup_{-\infty<x<\infty} \left| P_\theta\left(\frac{\sqrt{n}}{\sigma(\theta)}\,[Z_n - \mu(\theta)] \le x \right) - \Phi(x) \right| \to 0,$$

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where Φ is the c.d.f. of the standard normal distribution. More explicitly,

$$\forall_{\varepsilon>0}\exists_{n_0}\forall_{n\geq n_0}\forall_\theta\forall_x \qquad \left|P_\theta\left(\frac{\sqrt{n}}{\sigma(\theta)}\,[Z_n-\mu(\theta)]\leq x\right)-\Phi(x)\right|<\varepsilon.$$

We will then write

$$\frac{\sqrt{n}}{\sigma(\theta)}\,[Z_n-\mu(\theta)]\rightrightarrows_d N(0,1).$$

Uniform convergence in distribution was considered e.g. in Zieliński (2004), Salibian-Barrera and Zamar (2004), and Borovkov (1998). The definition given above may be considered as a special case of that given in Borovkov (1998) (Chapter II, par. 37, Def. 2).

THEOREM 2.2. *Let* $X=X_1,\ldots,X_n,\ldots$ *be i.i.d. with* $P_\theta(X=1)=\theta=1-P_\theta(X=0)$. *The parameter space is* $\Theta=\,]0,1[$.

(i) *There is no sequence of estimators* $\widehat{\theta}_n=\widehat{\theta}_n(X_1,\ldots,X_n)$ *such that*

$$\frac{\sqrt{n}}{\sigma(\theta)}\,[\widehat{\theta}_n-\theta]\rightrightarrows_d N(0,1).$$

(ii) *There is a sequence of stopping rules* T_r $(r=1,2,\ldots)$ *and sequential estimators* $\widetilde{\theta}_r=\widetilde{\theta}_r(X_1,\ldots,X_{T_r})$ *such that*

$$\frac{\sqrt{r}}{\sigma(\theta)}\,[\widetilde{\theta}_r-\theta]\rightrightarrows_d N(0,1).$$

Proof of (i). For every n there exists θ such that $P_\theta(X_1=\cdots=X_n=0)>1/2$. Clearly, for such θ the probability distribution of the random variable $(\sqrt{n}/\sigma(\theta))[\widehat{\theta}_n-\theta]$ has an atom which contains more than $1/2$ of the total probability mass. It follows immediately that

$$\sup_{-\infty<x<\infty}|P_\theta[(\sqrt{n}/\sigma(\theta))[\widehat{\theta}_n-\theta]\leq x]-\Phi(x)|\geq 1/4.$$

The proof of part (ii) requires some auxiliary facts and is presented in the next section.

3. Proofs. For the sake of our proofs the following version of uniform δ-method will be useful.

LEMMA 3.1. *Assume that* Z_n *is a UAN sequence of statistics, that is,* (1) *holds. Let* h *be a differentiable function defined on an open subset of the real line such that* $\mu(\theta)$ *belongs to the domain of* h *for every* θ. *If* $h'(\mu(\theta))\neq 0$ *for all* θ *and*

$$\frac{h(\mu(\theta)+\sigma(\theta)t)-h(\mu(\theta))}{h'(\mu(\theta))\sigma(\theta)t}\rightrightarrows 1 \qquad \text{as } t\to 0, \tag{2}$$

uniformly in θ, *then* $h(Z_n)$ *is also UAN*:

$$\frac{\sqrt{n}}{\sigma(\theta)h'(\mu(\theta))}\,[h(Z_n)-h(\mu(\theta))]\rightrightarrows_d N(0,1).$$

Proof. Let us write

$$J_n = J_n(\theta) = \frac{\sqrt{n}}{\sigma(\theta)}\,[Z_n - \mu(\theta)],$$
$$H_n = H_n(\theta) = \frac{\sqrt{n}}{\sigma(\theta)h'(\mu(\theta))}\,[h(Z_n) - h(\mu(\theta))].$$

Fix an $\varepsilon > 0$. In view of the uniform continuity of Φ, we can choose $\eta > 0$ such that

$$\sup_x [\Phi(x+\eta) - \Phi(x)] < \frac{1}{5}\,\varepsilon. \tag{3}$$

Next, we choose b such that

$$1 - \Phi(b) + \Phi(-b) < \frac{1}{5}\,\varepsilon. \tag{4}$$

By (2), there exists $\delta > 0$ such that for $|t| < \delta$ and for all θ,

$$\left|\frac{h(\mu(\theta) + \sigma(\theta)t) - h(\mu(\theta))}{h'(\mu(\theta))\sigma(\theta)} - t\right| < \frac{\eta}{b}\,|t|. \tag{5}$$

By assumption on Z_n, there exists n_0 such that for $n \geq n_0$, all x and θ,

$$|P_\theta(J_n \leq x) - \Phi(x)| < \frac{1}{5}\,\varepsilon. \tag{6}$$

We can assume additionally that $\sqrt{n_0} > b/\delta$.

We claim that for $n \geq n_0$ the following statements hold true for all θ. First, by (4) and (6) we have

$$\begin{aligned} P_\theta(|J_n| > b) &= P_\theta(J_n < -b) + P_\theta(J_n > b) \\ &\leq \Phi(-b) + \frac{1}{5}\,\varepsilon + 1 - \Phi(b) + \frac{1}{5}\,\varepsilon < \frac{3}{5}\,\varepsilon. \end{aligned} \tag{7}$$

We now apply (5) with $t = J_n/\sqrt{n}$. On the event $|J_n| \leq b$ we have $|J_n/\sqrt{n}| < \delta$ and consequently $|H_n - J_n| < (\eta/b)|J_n| \leq \eta$. Therefore

$$P_\theta(|H_n - J_n| > \eta) \leq P_\theta(|J_n| > b) < \frac{3}{5}\,\varepsilon. \tag{8}$$

It is now sufficient to combine (6), (8) and (3) to obtain

$$\begin{aligned} P_\theta(H_n \leq x) &\leq P_\theta(J_n \leq x + \eta) + P_\theta(|H_n - J_n| > \eta) \\ &\leq \Phi(x+\eta) + \frac{1}{5}\,\varepsilon + \frac{3}{5}\,\varepsilon \leq \Phi(x) + \varepsilon, \\ P_\theta(H_n \leq x) &\geq P_\theta(J_n \leq x - \eta) - P_\theta(|H_n - J_n| > \eta) \\ &\geq \Phi(x-\eta) - \frac{1}{5}\,\varepsilon - \frac{3}{5}\,\varepsilon \geq \Phi(x) - \varepsilon. \end{aligned}$$

Since ε is arbitrary, $H_n \rightrightarrows_d N(0,1)$ and the proof is complete. ■

REMARK. The strange looking assumption (2) is actually a kind of *uniform differentiability* condition. It is satisfied, for example, if

$$\frac{h'(\mu(\theta)+\sigma(\theta)t)}{h'(\mu(\theta))} \rightrightarrows 1 \qquad (\text{as } t \to 0, \text{ uniformly in } \theta).$$

By the standard Berry–Esséen theorem we have

THEOREM 3.2 (Berry–Esséen). *For i.i.d. random variables* $Y_1, \ldots, Y_n, \ldots,$ $S_n = \sum_{i=1}^n Y_i$, *and* $F_n(x) = P(n^{-1/2}\sigma^{-1}[S_n - n\mu] \le x)$ *we have*

$$|F_n(x) - \Phi(x)| \le C\frac{m_3}{\sigma^3\sqrt{n}},$$

where $m_3 = E|Y-\mu|^3$ *and* C *is an absolute constant.*

By the inequalities $m_3^{1/3} \le m_4^{1/4}$, $\sigma = m_2^{1/2} \le m_4^{1/4}$, and

$$\frac{m_3}{\sigma^3} \le \frac{m_4^{3/4}}{\sigma^3} = \frac{m_4^{3/4}}{\sigma^4}\,\sigma \le \frac{m_4^{3/4}}{\sigma^4}\,m_4^{1/4} = \frac{m_4}{\sigma^4}$$

we obtain

COROLLARY 3.3.

$$|F_n(x) - \Phi(x)| \le C\,\frac{m_4}{\sigma^4\sqrt{n}},$$

where $m_4 = E(Y-\mu)^4$.

Let us now consider the negative binomial scheme, that is, an i.i.d. sequence of random variables geometrically distributed with parameter θ. The central limit theorem for this scheme does not hold uniformly in $\theta \in \,]0,1[$ (Zieliński 2004): the normal approximation breaks down for θ approaching 1. In the following lemma we assume θ to be bounded away from 1.

LEMMA 3.4 (Central limit theorem for the negative binomial scheme). *Let* $Y = Y_1, \ldots, Y_r, \ldots$ *be i.i.d. and let* $P_\theta(Y = k) = \theta(1-\theta)^{k-1}$ *for* $k = 1, 2, \ldots$. *Let* $T_r = \sum_{i=1}^r Y_i$. *Assume that* $\theta \le 1-\kappa$, *i.e. the parameter space is* $\Theta = \,]0, 1-\kappa]$ *for some* $\kappa > 0$. *Then*

$$\frac{\theta\sqrt{r}}{\sqrt{1-\theta}}\left(\frac{T_r}{r} - \frac{1}{\theta}\right) \rightrightarrows_d N(0,1).$$

We will use the following elementary facts about the geometric distribution:

$$E_\theta(Y) = \frac{1}{\theta}, \qquad \sigma^2(\theta) = \mathrm{Var}_\theta(Y) = \frac{1-\theta}{\theta^2},$$

and

$$m_4(\theta) = E_\theta(Y - \mu(\theta))^4 = \frac{(1-\theta)(\theta^2 - 9\theta + 9)}{\theta^4}.$$

Consequently, for $\theta \le 1-\kappa$,

$$\frac{m_4(\theta)}{\sigma^4(\theta)} = \frac{\theta^2 - 9\theta + 9}{1-\theta} = \frac{\theta^2}{1-\theta} + 9 \le \frac{1}{\kappa} + 9.$$

From Corollary 3.3 it follows that

$$\sqrt{r}\,\frac{\theta}{\sqrt{1-\theta}}\left(\frac{T_r}{r} - \frac{1}{\theta}\right) \rightrightarrows_d N(0,1), \qquad \theta \in \,]0, 1-\kappa]. \ \blacksquare$$

LEMMA 3.5. *Under the assumptions of the previous lemma*,

$$\frac{\sqrt{r}}{\theta\sqrt{1-\theta}}\left(\frac{r}{T_r} - \theta\right) \rightrightarrows_d N(0,1).$$

Proof. It is enough to combine Lemma 3.4 with Lemma 3.1 (δ-method) with $h(x) = 1/x$, $\mu(\theta) = 1/\theta$, and $\sigma(\theta) = \sqrt{1-\theta}/\theta$; the function $h(x)$ obviously satisfies assumption (2) of Lemma 3.1. $\blacksquare$

LEMMA 3.6. *Let $X_1, \ldots, X_n, \ldots$ be the Bernoulli scheme with probability of success θ. Define the sequence of stopping rules $T'_r = \min\{n : S_n \ge r\}$, where $S_n = \sum_{i=1}^n X_i$. The sequence r/T'_r is UAN in $\theta \le 1-\kappa$, i.e. for the parameter space $\Theta = \,]0, 1-\kappa]$.*

Proof. This is a simple reformulation of Lemma 3.5. Indeed, it is easy to see that T'_r is a sum of i.i.d. geometrically distributed random variables. $\blacksquare$

Proof of Theorem 2.2(ii). We construct a sequence of stopping times T_r, $r = 1, 2, \ldots$, as follows. Define

$$\begin{aligned} T'_r &= \min\{n : S_n \ge r\}, \\ T''_r &= \min\{n : n - S_n \ge r\}, \\ \widetilde{T}_r &= \min\{n : S_n \ge r,\, n - S_n \ge r\} = \max(T'_r, T''_r), \\ T_r &= \widetilde{T}_r + r. \end{aligned}$$

We now construct a sequence of estimators $\widetilde{\theta}_r$ as follows. Define two auxiliary estimators $\widetilde{\theta}'_r = r/T'_r$ and $\widetilde{\theta}''_r = 1 - r/T''_r$, a random event

$$A_r = \left\{\frac{1}{r}\sum_{i=1}^{r} X_{\widetilde{T}_r + i} < \frac{1}{2}\right\},$$

and finally

$$\widetilde{\theta}_r = \begin{cases} \widetilde{\theta}'_r & \text{on } A_r, \\ \widetilde{\theta}''_r & \text{on } A_r^c. \end{cases}$$

We claim that $\widetilde{\theta}_r$ is UAN on $]0,1[$ with the asymptotic variance $\sigma^2(\theta)$ given by

$$\sigma^2(\theta) = \begin{cases} (1-\theta)\theta^2 & \text{for } \theta < 1/2, \\ (1-\theta)^2\theta & \text{for } \theta \ge 1/2. \end{cases}$$

To prove that, fix $\varepsilon > 0$ and choose $\delta > 0$ such that

$$\sup_{1/2-\delta<\theta<1/2+\delta} \sup_x \left| \varPhi\left(\frac{x}{\theta\sqrt{1-\theta}}\right) - \varPhi\left(\frac{x}{\sqrt{\theta}(1-\theta)}\right)\right| < \varepsilon/2.$$

Obviously $\delta < 1/2$. Choose r_1 such that for $r \ge r_1$ the inequality $P_\theta(A_r^c) < \varepsilon/2$ holds for all $\theta < 1/2 - \delta$ and $P_\theta(A_r) < \varepsilon/2$ holds for all $\theta > 1/2 + \delta$.

From Lemma 3.6 we conclude that

$$\frac{\sqrt{r}}{\theta\sqrt{1-\theta}}\,(\widetilde{\theta}'_r - \theta) \rightrightarrows_d N(0,1) \quad \text{on }]0, 1/2+\delta]$$

and

$$\frac{\sqrt{r}}{\sqrt{\theta}(1-\theta)}\,(\widetilde{\theta}''_r - \theta) \rightrightarrows_d N(0,1) \quad \text{on } [1/2-\delta, 1[.$$

Choose r_2 such that for $r \ge r_2$ and for all $\theta \le 1/2 + \delta$,

$$\sup_x \left| P_\theta\left(\sqrt{r}\,\frac{\widetilde{\theta}'_r - \theta}{\theta\sqrt{1-\theta}} \le x\right) - \varPhi(x)\right| = \sup_x \left| P_\theta(\sqrt{r}(\widetilde{\theta}'_r - \theta) \le x) - \varPhi\left(\frac{x}{\theta\sqrt{1-\theta}}\right)\right| < \varepsilon/2.$$

Then for $r \ge r_2$ and for all $\theta \ge 1/2 - \delta$ we also have

$$\sup_x \left| P_\theta\left(\sqrt{r}\,\frac{\widetilde{\theta}''_r - \theta}{\sqrt{\theta}(1-\theta)} \le x\right) - \varPhi(x)\right| = \sup_x \left| P_\theta(\sqrt{r}(\widetilde{\theta}''_r - \theta) \le x) - \varPhi\left(\frac{x}{\sqrt{\theta}(1-\theta)}\right)\right| < \varepsilon/2.$$

Define $r_0 = \max(r_1, r_2)$. For the estimator $\widetilde{\theta}_r$ we obtain

$$\sup_x \left| P_\theta(\sqrt{r}(\widetilde{\theta}_r - \theta) \le x) - \varPhi\left(\frac{x}{\sigma(\theta)}\right)\right| \le \sup_x \left| P_\theta(\sqrt{r}(\widetilde{\theta}_r - \theta) \le x, A_r) - P_\theta(A_r)\varPhi\left(\frac{x}{\sigma(\theta)}\right)\right| + \sup_x \left| P_\theta(\sqrt{r}(\widetilde{\theta}_r - \theta) \le x, A_r^c) - P_\theta(A_r^c)\varPhi\left(\frac{x}{\sigma(\theta)}\right)\right|.$$

Since $\widetilde{\theta}_r = \widetilde{\theta}'_r$ on A_r, and $\widetilde{\theta}'_r$ and A_r are independent, and similarly $\widetilde{\theta}_r = \widetilde{\theta}''_r$ on A_r^c, and $\widetilde{\theta}''_r$ and A_r^c are independent, the right hand side of the latter formula is equal to

$$P_\theta(A_r) \cdot \sup_x \left| P_\theta(\sqrt{r}(\widetilde{\theta}'_r - \theta) \le x) - \varPhi\left(\frac{x}{\sigma(\theta)}\right)\right| + P_\theta(A_r^c) \cdot \sup_x \left| P_\theta(\sqrt{r}(\widetilde{\theta}''_r - \theta) \le x) - \varPhi\left(\frac{x}{\sigma(\theta)}\right)\right|.$$

From now on we assume that $r \geq r_0$. For $\theta < 1/2 - \delta < 1/2$ we have $P_\theta(A_r^c) < \varepsilon/2$, $\sigma^2(\theta) = (1-\theta)\theta^2$, and

$$\left| P_\theta(\sqrt{r}(\widehat{\theta}_r' - \theta) \leq x) - \Phi\left(\frac{x}{\theta\sqrt{1-\theta}}\right)\right| < \varepsilon/2.$$

For $\theta > 1/2 + \delta > 1/2$ we have $P_\theta(A_r) < \varepsilon/2$, $\sigma^2(\theta) = (1-\theta)^2\theta$, and

$$\left| P_\theta(\sqrt{r}(\widetilde{\theta}_r'' - \theta) \leq x) - \Phi\left(\frac{x}{\sqrt{\theta}(1-\theta)}\right)\right| < \varepsilon/2.$$

For $1/2 - \delta < \theta < 1/2 + \delta$,

$$\left| P_\theta(\sqrt{r}(\widetilde{\theta}_r' - \theta) \leq x) - \Phi\left(\frac{x}{\sigma(\theta)}\right)\right|$$
$$< \left| P_\theta(\sqrt{r}(\widetilde{\theta}_r' - \theta) \leq x) - \Phi\left(\frac{x}{\theta\sqrt{1-\theta}}\right)\right| + \left|\Phi\left(\frac{x}{\theta\sqrt{1-\theta}}\right) - \Phi\left(\frac{x}{\sigma(\theta)}\right)\right| < \varepsilon$$

and similarly

$$\left| P_\theta(\sqrt{r}(\widetilde{\theta}_r'' - \theta) \leq x) - \Phi\left(\frac{x}{\sigma(\theta)}\right)\right|$$
$$< \left| P_\theta(\sqrt{r}(\widetilde{\theta}_r'' - \theta) \leq x) - \Phi\left(\frac{x}{\sqrt{\theta}(1-\theta)}\right)\right| + \left|\Phi\left(\frac{x}{\sqrt{\theta}(1-\theta)}\right) - \Phi\left(\frac{x}{\sigma(\theta)}\right)\right| < \varepsilon.$$

Finally, we obtain

$$\sup_x \left| P_\theta(\sqrt{r}(\widetilde{\theta}_r - \theta) \leq x) - \Phi\left(\frac{x}{\sigma(\theta)}\right)\right| < \varepsilon,$$

which ends the proof. ■

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Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
87-100 Toruń, Poland
E-mail: wniemiro@gmail.com

Institute of Mathematics
Polish Academy of Sciences
P.O. Box 21
00-956 Warszawa, Poland
E-mail: R.Zielinski@impan.gov.pl

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