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THE REGULARITY OF WEAK AND VERY WEAK SOLUTIONS OF THE POISSON EQUATION ON POLYGONAL DOMAINS WITH MIXED BOUNDARY CONDITIONS (PART II)

Abstract. We examine the regularity of weak and very weak solutions of the Poisson equation on polygonal domains with data in L^2. We consider mixed Dirichlet, Neumann and Robin boundary conditions. We also describe the singular part of weak and very weak solutions.

1. Introduction. In this paper we continue the investigation of the regularity of weak and very weak solutions of the Poisson equation, which were started in [5]. Our main goal is to complete the proofs of Theorems 1 and 2 formulated in [5]. Moreover, we examine the regularity of solutions of the Poisson equation with nonhomogeneous boundary conditions.

Let us recall that we consider the following problem for $f \in L^2(\Omega)$:

$$(1.1)\qquad \begin{cases} \Delta u = f & \text{in } \Omega,\\ \gamma_j u = 0 & \text{on } \Gamma_j \text{ for } j \in \mathbf{D},\\ \gamma_j \dfrac{\partial u}{\partial \nu_j} = 0 & \text{on } \Gamma_j \text{ for } j \in \mathbf{N},\\ \gamma_j \dfrac{\partial u}{\partial \nu_j} + \alpha_j \gamma_j u = 0 & \text{on } \Gamma_j \text{ for } j \in \mathbf{R}. \end{cases}$$

The relevant assumptions were formulated in Section 2 of [5]. We will also use the notation introduced there.

This paper is organized as follows. Sections 2 and 3 are devoted to the space $\mathcal{M}$ of very weak solutions of the homogeneous problem. In Section 2 we prove some smoothness results for elements of $\mathcal{M}$. In Section 3, we find

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a basis of $\mathcal{M}$. In the next two sections we apply the results of [5] to deduce Theorems 1 and 2 stated in [5]. In the last section we consider the mixed boundary value problem with nonhomogeneous boundary conditions. The most general result of the two papers is Corollary 7.

2. Smoothness of very weak solutions of the homogeneous problem. In the next section we will find an expansion of $v \in \mathcal{M}$. To justify it we have to establish some smoothness properties of $v \in \mathcal{M}$. We will need the following lemma.

LEMMA 1. (a) *If $v \in \mathcal{M}$ and $U \subseteq \mathbb{R}^2$ is a neighbourhood of the vertices S_j for $j = 1, \dots, N$, then v is smooth up to the boundary on $\Omega \setminus \overline{U}$, i.e. $v \in C^\infty(\overline{\Omega \setminus U})$.*

(b) *If $\rho(\cdot) := \operatorname{dist}(\cdot, \{S_j;\, j = 1, \dots, N\})$, then $\rho|\nabla v| \in L^2(\Omega)$ for all $v \in \mathcal{M}$.*

Proof. Every function $v \in \mathcal{M}$ is smooth in Ω, because it is harmonic in Ω. To prove (a), it is enough to show that v is smooth up to the boundary away from the vertices S_j, $j = 1, \dots, N$. If $j \in \mathbf{D}$ (resp. $j \in \mathbf{N}$), then we continue v across the Γ_j by odd (resp. even) reflection to get a function $\overline{v}$ which is harmonic and square integrable (see proof of Lemma 2.3.4 in [3]). Thus $\overline{v}_{|\Omega} = v$ is smooth up to the boundary. The proof of (a) will be finished if we show that for $j \in \mathbf{R}$ every $v \in \mathcal{M}$ can be continued harmonically across Γ_j. This will be deduced from the proof of (b). For the latter we have to introduce the following notation. For $\delta \geq 0$ such that $2\delta + \omega_j < 2\pi$, $j = 1, \dots, N$, we define $D_{\varrho,\delta,j} := D^+_{\varrho,\delta,j} \cup D^-_{\varrho,\delta,j}$, where

$$D^{\pm}_{\varrho,\delta,j} := \left\{ (r_j \cos\theta_j, r_j \sin\theta_j) \in \mathbb{R}^2;\ r_j \in (0, \varrho),\ \theta_j \in \left(\frac{-\delta \pm \delta}{2}, \omega_j + \frac{\delta \pm \delta}{2} \right) \right\}.$$

The proof of (b) consists of two steps. First, we prove

PROPOSITION 1. *Assume that $v \in \mathcal{M}$ and there exist numbers $\varrho, \delta > 0$ and a function $\overline{v}$ such that $\overline{v} \in L^2(D_{2\varrho,\delta,j})$ for $j = 1, \dots, N$ and $\overline{v}$ is a harmonic continuation of v. Then $r_j|\nabla v| \in L^2(D_{\varrho,0,j})$ for $j = 1, \dots, N$.*

Proof of Proposition 1. We fix a vertex S_j, let K be an integer with $K \geq 16\sin(\omega_j/2)/\sin\delta$ and set $\delta_1 := \omega_j/K$. Define

$$r_n := \left(1 - \frac{\sin\delta}{2}\right)^n \varrho, \qquad R_n := \frac{\sin\delta}{2}\, r_n \qquad \text{for } n \in \mathbb{N},$$

and consider the balls $B_{i,n,1,j}$, $B_{i,n,2,j}$ for $i = 0, \dots, K$, $n \in \mathbb{N}$, with center $(r_n, i\delta_1)$ (in polar coordinates attached to the vertex S_j) and radius R_n and $2R_n$, respectively. It can be easily verified that the family $\{B_{i,n,1,j};\ i = 0, \dots, K,\ n \in \mathbb{N}\}$ covers $D_{\varrho,0,j}$ and its order is less than $5K$. Furthermore, we have

$$D_{\varrho,0,j} \subseteq \bigcup_{n\in\mathbb{N}} \bigcup_{i=0}^{K} B_{i,n,1,j} \subseteq \bigcup_{n\in\mathbb{N}} \bigcup_{i=0}^{K} B_{i,n,2,j} \subseteq D_{2\varrho,\delta,j}. \tag{2.1}$$

Applying Theorem 8.2 of [1] we obtain a constant C, independent of i, j, n, such that

$$|\nabla \overline{v}(x,y)|^2 \le \frac{C}{R_n^4} \int_{B_{i,n,2,j}} |\overline{v}|^2\,dx\,dy \qquad \text{for } (x,y) \in B_{i,n,1,j}. \tag{2.2}$$

Hence, we have the following estimate of the $L^2(D_{\varrho,0,j})$ norm of $r_j|\nabla v|$:

$$\int_{D_{\varrho,0,j}} |r_j \nabla v(x,y)|^2\,dx\,dy \le \sum_{\substack{i=0,\dots,K\\ n\in\mathbb{N}}} \int_{B_{i,n,1,j}} |r_j|^2 |\nabla \overline{v}(x,y)|^2\,dx\,dy. \tag{2.3}$$

Thus, applying the Hölder inequality and the estimate (2.2), we conclude that (2.3) is less than or equal to

$$\sum_{\substack{i=0,\dots,K\\ n\in\mathbb{N}}} \frac{4C\pi\big(1+\frac{\sin\delta}{2}\big)^2}{\sin^2\delta} \int_{B_{i,n,2,j}} |\overline{v}|^2\,dx\,dy \le \frac{20KC\pi\big(1+\frac{\sin\delta}{2}\big)^2}{\sin^2\delta} \|\overline{v}\|^2_{L^2(D_{2\varrho,\delta,j})}.$$

Hence the proof of Proposition 1 is finished.

Now we shall verify the assumptions of Proposition 1, i.e. we will prove the following:

PROPOSITION 2. *If $v \in \mathcal{M}$, then there exist $\varrho, \delta > 0$ and a function $\overline{v}$ such that $\overline{v} \in L^2(D_{\varrho,\delta,j})$ for $j = 1,\dots,N$ and $\overline{v}$ is a harmonic continuation of v.*

Proof of Proposition 2. We already know that if $j \in \mathbf{D}$ (resp. $j \in \mathbf{N}$), then the odd (resp. even) continuation of $v \in \mathcal{M}$ across Γ_j has the desired properties. Hence we only have to show that if we have the Robin boundary condition on Γ_j, then there exists a harmonic and square integrable continuation of v across Γ_j (the idea of harmonic continuation across Γ_j for $j \in \mathbf{R}$ comes from [6]). To see this, assume that $v \in \mathcal{M}$, $j-1 \in \mathbf{R}$ and $w := -\frac{\partial}{\partial y_j} v + \alpha_{j-1} v$. The function w is harmonic in Ω and satisfies the homogeneous Dirichlet boundary condition on Γ_{j-1}. Thus the odd reflection of w across Γ_{j-1} gives a function W which is defined in $D^-_{\varrho,\delta,j}$ for some positive numbers ϱ, δ and is a harmonic continuation of w. Next, we will consider the equation

$$-\frac{\partial}{\partial y_j} h + \alpha_{j-1} h = W \quad \text{in } D^-_{\varrho,\delta,j}. \tag{2.4}$$

The solutions of (2.4) are of the form $h(x_j,y_j) = \exp(\alpha_{j-1}y_j)[c + f(x_j,y_j)]$, where $f(x_j,y_j)$ satisfies $\frac{\partial}{\partial y_j} f(x_j,y_j) = -\exp(-\alpha_{j-1}y_j) W(x_j,y_j)$. The function W is harmonic in $D^-_{\varrho,\delta,j}$, hence it is real analytic. Thus we can choose a real analytic f which satisfies the last equation. Then the function h is real analytic in $D^-_{\varrho,\delta,j}$. Set $g := v - h$ in $D_{\varrho,0,j}$. Clearly, g is real analytic in $D_{\varrho,0,j}$ and is the solution of the homogeneous equation (2.4), hence it is of the form $g(x_j, y_j) = q(x_j)\exp(\alpha_{j-1}y_j)$, where $q(x_j)$ is real analytic in $D_{\varrho,0,j}$. Define

$$G(x_j, y_j) := q(x_j)\exp(\alpha_{j-1}y_j), \qquad \overline{v} := G + h \quad \text{in } D^-_{\varrho,\delta,j}. \tag{2.5}$$

It is clear that G is real analytic in $D^-_{\varrho,\delta,j}$, hence so is $\overline{v}$. Furthermore, $\overline{v}$ is a continuation of v. The Laplacian of $\overline{v}$ is a real analytic function in $D^-_{\varrho,\delta,j}$ and vanishes in $D_{\varrho,0,j}$, so from the uniqueness principle for real analytic functions we obtain $\Delta\overline{v} = 0$ in $D^-_{\varrho,\delta,j}$.

The above means that we have defined a function $\overline{v}$ which is a harmonic continuation of $v \in \mathcal{M}$ across Γ_j for $j \in \mathbf{R}$. To sum up, we have defined a harmonic continuation across Γ_j in each case of boundary conditions. Hence $v \in \mathcal{M}$ is smooth up to the boundary away from the vertices S_j, $j = 1, \dots, N$. Thus the proof of (a) is complete.

To finish the proof of Proposition 2 we only have to show that $\overline{v}$ is square integrable in $D^-_{\varrho,\delta,j}$. From the definition, in $D^-_{\varrho,\delta,j}$ the function $\overline{v}$ is equal to

$$\begin{aligned} &q(x_j)\exp(\alpha_{j-1}y_j) \\ &\qquad + \exp(\alpha_{j-1}y_j)\Big[c + f(x_j,0) - \int_0^{y_j} \exp(-\alpha_{j-1}t)W(x_j,t)\,dt\Big], \end{aligned} \tag{2.6}$$

for some constant c. Clearly, we have $\overline{v}(x_j, 0) = v(x_j, 0)$ for $x_j \in (0, \varrho)$, hence from (2.6) after straightforward calculations we obtain the following equality in $D^-_{\varrho,\delta,j} \setminus D_{\varrho,0,j}$:

$$\overline{v}(x_j, y_j) = 2\exp(\alpha_{j-1}y_j)v(x_j, 0) - v(x_j, -y_j). \tag{2.7}$$

This gives an explicit formula for harmonic continuation across Γ_j for $j \in \mathbf{R}$. Obviously, $\overline{v} \in L^2(D^-_{\varrho,\delta,j})$ if and only if

$$\int_0^{\varrho} |v(r_j, 0)|^2 r_j\,dr_j < \infty. \tag{2.8}$$

Now we shall verify (2.8). Since on Γ_{j-1} the function v satisfies the Robin boundary condition and the Laplace equation, we get

$$\frac{1}{r_j^2}\frac{\partial^2}{\partial\theta_j^{\,2}}v = \alpha_{j-1}^2 v \quad \text{on } \Gamma_{j-1},$$

thus

$$\frac{\partial^2}{\partial r_j^2}v + \frac{1}{r_j}\frac{\partial}{\partial r_j}v + \alpha_{j-1}^2 v = 0 \quad \text{on } \Gamma_{j-1}. \tag{2.9}$$

The solutions of (2.9) are of the form $p(r_j) := AJ_0(\alpha_{j-1}r_j) + BY_0(\alpha_{j-1}r_j)$, where A, B are some constants and J_0, Y_0 are Bessel functions, of the first and second kind, respectively. It can be easily verified that for every $\varrho > 0$ the integral $\int_0^\varrho |p(r_j)|^2 r_j\,dr_j$ is finite, thus v satisfies (2.8). This shows that $\overline{v} \in L^2(D^-_{\varrho,\delta,j})$ and finishes the proof of Proposition 2.

Lemma 1 is an immediate consequence of Propositions 1 and 2. ∎

3. A basis of $\mathcal{M}$. In this section we will find a family of functions which span the space $\mathcal{M}$. To this end, we introduce the following notation:

$$(3.1)\qquad \widetilde{\chi}_j := \begin{cases} \psi_j^{-1} & \text{for } j-1, j \in \mathbf{N}\cup\mathbf{R},\\ \exp(-A_j x_j) & \text{for } j-1 \in \mathbf{D},\ j \in \mathbf{R},\\ \exp(-B_j y_j) & \text{for } j-1 \in \mathbf{R},\ j \in \mathbf{D},\\ 1 & \text{in the other cases,}\end{cases} \qquad \chi_j := \eta_j \widetilde{\chi}_j,$$

where the functions ψ_j, η_j and numbers A_j, B_j were defined in [5, (4.2)]. It is clear that each function χ_j for $j = 1, \ldots, N$ satisfies the boundary conditions of (1.1), where the constant α_j in the Robin boundary condition is replaced by $-\alpha_j$. Hence, if $v \in \mathcal{M}$, then for each $j = 1, \ldots, N$ the function $w_j := \chi_j v$ satisfies

$$\gamma_k w_j = 0 \quad \text{for } k \in \mathbf{D}, \qquad \gamma_k \frac{\partial}{\partial \nu_k} w_j = 0 \quad \text{for } k \in \mathbf{N}\cup\mathbf{R}.$$

Instead of looking for an expansion of $v \in \mathcal{M}$, we will find an expansion of w_j and then we will use the following property:

$$(3.2)\qquad v = \widetilde{\chi}_j^{-1} w_j \quad \text{on } B(S_j, \varepsilon).$$

Remark 1. Before we apply Grisvard's method (see [3, Proposition 2.3.5]) we have to get rid of the Robin boundary condition, because we have been unable to solve the equation

$$\frac{\partial^2}{\partial r^2} h + \frac{1}{r}\frac{\partial}{\partial r} h - \frac{\lambda^2(r)}{r^2}\, h = f,$$

where $\lambda = \lambda(r)$ satisfies $\lambda \tan(\lambda\omega_j) = \alpha_j r$. In this way we get a function w_j which is not harmonic and what is worse, it may not be square integrable on Ω. However applying Lemma 1 we get $\rho \Delta w_j \in L^2(\Omega)$.

Similarly to [2], we define for $j = 1, \ldots, N$ an unbounded operator Λ_j in $\mathcal{H}_j := L^2((0, \omega_j))$ defined on a domain $D(\Lambda_j)$ by $\Lambda_j \varphi = -\varphi''$. Here $D(\Lambda_j)$ is the subspace of $H^2((0, \omega_j))$ given by conditions: $\varphi'_{|\theta=0} = 0$ for $j-1 \in \mathbf{N}\cup\mathbf{R}$, $\varphi_{|\theta=0} = 0$ for $j-1 \in \mathbf{D}$, $\varphi'_{|\theta=\omega_j} = 0$ for $j \in \mathbf{N}\cup\mathbf{R}$ and $\varphi_{|\theta=\omega_j} = 0$ for $j \in \mathbf{D}$. The operator Λ_j is unbounded, selfadjoint and has a discrete spectrum. We denote by $\varphi_{j,m}$ the normalized eigenfunctions and by $\lambda^2_{j,m}$ the corresponding eigenvalues in increasing order. We are looking for an expansion of w_j of the form

$$w_j(r_j,\theta_j) = \sum_{m\in\mathbb{N}} w_{j,m}(r_j)\varphi_{j,m}(\theta_j), \tag{3.3}$$

where $w_{j,m}(r_j) = \int_0^{\omega_j} w_j(r_j,\theta_j)\varphi_{j,m}(\theta_j)\,d\theta_j$. Thus the function $w_{j,m}$ satisfies the equation

$$\frac{\partial^2}{\partial r_j^2}w_{j,m} + \frac{1}{r_j}\frac{\partial}{\partial r_j}w_{j,m} - \frac{\lambda_{j,m}^2}{r_j^2}\,w_{j,m} = f_{j,m}, \tag{3.4}$$

where $f_{j,m} = \int_0^{\omega_j} \Delta w_j\varphi_{j,m}\,d\theta_j$. The solutions of (3.4) are of the form

$$c_{j,m,1}r_j^{\lambda_{j,m}} + c_{j,m,2}r_j^{-\lambda_{j,m}} - \frac{1}{2\lambda_{j,m}}\,r_j^{-\lambda_{j,m}}\int_0^{r_j} s^{1+\lambda_{j,m}}f_{j,m}(s)\,ds - \frac{1}{2\lambda_{j,m}}\,r_j^{\lambda_{j,m}}\int_{r_j}^{\infty} s^{1-\lambda_{j,m}}f_{j,m}(s)\,ds \tag{3.5}$$

when $\lambda_{j,m}\neq 0$, and

$$c_{j,m,1} + c_{j,m,2}\ln r_j + \int_{r_j}^{\infty} s^{-1}\int_s^{\infty} tf_{j,m}(t)\,dt\,ds \tag{3.6}$$

when $\lambda_{j,m} = 0$, and $c_{j,m,1}$, $c_{j,m,2}$ are arbitrary constants. We will find the basis of $\mathcal{M}$ in two steps. First, we choose the coefficients in (3.5) and (3.6) in such a way that the functions $w_{j,m}$ are of the form (3.5) for $\lambda_{j,m}\neq 0$ and (3.6) for $\lambda_{j,m} = 0$ and then we indicate the expressions in the series (3.3), which are in H^1. This allows us to estimate the dimension of $\mathcal{M}$. Subsequently, we will define a linearly independent family in $\mathcal{M}$.

3.1. *Estimate of the dimension of* $\mathcal{M}$. First we establish some properties of the expressions (3.5) and (3.6).

PROPOSITION 3. *There exists a constant C such that for all $j=1,\dots,N$, $m\in\mathbb{N}$ the $L^2((0,\infty))$ norms of $r_j^{1/2}w_{j,m}$ and $r_j^{3/2}f_{j,m}$ are bounded by C.*

Proof. In view of Remark 1 we can write

$$\int_0^{\infty} r_j^3|f_{j,m}|^2\,dr_j \le \int_0^{\infty} r_j^3\int_0^{\omega_j}|\Delta w_j|^2\,d\theta_j\,dr_j = \int_{\Omega}|r_j\Delta w_j|^2\,dx\,dy < \infty.$$

Applying the Schwarz inequality we get

$$\int_0^{\infty} r_j|w_{j,m}|^2\,dr_j \le \int_0^{\infty}|w_j|^2 r_j\,dr_j\,d\theta_j \le c\|v\|^2_{L^2(\Omega)} < \infty. \ \blacksquare$$

REMARK 2. The expressions in (3.5) and (3.6) are well defined. Indeed, the supports of $f_{j,m}$ are bounded and applying the Schwarz inequality and

Proposition 3 we get

$$\Big|\int_0^{r_j} s^{1+\lambda_{j,m}} f_{j,m}(s)\,ds\Big| \le \int_0^{r_j} s^{\lambda_{j,m}-1/2}\cdot s^{3/2}|f_{j,m}(s)|\,ds$$
$$\le \frac{r_j^{\lambda_{j,m}}}{\sqrt{2\lambda_{j,m}}}\,\|r_j^{3/2} f_{j,m}\|_{L^2((0,\infty))}.$$

LEMMA 2. *For $j=1,\dots,N$ and for some $\varrho>0$ we have*

(a) $\displaystyle\sum_{\lambda_{j,m}\ne 0} \frac{1}{2\lambda_{j,m}}\, r_j^{-\lambda_{j,m}} \int_0^{r_j} s^{1+\lambda_{j,m}} f_{j,m}(s)\,ds\,\varphi_{j,m}(\theta_j) \in H^1(D_{\varrho,0,j})$,

(b) $\displaystyle\sum_{\lambda_{j,m}\ne 0} \frac{1}{2\lambda_{j,m}}\, r_j^{\lambda_{j,m}} \int_{r_j}^{\infty} s^{1-\lambda_{j,m}} f_{j,m}(s)\,ds\,\varphi_{j,m}(\theta_j) \in H^1(D_{\varrho,0,j})$.

Proof. It is clear that we will get (a) once we show that there exists a constant C such that

$$\Big\|r_j^{-(\lambda_{j,m}+1/2)+k} \int_0^{r_j} s^{1+\lambda_{j,m}} f_{j,m}(s)\,ds\Big\|_{L^2((0,\varrho))} \le \frac{C}{\lambda_{j,m}} \quad \text{for } k=0,1. \tag{3.7}$$

We will only examine the case of $k=0$, because using the Hölder inequality, the case of $k=1$ can be reduced to the former. To show (3.7) we apply the Hardy inequality ([4, Theorem 330]) to obtain

$$\int_0^{\varrho} r_j^{-(2\lambda_{j,m}+1)} \Big|\int_0^{r_j} s^{1+\lambda_{j,m}} f_{j,m}(s)\,ds\Big|^2\,dr_j$$
$$\le \frac{1}{\lambda_{j,m}^2}\int_0^{\infty} r_j^3 |f_{j,m}|^2\,dr_j = \frac{1}{\lambda_{j,m}^2}\,\|r_j^{3/2} f_{j,m}\|^2_{L^2((0,\infty))}.$$

Hence from Proposition 3 we get (3.7), proving (a). For (b) we proceed similarly. ∎

The functions $w_{j,m}$ are solutions of (3.4) for $r_j>0$, thus for each $j=1,\dots,N$ there exist unique constants $c_{j,m,1}$ and $c_{j,m,2}$ such that $w_{j,m}$ is of the form (3.5) for $\lambda_{j,m}\ne 0$ and of the form (3.6) for $\lambda_{j,m}=0$. Hence the constants $c_{j,m,2}$ defined in this way have the following properties.

PROPOSITION 4. *For each $j=1,\dots,N$ and $m\in\mathbb{N}$, if $\lambda_{j,m}\ge 1$, then $c_{j,m,2}=0$.*

Proof. As in the proof of Lemma 2 we can show that for $j=1,\dots,N$ and $m\in\mathbb{N}$ the functions

$$r_j^{-\lambda_{j,m}+1/2}\int_0^{r_j} s^{1+\lambda_{j,m}} f_{j,m}(s)\,ds \quad\text{and}\quad r_j^{\lambda_{j,m}+1/2}\int_{r_j}^{\infty} s^{1-\lambda_{j,m}} f_{j,m}(s)\,ds$$

are square integrable on $D_{\varrho,0,j}$ for some $\varrho > 0$. Hence from Proposition 3 and expansion of $w_{j,m}$ in the form (3.5) we get $c_{j,m,2} r_j^{-\lambda_{j,m}+1/2} \in L^2(D_{\varrho,0,j})$. But if $\lambda_{j,m} \ge 1$, then $r_j^{-\lambda_{j,m}+1/2}$ is not square integrable on any neighbourhood of S_j, and so $c_{j,m,2} = 0$. ■

PROPOSITION 5. *If $\varrho < 1/2$, then for each $j = 1, \ldots, N$ we have*

$$\sum_{m\in\mathbb{N}} c_{j,m,1} r_j^{\lambda_{j,m}} \varphi_{j,m} \in H^1(D_{\varrho,0,j}).$$

Proof. First we find an estimate for the coefficients $c_{j,m,1}$. If $m \ge 3$ and $\varrho < 1$, then applying the expansion of $w_{j,m}$ in the form (3.5), next the estimate (3.7) and Propositions 3 and 4 we get

$$\|c_{j,m,1} r_j^{1/2+\lambda_{j,m}}\|_{L^2((0,\varrho))} \le C_1 + \frac{C_2}{\lambda_{j,m}}, \tag{3.8}$$

where C_1 and C_2 are some constants independent of j and m. By the definition of $\lambda_{j,m}$ the right hand side of (3.8) is bounded by $C_1 + C_2$ if $m \ge 3$. As the left hand side of (3.8) equals $|c_{j,m,1}| \varrho^{\lambda_{j,m}+1} / \sqrt{2(\lambda_{j,m}+1)}$, we get

$$|c_{j,m,1}| \le \frac{C_3 \sqrt{\lambda_{j,m}}}{\varrho^{\lambda_{j,m}}} \quad \text{for } m \ge 3, \tag{3.9}$$

where the constant C_3 is independent of j and m. This leads to the following bound on $\|c_{j,m,1} r_j^{\lambda_{j,m}} \varphi_{j,m}\|^2_{H^1(D_{\varrho/2,0,j})}$:

$$\begin{aligned}|c_{j,m,1}|^2 \Big(\int_0^{\varrho/2} r_j^{2\lambda_{j,m}+1}\, dr_j + (1+\lambda_{j,m}^2) \int_0^{\varrho/2} r_j^{2\lambda_{j,m}-1}\, dr_j \Big) \\ = |c_{j,m,1}|^2 \left(\frac{\varrho}{2}\right)^{2\lambda_{j,m}} \left[\frac{\varrho^2}{8(\lambda_{j,m}+1)} + \frac{1+\lambda_{j,m}^2}{2\lambda_{j,m}} \right] \\ \le C_3^2 \lambda_{j,m} \left(\frac{1}{2}\right)^{2\lambda_{j,m}} \left[\frac{\varrho^2}{8(\lambda_{j,m}+1)} + \frac{1+\lambda_{j,m}^2}{2\lambda_{j,m}} \right].\end{aligned}$$

Thus the series $\sum_{\lambda_{j,m}\ne 0} c_{j,m,1} r_j^{\lambda_{j,m}} \varphi_{j,m}$ converges in $H^1(D_{\varrho/2,0,j})$. ■

COROLLARY 1. *For each $j = 1, \ldots, N$ there exists $\varrho > 0$ such that the function w_j has the following expansion in $D_{\varrho,0,j}$:*

$$\begin{aligned}\sum_{\lambda_{j,m}\in(0,1)} c_{j,m,2} r_j^{-\lambda_{j,m}} \varphi_{j,m} \\ + \sum_{\lambda_{j,m}=0} \Big(c_{j,m,2} \ln r_j + \int_{r_j}^{\infty} s^{-1} \int_s^{\infty} t f_{j,m}(t)\, dt\, ds \Big) \varphi_{j,m} + h_j,\end{aligned} \tag{3.10}$$

where $h_j \in H^1(D_{\varrho,0,j})$.

Proof. Using (3.3), then applying the expansion of $w_{j,m}$ in the form (3.5) and (3.6) we get the assertion from Lemma 2 and Propositions 4, 5. ■

Unfortunately, the expansion of w_j in the form (3.10) does not allow us to deduce the desired property, because we cannot prove that the function $\int_{r_j}^{\infty} s^{-1} \int_s^{\infty} t f_{j,m}(t)\, dt\, ds$ is in H^1. To overcome this difficulty we have to modify the expansion of $w_{j,m}$ for $\lambda_{j,m} = 0$. Let $p \in (1,2)$ be such that

$$p < \frac{2}{1+\lambda_{j,m}} \quad \text{whenever} \quad \lambda_{j,m} \in (0,1). \tag{3.11}$$

PROPOSITION 6. *For each $j = 1, \ldots, N$ we have $w_j \in W^{1,p}(\Omega)$.*

Proof. In view of the definition of w_j, Lemma 1 and Corollary 1, it is enough to show that the function (3.10) is in $W^{1,p}(D_{\varrho,0,j})$ for some $\varrho > 0$. It can be easily verified that the functions $r_j^{-\lambda_{j,m}} \varphi_{j,m}$ and $\ln r_j \varphi_{j,m}$ are in $W^{1,p}(D_{\varrho,0,j})$. We will prove that $\int_{r_j}^{\infty} s^{-1} \int_s^{\infty} t f_{j,m}(t)\, dt\, ds\, \varphi_{j,m}$ is in $W^{1,p}(D_{\varrho,0,j})$, that is,

$$r_j^{1/p} \int_{r_j}^{\infty} s^{-1} \int_s^{\infty} t f_{j,m}(t)\, dt\, ds,\ r_j^{1/p-1} \int_{r_j}^{\infty} t f_{j,m}(t)\, dt \in L^p((0,\varrho)). \tag{3.12}$$

To justify (3.12), we shall check that $t f_{j,m} \in L_{p,1/p}(\mathbb{R}_+)$. The function $f_{j,m}$ vanishes outside $B(S_j, \delta)$ for some $\delta > 0$, hence after applying the Hölder inequality with exponents $2/p$ and $2/(2-p)$ we get

$$\begin{aligned}\int_0^{\infty} |t f_{j,m}(t) \cdot t^{1/p}|^p\, dt &= \int_0^{\delta} t^{1-p/2} \cdot |t^{3/2} f_{j,m}(t)|^p\, dt \\ &\le \delta^{2-p} \|t^{3/2} f_{j,m}\|^p_{L^2((0,\infty))} < \infty.\end{aligned}$$

Applying twice the Hardy inequality ([4, Theorem 330]), and then the Hölder inequality, we obtain

$$\begin{aligned}\int_0^{\infty} r_j \Big[\int_{r_j}^{\infty} s^{-1} \int_s^{\infty} t f_{j,m}(t)\, dt\, ds\Big]^p dr_j &\le \Big(\frac{p}{2}\Big)^p \int_0^{\infty} r_j \Big[\int_{r_j}^{\infty} t f_{j,m}(t)\, dt\Big]^p dr_j \\ &\le \Big(\frac{p}{2}\Big)^{2p} \int_0^{\infty} r_j^{2p+1} |f_{j,m}(r_j)|^p\, dr_j = \Big(\frac{p}{2}\Big)^{2p} \int_0^{\delta} t^p |t^{1+1/p} f_{j,m}(t)|^p\, dt \\ &\le \Big(\frac{p}{2}\Big)^{2p} \delta^p \|t f_{j,m}\|_{L_{p,1/p}(\mathbb{R}_+)} < \infty.\end{aligned}$$

Using again the Hardy inequality we get

$$\int_0^\infty r_j^{1-p}\Big[\int_{r_j}^\infty tf_{j,m}(t)\,dt\Big]^p\,dr_j \le \Big(\frac{2}{2-p}\Big)^p\int_0^\infty t^{1+p}|f_{j,m}(t)|^p\,dt$$
$$= \Big(\frac{2}{2-p}\Big)^p\|tf_{j,m}\|_{L_{p,1/p}(\mathbb{R}_+)} < \infty.$$

Hence we have (3.12), and the proof is finished. ∎

Corollary 2. *If* $p\in(1,2)$ *satisfies* (3.11), *then* $\mathcal{M}\subseteq W^{1,p}(\Omega)$.

Proof. From Lemma 1 we have the smoothness of $v\in\mathcal{M}$ away from the vertices S_j, $j=1,\ldots,N$. Proposition 6 and (3.2) give us the desired smoothness in a neighbourhood of the vertices S_j, $j=1,\ldots,N$. ∎

Now we are able to define another solution of (3.4) for $\lambda_{j,m}=0$.

Proposition 7. *The function* $\int_0^{r_j}s^{-1}\int_0^s tf_{j,m}(t)\,dt\,ds$ *is in* $H^1(D_{\varrho,0,j})$ *for some* $\varrho>0$ *and satisfies* (3.4) *for* $\lambda_{j,m}=0$.

Proof. It is clear that $\int_0^{r_j}s^{-1}\int_0^s tf_{j,m}(t)\,dt\,ds$ satisfies (3.4). We will show that

$$r_j^{1/2}\int_0^{r_j}s^{-1}\int_0^s tf_{j,m}(t)\,dt\,ds,\ r_j^{-1/2}\int_0^{r_j}tf_{j,m}(t)\,dt\in L^2((0,\varrho)),\tag{3.13}$$

for some $\varrho>0$. First we prove that $t^{1/p}f_{j,m}\in L^p((0,\infty))$. From Corollary 2 and the definition of w_j we have $\Delta w_j\in L^p(\Omega)$, hence

$$\int_0^\infty|t^{1/p}f_{j,m}(t)|^p\,dt\le\int_0^\infty\int_0^{\omega_j}|\Delta w_j\varphi_{j,m}|^p t\,dt\,d\theta_j\le C\|\Delta w_j\|^p_{L^p(\Omega)}<\infty.$$

Applying the Hölder inequality with exponents p and p^*, where $1/p+1/p^*=1$, we get

$$\Big|\int_0^s tf_{j,m}(t)\,dt\Big|\le\frac{1}{2}s^{2/p^*}\|t^{1/p}f_{j,m}\|_{L^p((0,\infty))}.\tag{3.14}$$

Thus we have

$$\int_0^\varrho r_j\Big[\int_0^{r_j}s^{-1}\int_0^s tf_{j,m}(t)\,dt\,ds\Big]^2dr_j\le\frac{1}{4}\|t^{1/p}f_{j,m}\|^2_{L^p((0,\infty))}\int_0^\varrho r_j\Big[\int_0^{r_j}s^{2/p^*-1}\,ds\Big]^2dr_j$$
$$=\frac{p^3\|t^{1/p}f_{j,m}\|^2_{L^p((0,\infty))}}{64(p-1)^2(3p-2)}\,\varrho^{6-4/p}<\infty.$$

As above, using the estimate (3.14) we get

$$\int_0^\varrho r_j^{-1}\Big[\int_0^{r_j}tf_{j,m}(t)\,dt\Big]^2\,dr_j\le\frac{\varrho^{4p/(p-1)}}{16(p-1)}\,\|t^{1/p}f_{j,m}\|^2_{L^p((0,\infty))}<\infty.$$

Hence we have shown (3.13), and the proof is finished. ∎

In view of Proposition 7 we can find unique constants $\widetilde{c}_{j,m,1}$ and $\widetilde{c}_{j,m,2}$ such that if $\lambda_{j,m} = 0$ then

$$w_{j,m} = \widetilde{c}_{j,m,1} + \widetilde{c}_{j,m,2} \ln r_j + \int_0^{r_j} s^{-1} \int_0^s t f_{j,m}(t)\, dt\, ds. \tag{3.15}$$

Now we are in a position to describe the singular part of w_j:

COROLLARY 3. *For each $j = 1, \ldots, N$ there exists $\varrho > 0$ such that in $D_{\varrho,0,j}$,*

$$w_j = \sum_{\lambda_{j,m} \in (0,1)} c_{j,m,2} r_j^{-\lambda_{j,m}} \varphi_{j,m} + \sum_{\lambda_{j,m}=0} \widetilde{c}_{j,m,2} \ln r_j \varphi_{j,m} + h_j, \tag{3.16}$$

where $h_j \in H^1(D_{\varrho,0,j})$.

Proof. This is an immediate consequence of Corollary 1, Proposition 7 and (3.15). ∎

Let us introduce the following notation: for $j = 1, \ldots, N$ we write

$$\begin{aligned} &N_j := \{m;\, \lambda_{j,m} \in (0,1)\}, \qquad M_j := \{m;\, \lambda_{j,m} = 0\},\\ &n_j := |N_j|, \qquad m_j := |M_j|. \end{aligned} \tag{3.17}$$

We define an operator $\mathcal{P}$ by

$$\mathcal{P}\colon \mathcal{M} \to \prod_{j=1}^N \mathbb{R}^{n_j} \times \mathbb{R}^{m_j}, \qquad \mathcal{P}(v) = ((c_{j,m,2})_{m\in N_j}, (\widetilde{c}_{j,m,2})_{m \in M_j})_{j=1}^N, \tag{3.18}$$

where $c_{j,m,2}$ and $\widetilde{c}_{j,m,2}$ are the unique coefficients from the expansion of w_j of the form (3.16). It is clear that $\mathcal{P}$ is well defined and is a linear operator.

THEOREM 1. *The operator $\mathcal{P}$ is an injection.*

Proof. Suppose that $\mathcal{P}(v) = 0$. Then the coefficients $c_{j,m,2}$ and $\widetilde{c}_{j,m,2}$ in (3.16) vanish. Hence $w_j \in H^1(D_{\varrho,0,j})$ for each $j = 1, \ldots, N$ and some $\varrho > 0$, and from (3.2) we obtain $v \in H^1(U)$, where U is some neighbourhood of the vertices S_j, $j = 1, \ldots, N$. From Lemma 1 we conclude that $v \in H^1(\Omega)$, i.e. $v \in E(\Delta, L^2(\Omega))$. Applying Lemmas 1.5.4 and 2.1.2 of [3] we infer that v is a variational solution of the homogeneous problem (1.1), i.e. v satisfies the equality (3.3) of [5] with right hand side zero. Thus, from Proposition 1 of [5] we conclude that $v = 0$. ∎

Finally, from the definition of n_j, m_j and Theorem 1 we get the following estimate on the dimension on $\mathcal{M}$.

COROLLARY 4. *The dimension of $\mathcal{M}$ does not exceed $\sum_{j=1}^N (m_j + n_j)$.* ∎

Now we will find a basis of $\mathcal{M}$.

3.2. *A linearly independent family.* We are going to prove the following theorem.

THEOREM 2. *The dimension of* $\mathcal{M}$ *is equal to the number of eigenvalues lying in* $[0,1)$, *i.e.*

$$\dim \mathcal{M} = |\{\lambda_{j,m};\ \lambda_{j,m} \in [0,1)\}|. \tag{3.19}$$

Proof. In view of Corollary 4 it is enough to find a linearly independent family $\{\sigma_{j,m}\}_{\lambda_{j,m}\in[0,1)}$ in $\mathcal{M}$. Define

$$u_{j,m} := \begin{cases} \eta_j \widetilde{\chi}_j^{-1} r_j^{-\lambda_{j,m}} \varphi_{j,m} & \text{if } \lambda_{j,m} \in (0,1), \\ \eta_j \widetilde{\chi}_j^{-1} \ln r_j \varphi_{j,m} & \text{if } \lambda_{j,m} = 0. \end{cases} \tag{3.20}$$

The functions $\widetilde{\chi}_j$ were defined in (3.1). We will prove the following proposition.

PROPOSITION 8. *For any* $\delta > 0$ *and* $\lambda_{j,m} \in [0,1)$ *there exists a constant* $C > 0$ *such that*

$$\Big|\int_0^\delta r_j^{-\lambda_{j,m}} \varphi\, dr_j\Big| \le C\|\varphi\|_{H^{1/2}((0,\delta))} \quad \textit{for } \varphi \in H^{1/2}((0,\delta)).$$

Proof of Proposition 8. Let $s_0 \in (1/2,1)$ satisfy $3/2 - \lambda_{j,m} > s_0$. Then by Theorem 1.2.18 of [3] we conclude that $r_j^{1-\lambda_{j,m}} \in H^{s_0+1/2}(\Omega \cap B(0,\delta))$. Hence there exists a constant C_1 such that

$$\|r_j^{1-\lambda_{j,m}}\|_{H^{s_0}((0,\delta))} \le C_1 \|r_j^{1-\lambda_{j,m}}\|_{H^{s_0+1/2}(\Omega\cap B(0,\delta))}.$$

From the continuity of the differentiation operator ([2, Theorem 1.4.4.6]) we get a constant C_2 such that

$$\|r_j^{-\lambda_{j,m}}\|_{H^{s_0-1}((0,\delta))} \le C_2 \|r_j^{1-\lambda_{j,m}}\|_{H^{s_0}((0,\delta))}.$$

We have $H^{s_0-1}((0,\delta)) = H_0^{1-s_0}((0,\delta))^* = H^{1-s_0}((0,\delta))^*$, because $1 - s_0 \in (0,1/2)$. Hence

$$\Big|\int_0^\delta r_j^{-\lambda_{j,m}} \varphi\, dr_j\Big| \le \|r_j^{-\lambda_{j,m}}\|_{H^{s_0-1}((0,\delta))} \cdot \|\varphi\|_{H^{1-s_0}((0,\delta))}.$$

Combining the above inequalities and applying the continuity of the imbedding $H^{1/2}((0,\delta)) \hookrightarrow H^{1-s_0}((0,\delta))$ we obtain the desired conclusion.

PROPOSITION 9. *Under the notations of* [5, (3.1) *and* (3.2)], *the Laplacian of* $u_{j,m}$ *is an element of the dual of* V, *i.e. there exists a constant* C *such that*

$$\Big|\int_\Omega \Delta u_{j,m} v\, dx\, dy\Big| \le C\|v\|_{H^1(\Omega)} \quad \textit{for all } v \in V. \tag{3.21}$$

Proof of Proposition 9. It is easy to calculate that $\Delta u_{j,m} = p_{j,m} r_j^{-(1+\lambda_{j,m})} + h_{j,m}$, where $p_{j,m}$ is a smooth function and $h_{j,m} \in L^2(\Omega)$. Hence the proof will be finished if we show that $r_j^{-(1+\lambda_{j,m})} \in V^*$.

REMARK 3. The last statement may look obvious, but it is not quite so. It is well known that if $\lambda_{j,m} \in (0,1)$, then $r_j^{-(1+\lambda_{j,m})}$ is in $H_0^1(\Omega)^*$. However, it is not clear whether it is in the smaller space V^*.

We will show that there exists a constant C such that for some $\delta > 0$,

$$\Big| \int_{\Omega\cap B(S_j,\delta)} r_j^{-(1+\lambda_{j,m})} v\,dx\,dy \Big| \le C\|v\|_{H^1(\Omega)} \quad \text{for } v \in V \cap \mathcal{D}(\overline{\Omega}). \tag{3.22}$$

From Proposition 8 and the trace theorem we get the estimate

$$\begin{aligned}\Big| \int_{\Omega\cap B(S_j,\delta)} r_j^{-(1+\lambda_{j,m})} v\,dx\,dy \Big| &= \Big| \int_0^{\omega_j}\int_0^{\delta} r_j^{-\lambda_{j,m}} v(r_j,\theta_j)\,dr_j\,d\theta_j \Big| \\ &\le C \int_0^{\omega_j} \|v(\cdot,\theta_j)\|_{H^{1/2}((0,\delta))}\,d\theta_j \le \overline{C}\|v\|_{H^1(\Omega)}.\end{aligned}$$

Applying the Hahn–Banach Theorem we conclude that $r_j^{-(1+\lambda_{j,m})}$ is in V^*. Hence the proof of Proposition 9 is finished.

From the Riesz Theorem and Proposition 9 we get the following corollary.

COROLLARY 5. *For each $\lambda_{j,m} \in [0,1)$ there exists a unique $v_{j,m} \in V$ such that*

$$(v_{j,m}, w)_V = \langle w, \Delta u_{j,m}\rangle \quad \textit{for } w \in V. \ \blacksquare \tag{3.23}$$

Assume that $\lambda_{j,m} \in [0,1)$ and let $v_{j,m}$ be provided by Corollary 5. Set

$$\sigma_{j,m} := u_{j,m} - v_{j,m}. \tag{3.24}$$

We show that $\sigma_{j,m} \in \mathcal{M}$. It is clear that $\sigma_{j,m} \in L^2(\Omega)$ and $\Delta\sigma_{j,m} = 0$ in the sense of distributions, hence $\sigma_{j,m} \in D(\Delta, L^2(\Omega))$. Clearly, $u_{j,m}$ satisfies the boundary condition (1.1). We need to check that so does $v_{j,m}$. By straightforward calculations we get $\Delta u_{j,m} = \Delta v_{j,m} \in L^p(\Omega)$ for some $p \in (1,2)$ with $p < 2/(1+\lambda_{j,m})$. Hence $v_{j,m} \in E(\Delta, L^p(\Omega))$ and applying the Green formula ([2, Theorem 1.5.3.11]) to (3.23) we obtain

$$-\sum_{k\in\mathbf{R}} \alpha_k \int_{\Gamma_k} \gamma_k v_{j,m} \gamma_k w\,d\sigma = \sum_{k\in\mathbf{N}\cup\mathbf{R}} \Big\langle \gamma_k \frac{\partial}{\partial\nu_k} v_{j,m}, \gamma_k w \Big\rangle \quad \text{for } w \in V_s \cap \mathcal{D}(\overline{\Omega}). \tag{3.25}$$

From (3.25) and the trace theorem [2, 1.5.2.3] we conclude that $v_{j,m}$ satisfies the desired boundary conditions. Thus, we have proved that

$$\sigma_{j,m} \in \mathcal{M} \quad \text{for } \lambda_{j,m} \in [0,1). \tag{3.26}$$

The family $\{\sigma_{j,m}\}_{\lambda_{j,m}\in[0,1)}$ is linearly independent, because for different j's the supports of $\sigma_{j,m}$ are disjoint, and for the same j the functions $\sigma_{j,m}$ belong to different Sobolev spaces $W^{1,p}(\Omega)$. Hence from Corollary 4 we have

$$\mathcal{M} = \operatorname{span}\{\sigma_{j,m};\ \lambda_{j,m} \in [0,1)\}. \tag{3.27}$$

Thus the proof of Theorem 2 is finished. ∎

Now we are going to describe the space $\mathcal{N}$ of [5, (4.1)].

4. A basis of $\mathcal{N}$.

Theorem 4 of [5] and Theorem 2 yield

COROLLARY 6. *The annihilator $\mathcal{N}$ is described by*

$$\sigma_{j,m} \in \mathcal{N} \ \Leftrightarrow\ \int_\Omega \sigma_{j,m} \Delta\psi_j \, dx\, dy = 0. \ ∎ \tag{4.1}$$

In view of Corollary 6, after integrating by parts and applying (3.23) and (3.24) we get

$$\sigma_{j,m} \in \mathcal{N} \ \Leftrightarrow\ \int_\Omega u_{j,m} \Delta\psi_j \, dx\, dy = \langle \psi_j, \Delta u_{j,m}\rangle. \tag{4.2}$$

To verify the right hand side of (4.2) we need some more notation. For $h \in \mathcal{D}(\overline{\Omega})$ we define

$$P_{\varepsilon,j,m}(h) := \int_{\Omega_\varepsilon} u_{j,m} \Delta h \, dx\, dy, \qquad P_{j,m}(h) := \lim_{\varepsilon\to 0^+} P_{\varepsilon,j,m}(h).$$

REMARK 4. Using Lebesgue's theorem it can be shown that $P_{j,m}(h)$ is well defined for all $h \in \mathcal{D}(\overline{\Omega})$, because $u_{j,m} \in L^1(\Omega)$.

From the proof of Lemma 2.1.2 of [3] we know that there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathcal{D}_S$ and $\delta > 0$ such that

$$1 - f_n \xrightarrow[n\to\infty]{H^1(\Omega)} 0, \qquad 1 - f_n = 0 \quad \text{outside } \bigcup_{j=1}^{N} B(S_j, \delta),$$
$$f_n = f_n(r_j) \quad \text{on } B(S_j, \delta).$$

Hence, $f_n\psi_j \in \mathcal{D}_S$ and $f_n\psi_j \xrightarrow[n\to\infty]{H^1(\Omega)} \psi_j$. Thus (4.2) may be rewritten as

$$\sigma_{j,m} \in \mathcal{N} \ \Leftrightarrow\ \int_\Omega u_{j,m} \Delta\psi_j \, dx\, dy = \lim_{n\to\infty} \int_\Omega f_n \psi_j \Delta u_{j,m} \, dx\, dy. \tag{4.3}$$

The supports of $f_n\psi_j\Delta u_{j,m}$ do not contain the vertex S_j, hence we can apply the Green formula. Thus (4.3) can be reduced to the identity

$$\sigma_{j,m} \in \mathcal{N} \ \Leftrightarrow\ \lim_{n\to\infty} P_{j,m}((1-f_n)\psi_j) = 0. \tag{4.4}$$

Now we are able to prove the key result of this section.

THEOREM 3. *Each function $v \in \mathcal{N}$ is a linear combination of the functions $\sigma_{j,m}$ for $\lambda_{j,m} \in (0,1)$, i.e.*

$$\mathcal{N} = \operatorname{span}\{\sigma_{j,m};\ \lambda_{j,m} \in (0,1)\}. \tag{4.5}$$

Proof. By Theorem 4 of [5] and (3.27), (4.4) we only need to check that

$$\lim_{n\to\infty} P_{j,m}((1-f_n)\psi_j) = 0 \quad \text{for } \lambda_{j,m} \in (0,1), \tag{4.6}$$

$$\lim_{n\to\infty} P_{j,m}((1-f_n)\psi_j) \neq 0 \quad \text{for } \lambda_{j,m} = 0. \tag{4.7}$$

First, we deal with the case $\lambda_{j,m} \in (0,1)$, and calculate $P_{\varepsilon,j,m}(h)$ for $h \in \mathcal{D}(\overline{\Omega})$. Without loss of generality we may assume that $\operatorname{supp} h \subseteq \bigcup_{j=1}^N \{(x,y);\ \eta_j = 1\} \subseteq \bigcup_{j=1}^N B(S_j,\delta)$ for some $\delta > 0$. Then after integrating by parts, twice with respect to r_j and twice with respect to θ_j we get

$$\begin{aligned}
&P_{\varepsilon,j,m}(h)\\
&= -\varepsilon^{1-\lambda_{j,m}} \int_0^{\omega_j} \varphi_{j,m} \frac{\partial}{\partial\theta_j} h(\varepsilon,\theta_j)\theta_j - \frac{1}{\lambda_{j,m}} \varepsilon^{-\lambda_{j,m}} \int_0^{\omega_j} \varphi_{j,m} \frac{\partial^2}{\partial{\theta_j}^2} h(\varepsilon,\theta_j)\, d\theta_j \\
&\quad + \int_\varepsilon^\delta \frac{1}{\lambda_{j,m}} r_j^{-\lambda_{j,m}} \left[\varphi_{j,m} \frac{\partial}{\partial r_j}\frac{\partial}{\partial\theta_j} h \Big|_0^{\omega_j} - \frac{\partial}{\partial\theta_j}\varphi_{j,m} \frac{\partial}{\partial r_j} h \Big|_0^{\omega_j} \right] dr_j.
\end{aligned}$$

Thus, [5, (4.6)] implies that $\lim_{n\to\infty} P_{j,m}((1-f_n)\psi_j)$ is equal to

$$\begin{aligned}
\lim_{n\to\infty} \lim_{\varepsilon\to 0^+} \int_\varepsilon^\delta \frac{1}{\lambda_{j,m}} r_j^{-\lambda_{j,m}} \Big[&\varphi_{j,m} \frac{\partial}{\partial r_j}\frac{\partial}{\partial\theta_j}((1-f_n)\widetilde{\chi}_j) \\
&- \frac{\partial}{\partial\theta_j}\varphi_{j,m} \frac{\partial}{\partial r_j}(1-f_n)\widetilde{\chi}_j \Big|_0^{\omega_j} \Big] dr_j.
\end{aligned} \tag{4.8}$$

If $j-1, j \in \mathbf{N} \cup \mathbf{R}$ and $j-1, j \notin \mathbf{N}$, then $\varphi_{j,m}$ satisfies the Neumann boundary condition, hence (4.8) reduces to

$$\lim_{n\to\infty} \lim_{\varepsilon\to 0^+} \int_\varepsilon^\delta \frac{1}{\lambda_{j,m}} r_j^{-\lambda_{j,m}} \left[\varphi_{j,m} \frac{\partial}{\partial r_j}\frac{\partial}{\partial\theta_j}((1-f_n)\widetilde{\chi}_j) \Big|_0^{\omega_j} \right] dr_j.$$

Applying [5, (4.6)], after a straightforward calculation we get

$$\lim_{n\to\infty} \lim_{\varepsilon\to 0^+} P_{\varepsilon,j,m}((1-f)\psi_j) = 0.$$

If $\omega_j = 3\pi/2$ and $j-1 \in \mathbf{D}$, $j \in \mathbf{N} \cup \mathbf{R}$ or $j-1 \in \mathbf{N} \cup \mathbf{R}$, $j \in \mathbf{D}$, then the integrand in (4.8) is equal to $c_0 r_j^{-\lambda_{j,m}}[(1-f_n) r_j \widetilde{\chi}_j]$ on the side where we have the Robin condition and zero elsewhere. Hence, as previously, the limit is zero. In the remaining cases we have

$$\varphi_{j,m} \frac{\partial}{\partial\theta_j}\widetilde{\chi}_j \Big|_0^{\omega_j} = 0 \quad \text{and} \quad \frac{\partial}{\partial\theta_j}\varphi_{j,m}\widetilde{\chi}_j \Big|_0^{\omega_j} = 0,$$

hence (4.8) is also zero. Thus, we have proved (4.6).

Now, we show (4.7). First, we calculate $P_{\varepsilon,j,m}(h)$ for $h \in \mathcal{D}(\overline{\Omega})$ such that $\operatorname{supp} h \subseteq \bigcup_{j=1}^N \{(x,y);\ \eta_j = 1\} \subseteq \bigcup_{j=1}^N B(S_j,\delta)$ for some $\delta > 0$. After integrating by parts we have

$$P_{\varepsilon,j,m}(h) = \frac{1}{\sqrt{\omega_j}} \int_0^{\omega_j} h(\varepsilon,\theta_j)\,d\theta_j + \frac{1}{\sqrt{\omega_j}}\,\varepsilon \ln \varepsilon \int_0^{\omega_j} \frac{\partial}{\partial r_j} h(\varepsilon,\theta_j)\,d\theta_j + \frac{1}{\sqrt{\omega_j}} \int_\varepsilon^\delta r_j^{-1} \ln r_j \,\frac{\partial}{\partial \theta_j} h\,dr_j\Big|_0^{\omega_j}. \tag{4.9}$$

If we replace h by $(1-f_n)\psi_j$, the last two terms in (4.9) converge to zero as $n \to \infty$ and $\varepsilon \to 0^+$. Hence

$$\lim_{n\to\infty} P_{j,m}((1-f_n)\psi_j) = \lim_{n\to\infty} \lim_{\varepsilon\to 0^+} \frac{1}{\sqrt{\omega_j}} \int_0^{\omega_j} (1-f_n)\psi_j(\varepsilon,\theta_j)\,d\theta_j = \lim_{\varepsilon\to 0^+} \frac{1}{\sqrt{\omega_j}} \int_0^{\omega_j} \psi_j(\varepsilon,\theta_j)\,d\theta_j \neq 0,$$

because there exist constants $c > 0$ and $\delta > 0$ such that $\psi_j(\varepsilon,\theta_j) \geq c$ for all $\varepsilon \in (0,\delta)$ and $\theta_j \in [0,\omega_j]$. Thus we have shown (4.7) and finished the proof of Theorem 3. ∎

5. Proofs of the main results

Proof of Theorem 1 of [5]. Let K denote the dimension of $\mathcal{N}$, i.e. $K = \sum_{j=1}^N n_j$ (see Theorem 3 and the definition of n_j in (3.17)). Let $\bar{S}_k$ for $k = 1,\dots,K$ enumerate the elements of the family $\{\eta_j \chi_j^{-1} r_j^{\lambda_{j,m}} \varphi_{j,m};\ \lambda_{j,m} \in (0,1)\}$ (by Theorem 3 the dimension of $\mathcal{N}$ equals the number of eigenvalues $\lambda_{j,m} \in (0,1)$). It is clear that the family $\{\bar{S}_k\}_{k=1}^K$ is linearly independent. The Laplacian of $\bar{S}_k$ is not orthogonal to $\mathcal{N}$, because from the definition of $\mathcal{N}$ and Proposition 1 of [5] we have

$$\text{if } w \in E(\Delta, L^2(\Omega)) \text{ satisfies (1.1), then } \Delta w \perp \mathcal{N} \Leftrightarrow w \in H^2(\Omega). \tag{5.1}$$

Let F_k be the projection of $\Delta\bar{S}_k$ onto $\mathcal{N}$. The family $\{F_k\}_{k=1}^K$ is linearly independent, because $\operatorname{span}\{\bar{S}_k;\ k = 1,\dots,K\} \cap H^2(\Omega) = \{0\}$. We denote by S_k the variational solution of (1.1) for $f = F_k$. From the definition of S_k and (5.1) we have $S_k \in H^1(\Omega) \setminus H^2(\Omega)$ for $k = 1,\dots,K$ and the Laplacians of S_k for $k = 1,\dots,K$ are a basis of $\mathcal{N}$. From Theorem 3 of [5] we know that each $h \in L^2(\Omega)$ has a unique decomposition $h = h_r + h_s$, where $h_r \in \Delta T^2(\Omega)$ and $h_s \in \mathcal{N}$. Thus, there exist unique $u_r \in T^2(\Omega)$ and $a_k \in \mathbb{R}$ such that u_r is the solution of (1.1) with $f = h_r$ and $h_s = \sum_{k=1}^K a_k \Delta S_k$. To sum up, the function $u := u_r + \sum_{k=1}^K a_k S_k$ is a variational solution of (1.1), hence we have [5, (2.2)]. The estimate [5, (2.3)] is a simple consequence of [5, (3.4)]. Thus, we have finished the proof of Theorem 1 of [5]. ∎

Remark 5. The integer K which appears in Theorem 1 of [5] is equal to $\sum_{j=1}^{N} n_j$. The numbers n_j can be easily calculated from (3.17) because for $m \geq 1$ we have

$$\lambda_{j,m} = \begin{cases} \dfrac{m\pi}{\omega_j} & \text{for } j-1, j \in \mathbf{D}, \\ \dfrac{(m-1)\pi}{\omega_j} & \text{for } j-1, j \in \mathbf{N} \cup \mathbf{R}, \\ \dfrac{(m-1/2)\pi}{\omega_j} & \text{for } j-1 \in \mathbf{D},\ j \in \mathbf{N} \cup \mathbf{R}, \\ & \text{and } j-1 \in \mathbf{N} \cup \mathbf{R},\ j \in \mathbf{D}. \end{cases} \tag{5.2}$$

Furthermore, the singular part of the solutions in a neighbourhood of the vertex S_j has the form $\sum_{m\in N_j} d_j \chi_j^{-1} r_j^{\lambda_{j,m}} \varphi_{j,m}$, where the d_j are constants which depend only on the data.

Now we prove a regularity result for the mixed boundary value problem (1.1) in the maximal domain of the Laplace operator in $L^2(\Omega)$, i.e. in $D(\Delta, L^2(\Omega))$.

Proof of Theorem 2 of [5]. Assume that $v \in D(\Delta, L^2(\Omega))$ satisfies (1.1). We denote by u a variational solution of problem (1.1) with $f = \Delta v$ (see [5, Proposition 1]). Then $u \in H^1(\Omega)$ and $v-u \in \mathcal{M}$. Set $\overline{K} = \sum_{j=1}^{N} (n_j + m_j)$ (the numbers n_j and m_j were defined in (3.17) and can be computed from (5.2)). Let F_k for $k = 1, \ldots, \overline{K}$ be the basis $\{\sigma_{j,m};\ \lambda_{j,m} \in [0,1)\}$ of $\mathcal{M}$ (see (3.27)). Hence $v - u = \sum_{k=1}^{\overline{K}} c_k F_k$ for some numbers c_k, i.e. [5, (2.4)] holds. ∎

Remark 6. The numbers n_j, m_j may be computed from (3.17) and (5.2). Applying the definition of $\sigma_{j,m}$ (see (3.24)) we can describe the singular part (i.e. the components which are not in H^1) of a very weak solution of (1.1). Hence in a neighbourhood of the vertex S_j the singular part of the solution $v \in D(\Delta, L^2(\Omega))$ of (1.1) has the form $\sum_{m\in N_j} d_j \chi_j^{-1} r_j^{-\lambda_{j,m}} \varphi_{j,m}$, where d_j are some constants. In particular (see Corollary 2), each very weak solution of problem (1.1) is in $W^{1,p}(\Omega)$, where $p \in (1,2)$ satisfies (3.11).

6. The nonhomogeneous boundary conditions. Finally, we shall consider the mixed boundary value problem (1.1) with nonhomogeneous boundary conditions. First, we have to describe the space of data on the boundary. For this we introduce the following notations. Let H_j denote the space $H^{3/2}(\Gamma_j)$ for $j \in \mathbf{D}$ and $H^{1/2}(\Gamma_j)$ for $j \in \mathbf{N} \cup \mathbf{R}$, and $H := \prod_{j=1}^{N} H_j$.

We define an operator γ_Γ: $H^2(\Omega) \to H$ by

$$(\gamma_\Gamma u)_j := \begin{cases} \gamma_j u & \text{for } j \in \mathbf{D}, \\ \gamma_j \dfrac{\partial u}{\partial \nu_j} & \text{for } j \in \mathbf{N}, \\ \gamma_j \dfrac{\partial u}{\partial \nu_j} + \alpha_j \gamma_j u & \text{for } j \in \mathbf{R}, \end{cases} \qquad B_\Gamma := \gamma_\Gamma(H^2(\Omega)).$$

We let $\overline{\alpha}_j$ be α_j for $j \in \mathbf{R}$ and zero otherwise. Set $M_1 := \{j;\ j-1 \in \mathbf{D},\ j \in \mathbf{N} \cup \mathbf{R} \text{ and } \omega_j = \pi/2, 3\pi/2\}$, $M_2 := \{j;\ j-1 \in \mathbf{N} \cup \mathbf{R},\ j \in \mathbf{D} \text{ and } \omega_j = \pi/2, 3\pi/2\}$. Applying Theorem 1.6.3 of [3] we can characterize the set B_Γ in the following way (see [5, (3.11)] for the definition of the relation $\equiv_{S_j}$):

$$\begin{aligned} B_\Gamma = \Big\{ (h_j)_{j=1}^N \in H;\quad & h_{j-1}(S_j) = h_j(S_j) \quad \text{for } j-1, j \in \mathbf{D}, \\ & h_j \equiv_{S_j} \sin\omega_j \frac{\partial}{\partial \tau_{j-1}} h_{j-1} + \overline{\alpha}_j h_{j-1} \quad \text{for } j \in M_1, \\ & h_{j-1} \equiv_{S_j} \overline{\alpha}_{j-1} h_j - \sin\omega_j \frac{\partial}{\partial \tau_j} h_j \quad \text{for } j \in M_2 \Big\}. \end{aligned}$$

It can be easily verified that B_Γ is a Banach space with the norm (see [5, (3.11)] for the definition of $(\cdot,\cdot)_{\mathfrak{R}_\delta}$)

$$\begin{aligned} \text{(6.1)} \qquad \|\cdot\|_{B_\Gamma} = \Big\{ \sum_{j=1}^N \|h_j\|_{H_j}^2 &+ \sum_{j \in M_1} \Big(h_j, \sin\omega_j \frac{\partial h_{j-1}}{\partial \tau_{j-1}} + \overline{\alpha}_j h_{j-1} \Big)_{\mathfrak{R}_\delta} \\ &+ \sum_{j \in M_1} \Big(h_{j-1}, \overline{\alpha}_{j-1} h_j - \sin\omega_j \frac{\partial h_j}{\partial \tau_j} \Big)_{\mathfrak{R}_\delta} \Big\}^{1/2}. \end{aligned}$$

Here δ is a positive number such that $\delta < \min_j |\Gamma_j|$.

REMARK 7. The expressions $(\cdot,\cdot)_{\mathfrak{R}_\delta}$ in the norm $\|\cdot\|_{B_\Gamma}$ are essential, because B_Γ is not a closed subspace of H if $M_1 \cup M_2 \neq \emptyset$. The topology of the norm (6.1) is strictly stronger than the subspace topology. However, the operator γ_Γ is continuous with respect to the norm $\|\cdot\|_{B_\Gamma}$.

Thus, the operator γ_Γ is linear and continuous from $H^2(\Omega)$ onto the Banach space $(B_\Gamma, \|\cdot\|_{B_\Gamma})$, hence γ_Γ is an open mapping. We define a relation $\varrho_\Gamma \subseteq B_\Gamma \times H^2(\Omega)$ by

$$(h, w) \in \varrho_\Gamma \ \Leftrightarrow\ \text{there exists } u \in H^2(\Omega) \text{ such that } \gamma_\Gamma u = h,\ (I - \Pi)u = w,$$

where I is the identity mapping and Π denotes a linear and continuous projection of $H^2(\Omega)$ onto its closed subspace $\ker \gamma_\Gamma$. By a straightforward calculation it can be shown that ϱ_Γ is a linear operator defined on B_Γ

with values in $H^2(\Omega)$. Furthermore, ϱ_Γ is a right inverse of γ_Γ and $\varrho_\Gamma^{-1}(U) = \gamma_\Gamma((I - \Pi)^{-1}(U))$ for each $U \subseteq H^2(\Omega)$. Thus ϱ_Γ is continuous. Now we can formulate our result on the regularity of weak solutions of the mixed boundary value problem (1.1) with nonhomogeneous boundary conditions.

THEOREM 4. *There exist a constant C, an integer K and a family of functions $\{S_k\}_{k=1}^K \subseteq H^1(\Omega) \setminus H^2(\Omega)$ such that for each $f \in L^2(\Omega)$ and $h \in B_\Gamma$ there exists a unique $u \in E(\Delta, L^2(\Omega))$ which satisfies*

$$(6.2) \qquad \begin{cases} \Delta u = f & \text{in } \Omega, \\ \gamma_j u = h_j & \text{on } \Gamma_j \text{ for } j \in \mathbf{D}, \\ \gamma_j \dfrac{\partial u}{\partial \nu_j} = h_j & \text{on } \Gamma_j \text{ for } j \in \mathbf{N}, \\ \gamma_j \dfrac{\partial u}{\partial \nu_j} + \alpha_j \gamma_j u = h_j & \text{on } \Gamma_j \text{ for } j \in \mathbf{R}. \end{cases}$$

Furthermore, there exists unique $u_r \in H^2(\Omega)$ and a sequence $\{a_k\}_{k=1}^K$ such that

$$u = u_r + \sum_{k=1}^K a_k S_k,$$

$$\|u_r\|_{H^2(\Omega)} + \sum_{k=1}^K |a_k| \le C\{\|f\|_{L^2(\Omega)} + \|h\|_{B_\Gamma}\}.$$

Proof. Set $w = \varrho_\Gamma(h)$. Applying Theorem 1 of [5] to problem (1.1) with f replaced by $f - \Delta w \in L^2(\Omega)$, we get an integer K and a family $\{S_k;\ k = 1, \ldots, K\} \subseteq H^1(\Omega) \setminus H^2(\Omega)$ such that the solution $v \in E(\Delta, L^2(\Omega))$ has the form $v = v_r + \sum_{k=1}^K a_k S_k$. Here $v_r \in H^2(\Omega)$ and a_k are some numbers (see [5, (2.2)]). Thus $u := v + w$ satisfies (6.2) and belongs to $E(\Delta, L^2(\Omega))$. Clearly $u_r := v_r + w \in H^2(\Omega)$ and we have a unique expansion $u = u_r + \sum_{k=1}^K a_k S_k$, because the family $\{S_k\}_{k=1}^K \subseteq H^1(\Omega) \setminus H^2(\Omega)$ is linearly independent. The uniqueness of u follows from Proposition 1 of [5]. Applying inequality [5, (2.3)] and the continuity of ϱ_Γ we obtain the estimate

$$\|u_r\|_{H^2(\Omega)} \le C\|f\|_{L^2(\Omega)} + (C + 1)C_1\|h\|_{B_\Gamma},$$

where C is the constant from [5, (2.3)] and C_1 is a Lipschitz constant of ϱ_Γ. Hence the proof of Theorem 4 is finished. ■

From Theorems 1 and 2 of [5] and Theorem 4 we can deduce the following corollary.

COROLLARY 7. *There exist a constant C, integers K, $\overline{K}$, a family of harmonic functions $\{F_k\}_{k=1}^{\overline{K}} \subseteq L^2(\Omega) \setminus H^1(\Omega)$ and a family $\{S_k\}_{k=1}^K \subseteq H^1(\Omega) \setminus H^2(\Omega)$ with the following property. If $h \in B_\Gamma$ and $u \in D(\Delta, L^2(\Omega))$ satisfies* (6.2), *then there exist unique numbers a_k, $k = 1, \ldots, K$, c_k, $k =$*

$1, \dots, \overline{K}$, *and* $w \in H^2(\Omega)$ *such that*

$$u = w + \sum_{k=1}^{K} a_k S_k + \sum_{k=1}^{\overline{K}} c_k F_k,$$

$$\|w\|_{H^2(\Omega)} + \sum_{k=1}^{K} |a_k| \le C\{\|\Delta u\|_{L^2(\Omega)} + \|h\|_{B_\Gamma}\}.$$

The integers K, $\overline{K}$ *and the asymptotic behaviour of the functions* S_k, F_k *are described in Remarks* 5 *and* 6. ■

The last result answers the following question: how regular are the functions from the space $L^2(\Omega)$ which satisfy the Poisson equation on a polygonal domain with regular data on the boundary? Regular data on the boundary means here that the data is the value of a boundary operator on an $H^2(\Omega)$ function. A similar problem in the case of a smooth domain was examined in [2, Section 2.5.2]. Our Corollary 7 may be treated as a generalization of Propositions 2.5.2.3 and 2.5.2.4 of [2] to the mixed boundary value problem on a polygonal domain.

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