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# EXISTENCE OF RENORMALIZED SOLUTIONS FOR PARABOLIC EQUATIONS WITHOUT THE SIGN CONDITION AND WITH THREE UNBOUNDED NONLINEARITIES

*Abstract.* We study the problem

$$\frac{\partial b(x,u)}{\partial t} - \operatorname{div}(a(x,t,u,Du)) + H(x,t,u,Du) = \mu \quad \text{in } Q = \Omega \times (0,T),$$
$$b(x,u)|_{t=0} = b(x,u_0) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{in } \partial\Omega \times (0,T).$$

The main contribution of our work is to prove the existence of a renormalized solution without the sign condition or the coercivity condition on $H(x,t,u,Du)$. The critical growth condition on $H$ is only with respect to $Du$ and not with respect to $u$. The datum $\mu$ is assumed to be in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $b(x,u_0) \in L^1(\Omega)$.

**1. Introduction.** In the present paper we establish the existence of a renormalized solution for a class of nonlinear parabolic equations of the type

$$(1.1)\qquad \begin{aligned} &\frac{\partial b(x,u)}{\partial t} + \operatorname{div}(a(x,t,u,Du)) + H(x,t,u,Du) = \mu \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{in } Q = \Omega \times (0,T),\\ &u = 0 \quad \text{on } \partial\Omega \times (0,T),\\ &b(x,u)|_{t=0} = b(x,u_0) \quad \text{on } \Omega. \end{aligned}$$

In problem (1.1), $\Omega$ is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, $T$ is a positive real

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number, while the data $\mu$ and $b(x,u_0)$ are in $L^1(Q)+L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $L^1(\Omega)$. The operator $-\operatorname{div}(a(x,t,u,Du))$ is a Leray–Lions operator which is coercive, $b(x,u)$ is an unbounded function of $u$, $H$ is a nonlinear lower order term and $\mu=f-\operatorname{div}F$ with $f\in L^1(Q)$, $F\in(L^{p'}(Q))^N$.

Dall'Aglio–Orsina [8] and Porretta [13] proved the existence of solutions for the problem (1.1), where $b(x,u)=u$ and $H$ is a nonlinearity with the following "natural" growth condition (of order $p$):

$$|H(x,t,s,\xi)|\le b(s)(|\xi|^p+c(x,t)), \tag{1.2}$$

and which satisfies the classical sign condition

$$H(x,t,s,\xi)s\ge 0. \tag{1.3}$$

The right hand side $\mu$ is assumed to belong to $L^1(Q)$. This result generalizes an analogous one of Boccardo–Gallouët [4] (see also [6, 7] for related topics).

It is our purpose to prove the existence of a renormalized solution for the problem (1.1) in the Sobolev space setting without the sign condition (1.3) and without the coercivity condition

$$|H(x,t,s,\xi)|\ge\beta|\xi|^p\quad\text{ for }|s|\ge\gamma. \tag{1.4}$$

Our growth condition on $H$ is simpler than (1.2): it only concerns growth with respect to $Du$ and not with respect to $u$ (see assumption (H2)). The term $\mu$ belongs to $L^1(Q)$. Note that our result generalizes that of Porretta [13].

The notion of renormalized solution was introduced by J. DiPerna and P.-L. Lions [10] in their study of the Boltzmann equation. This notion was then adapted to an elliptic version of (1.1) by L. Boccardo et al. [5] when the right hand side is in $W^{-1,p'}(\Omega)$, by J. M. Rakotoson [15] when the right hand side is in $L^1(\Omega)$, and finally by G. Dal Maso, F. Murat, L. Orsina and A. Prignet [9] for the case of the right hand side being general measure data.

The plan of the paper is as follows. In Section 2 we make precise all the assumptions on $b$, $a$, $H$, $f$ and $b(x,u_0)$, and give the definition of a renormalized solution of (1.1). In Section 3 we establish the existence of such a solution (Theorem 3.1). Section 4 is devoted to an example which illustrates our abstract result.

**2. Assumptions on data and definition of a renormalized solution.** Throughout the paper, we assume that the following assumptions hold true.

Assumption (H1). $\Omega$ is a bounded open set in $\mathbb{R}^N$ ($N\ge 1$), $T>0$ is given, we set $Q=\Omega\times(0,T)$, and

$$b:\Omega\times\mathbb{R}\to\mathbb{R}\quad\text{ is a Carathéodory function} \tag{2.1}$$

such that for every $x \in \Omega$, $b(x,\cdot)$ is a strictly increasing $C^1$-function with $b(x,0)=0$.

Next, for any $k>0$, there exist $\lambda_k>0$ and functions $A_k \in L^\infty(\Omega)$ and $B_k \in L^p(\Omega)$ such that

$$\lambda_k \le \frac{\partial b(x,s)}{\partial s} \le A_k(x) \quad \text{and} \quad \left| D_x\left(\frac{\partial b(x,s)}{\partial s}\right)\right| \le B_k(x) \tag{2.2}$$

for almost every $x \in \Omega$, and every $s$ such that $|s| \le k$; we denote by $D_x(\partial b(x,s)/\partial s)$ the gradient of $\partial b(x,s)/\partial s$ in the sense of distributions.

There exist $k \in L^{p'}(Q)$ and $\alpha > 0$, $\beta > 0$ such that for almost every $(x,t) \in Q$ all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$|a(x,t,s,\xi)| \le \beta[k(x,t) + |s|^{p-1} + |\xi|^{p-1}], \tag{2.3}$$

$$[a(x,t,s,\xi) - a(x,t,s,\eta)](\xi-\eta) > 0 \quad \text{for all } \xi \ne \eta, \tag{2.4}$$

$$a(x,t,s,\xi).\xi \ge \alpha|\xi|^p. \tag{2.5}$$

Assumption (H2). Let $H : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for a.e. $(x,t) \in Q$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the growth condition

$$|H(x,t,s,\xi)| \le \gamma(x,t) + g(s)|\xi|^p \tag{2.6}$$

is satisfied, where $g : \mathbb{R} \to \mathbb{R}^+$ is a bounded continuous positive function that belongs to $L^1(\mathbb{R})$, while $\gamma \in L^1(Q)$.

We recall that, for $k>1$ and $s$ in $\mathbb{R}$, the truncation is defined as

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k, \\ ks/|s| & \text{if } |s| > k. \end{cases}$$

Definition 2.1. Let $f \in L^1(Q)$, $F \in (L^{p'}(Q))^N$ and $b(\cdot,u_0) \in L^1(\Omega)$. A real-valued function $u$ defined on $Q$ is a *renormalized solution* of problem (1.1) if

$$T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega)) \text{ for all } k \ge 0,\ b(x,u) \in L^\infty(0,T;L^1(\Omega)), \tag{2.7}$$

$$\int_{\{m \le |u| \le m+1\}} a(x,t,u,Du)Du\,dx\,dt \to 0 \quad \text{as } m \to \infty, \tag{2.8}$$

$$\begin{aligned} &\frac{\partial B_S(x,u)}{\partial t} - \operatorname{div}(S'(u)a(x,t,u,Du)) + S''(u)a(x,t,u,Du)Du \\ &\quad + H(x,t,u,Du)S'(u) = fS'(u) - \operatorname{div}(S'(u)F) + S''(u)FDu \quad \text{in } \mathcal{D}'(Q), \end{aligned} \tag{2.9}$$

for all $S \in W^{2,\infty}(\mathbb{R})$ which are piecewise $C^1$ and such that $S'$ has a compact support in $\mathbb{R}$, where $B_S(x,z) = \int_0^z \frac{\partial b(x,r)}{\partial r} S'(r)\,dr$ and

$$B_S(x,u)|_{t=0} = B_S(x,u_0) \quad \text{in } \Omega. \tag{2.10}$$

Remark 2.2. Equation (2.9) is formally obtained through pointwise multiplication of (1.1) by $S'(u)$. However, while $a(x,t,u,Du)$ and $H(x,t,u,Du)$

do not in general make sense in (1.1), all the terms in (2.9) have a meaning in $\mathcal{D}'(Q)$.

Indeed, if $M$ is such that $\operatorname{supp} S' \subset [-M, M]$, the following identifications are made in (2.9):

- $S(u)$ belongs to $L^\infty(Q)$ since $S$ is a bounded function.
- $S'(u)a(x,t,u,Du)$ identifies with $S'(u)a(x,t,T_M(u),DT_M(u))$ a.e. in $Q$. Since $|T_M(u)| \le M$ a.e. in $Q$ and $S'(u) \in L^\infty(Q)$, we deduce from (2.3) and (2.7) that
$$S'(u)a(x,t,T_M(u),DT_M(u)) \in (L^{p'}(Q))^N.$$
- $S''(u)a(x,t,u,Du)Du$ identifies with $S''(u)a(x,t,T_M(u),DT_M(u)) \cdot DT_M(u)$ and
$$S''(u)a(x,t,T_M(u),DT_M(u))DT_M(u) \in L^1(Q).$$
- $S'(u)H(x,t,u,Du)$ identifies with $S'(u)H(x,t,T_M(u),DT_M(u))$ a.e. in $Q$. Since $|T_M(u)| \le M$ a.e. in $Q$ and $S'(u) \in L^\infty(Q)$, we see from (2.3) and (2.6) that
$$S'(u)H(x,t,T_M(u),DT_M(u)) \in L^1(Q).$$
- $S'(u)f$ belongs to $L^1(Q)$ while $S'(u)F$ belongs to $(L^{p'}(Q))^N$.
- $S''(u)FDu$ identifies with $S''(u)FDT_M(u)$, which belongs to $L^1(Q)$.

The above considerations show that equation (2.9) holds in $\mathcal{D}'(Q)$ and that
$$\frac{\partial B_S(x,u)}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q).$$
Due to the properties of $S$ and (2.9), $\partial S(u)/\partial t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$, which implies that $S(u) \in C^0([0,T];L^1(\Omega))$ so that the initial condition (2.10) makes sense, since, due to the properties of $S$ (increasing) and (2.2), we have
$$|B_S(x,r) - B_S(x,r')| \le A_k(x)|S(r) - S(r')| \quad \text{for all } r, r' \in \mathbb{R}. \tag{2.11}$$

**3. Existence results.** In this section we establish the following existence theorem:

THEOREM 3.1. *Let $f \in L^1(Q)$, $F \in (L^{p'}(Q))^N$ and suppose $u_0$ is a measurable function such that $b(\cdot,u_0) \in L^1(\Omega)$. Assume that* (H1) *and* (H2) *hold true. Then there exists a renormalized solution $u$ of problem* (1.1) *in the sense of Definition* 2.1.

*Proof.* The proof is in five steps.

STEP 1: *Approximate problem and a priori estimates.* For $n > 0$, we define approximations of $b$, $H$, $f$ and $u_0$. First, set
$$b_n(x,r) = b(x,T_n(r)) + \frac{1}{n}r. \tag{3.1}$$

In view of (3.1), $b_n$ is a Carathéodory function and satisfies (2.2): there exist $\lambda_n > 0$ and functions $A_n \in L^\infty(\Omega$ and $B_n \in L^p(\Omega)$ such that

$$\lambda_n \le \frac{\partial b_n(x,s)}{\partial s} \le A_n(x) \text{ and } \left| D_x\left(\frac{\partial b_n(x,s)}{\partial s}\right)\right| \le B_n(x) \text{ a.e. in } \Omega,\ s \in \mathbb{R}.$$

Next, set

$$H_n(x,t,s,\xi) = \frac{H(x,t,s,\xi)}{1+\frac{1}{n}|H(x,t,s,\xi)|},$$

and select $f_n$, $u_{0n}$ and $b_n$ so that

$$\text{(3.2)} \quad \begin{aligned} &f_n \in L^{p'}(Q) \text{ and } f_n \to f \text{ a.e. in } Q \text{ and strongly in } L^1(Q) \text{ as } n \to \infty,\\ &u_{0n} \in \mathcal{D}(\Omega), \quad \|b_n(x,u_{0n})\|_{L^1} \le \|b(x,u_0)\|_{L^1},\end{aligned}$$

$$\text{(3.3)} \quad b_n(x,u_{0n}) \to b(x,u_0) \text{ a.e. in } \Omega \text{ and strongly in } L^1(\Omega).$$

Let us now consider the approximate problem

$$\text{(3.4)} \quad \begin{aligned} &\frac{\partial b_n(x,u_n)}{\partial t} - \operatorname{div}(a(x,t,u_n,Du_n)) + H_n(x,t,u_n,Du_n)\\ &\qquad\qquad\qquad = f_n - \operatorname{div} F \text{ in } \mathcal{D}'(Q),\\ &u_n = 0 \quad \text{in } (0,T)\times\partial\Omega,\\ &b_n(x,u_n)|_{(t=0)} = b_n(x,u_{0n}).\end{aligned}$$

Note that

$$|H_n(x,t,s,\xi)| \le H(x,t,s,\xi) \quad \text{and} \quad |H_n(x,t,s,\xi)| \le n$$

for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$.

Moreover, since $f_n \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, proving existence of a weak solution $u_n \in L^p(0,T;W_0^{1,p}(\Omega))$ of (3.4) is an easy task (see e.g. [12]).

Let $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ with $\varphi > 0$. Choosing $v = \exp(G(u_n))\varphi$ as a test function in (3.4) where $G(s) = \int_0^s (g(r)/\alpha)\,dr$ (the function $g$ appears in (2.6)), we have

$$\begin{aligned}\int_Q \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))\varphi\,dx\,dt + \int_Q a(x,t,u_n,Du_n)D(\exp(G(u_n))\varphi)\,dx\,dt\\ + \int_Q H_n(x,t,u_n,Du_n)\exp(G(u_n))\varphi\,dx\,dt\\ = \int_Q f_n\exp(G(u_n))\varphi\,dx\,dt + \int_Q FD(\exp(G(u_n))\varphi)\,dx\,dt.\end{aligned}$$

In view of (2.6) we obtain

$$\begin{aligned}\int_Q \frac{\partial b_n(x,u_n)}{\partial t}&\exp(G(u_n))\varphi\,dx\,dt\\&+\int_Q a(x,t,u_n,Du_n)Du_n\frac{g(u_n)}{\alpha}\exp(G(u_n))\varphi\,dx\,dt\\&+\int_Q a(x,t,u_n,Du_n)\exp(G(u_n))D\varphi\,dx\,dt\\ \le \int_Q &\gamma(x,t)\exp(G(u_n))\varphi\,dx\,dt+\int_Q g(u_n)|Du_n|^p\exp(G(u_n))\varphi\,dx\,dt\\&+\int_Q f_n\exp(G(u_n))\varphi\,dx\,dt+\int_Q FD(\exp(G(u_n))\varphi)\,dx\,dt.\end{aligned}$$

By using (2.5) we obtain

$$\begin{aligned}\int_Q \frac{\partial b_n(x,u_n)}{\partial t}&\exp(G(u_n))\varphi\,dx\,dt\\&+\int_Q a(x,t,u_n,Du_n)\exp(G(u_n))D\varphi\,dx\,dt\\ \le \int_Q &\gamma(x,t)\exp(G(u_n))\varphi\,dx\,dt+\int_Q f_n\exp(G(u_n))\varphi\,dx\,dt\\&+\int_Q F\exp(G(u_n))D\varphi\,dx\,dt+\int_Q FD(\exp(G(u_n)))\varphi\,dx\,dt\end{aligned}\tag{3.5}$$

for all $\varphi\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$ with $\varphi>0$.

On the other hand, taking $v=\exp(-G(u_n))\varphi$ as a test function in (3.4) we deduce as in (3.5) that

$$\begin{aligned}\int_Q \frac{\partial b_n(x,u_n)}{\partial t}&\exp(-G(u_n))\varphi\,dx\,dt\\&+\int_Q a(x,t,u_n,Du_n)\exp(-G(u_n))D\varphi\,dx\,dt\\&+\int_Q \gamma(x,t)\exp(-G(u_n))\varphi\,dx\,dt\\ \ge \int_Q &f_n\exp(-G(u_n))\varphi\,dx\,dt+\int_Q F\exp(-G(u_n))D\varphi\,dx\,dt\\&+\int_Q FD(\exp(-G(u_n)))\varphi\,dx\,dt\end{aligned}\tag{3.6}$$

for all $\varphi\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$ with $\varphi>0$.

Letting $\varphi = T_k(u_n)^+\chi_{(0,\tau)}$, for every $\tau \in [0,T]$, in (3.5), we have

$$\begin{aligned}(3.7)\qquad &\int_\Omega B^n_{k,G}(x,u_n(\tau))dx + \int_{Q_\tau} a(x,t,u_n,Du_n)\exp(G(u_n))DT_k(u_n)^+\,dx\,dt \\ &\le \int_{Q_\tau} \gamma(x,t)\exp(G(u_n))T_k(u_n)^+\,dx\,dt + \int_{Q_\tau} f_n\exp(G(u_n))T_k(u_n)^+\,dx\,dt \\ &\quad + \int_Q FD(T_k(u_n)^+)\exp(G(u_n))\,dx\,dt \\ &\quad + \int_Q FT_k(u_n)^+\exp(G(u_n))Du_n\frac{g(u_n)}{\alpha}\,dx\,dt + \int_\Omega B^n_{k,G}(x,u_{0n})\,dx,\end{aligned}$$

where $B^n_{k,G}(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}T_k(s)^+\exp(G(s))ds$. Due to the definition of $B^n_{k,G}$ and $|G(u_n)| \le \exp(\|g\|_{L^1(\mathbb{R})}/\alpha)$ we have

$$(3.8)\qquad 0 \le \int_\Omega B^n_{k,G}(x,u_{0n})\,dx \le k\exp(\|g\|_{L^1(\mathbb{R})}/\alpha)\|b(.,u_0)\|_{L^1(\Omega)}.$$

Using (3.8), $B^n_{k,G}(x,u_n) \ge 0$ and Young's inequality, we obtain

$$\begin{aligned}&\int_{Q_\tau} a(x,t,u_n,DT_k(u_n)^+)DT_k(u_n)^+\exp(G(u_n))\,dx\,dt \\ &\le k\exp(\|g\|_{L^1(\mathbb{R})}/\alpha)\bigg(\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \frac{1}{p'\alpha^{p'/p}}\|F\|_{(L^{p'}(Q))^N} \\ &\quad + \|b_n(x,u_{0n})\|_{L^1(\Omega)}\bigg) + \frac{\alpha}{p}\int_{Q_\tau} |DT_k(u_n)^+|^p\exp(G(u_n))\,dx\,dt \\ &\quad + \frac{1}{\alpha}\int_{Q_\tau} Fg(u_n)\exp(G(u_n))Du_nT_k(u_n)^+\,dx\,dt.\end{aligned}$$

Thanks to (2.5) we have

$$\begin{aligned}(3.9)\qquad &\alpha\left(\frac{p-1}{p}\right)\int_{Q_\tau} |DT_k(u_n)^+|^p\exp(G(u_n))\,dx\,dt \\ &\le k\exp(\|g\|_{L^1(\mathbb{R})}/\alpha)\bigg(\|f_n\|_{L^1(Q)} + \|\gamma\|_{L^1(Q)} + \frac{1}{p'\alpha^{p'/p}}\|F\|_{(L^{p'}(Q))^N} \\ &\quad + \|b_n(x,u_{0n})\|_{L^1(\Omega)}\bigg) + \frac{1}{\alpha}\int_{Q_\tau} Fg(u_n)\exp(G(u_n))Du_n\chi_{\{u_n>0\}}\,dx\,dt.\end{aligned}$$

Let us observe that if we take $\varphi = \rho(u_n) = \int_0^{u_n} g(s)\chi_{\{s>0\}}ds$ in (3.5) and use

(2.5) we obtain

$$\Big[\int_\Omega B_g^n(x,u_n)\,dx\Big]_0^T + \alpha\int_Q |Du_n|^p g(u_n)\chi_{\{u_n>0\}}\exp(G(u_n))\,dx\,dt$$
$$\le \Big(\int_0^\infty g(s)\,ds\Big)\exp(\|g\|_{L^1(\mathbb{R})}/\alpha)(\|\gamma\|_{L^1(Q)}+\|f_n\|_{L^1(Q)})$$
$$+\int_Q FDu_n g(u_n)\chi_{\{u_n>0\}}\exp(G(u_n))\,dx\,dt$$
$$+\Big(\int_0^\infty g(s)\,ds\Big)\int_Q |FDu_n|\frac{g(u_n)}{\alpha}\exp(G(u_n))\chi_{\{u_n>0\}}\,dx\,dt,$$

where $B_g^n(x,r)=\int_0^r \frac{\partial b_n(x,s)}{\partial s}\rho(s)\exp(G(s))ds$, which implies, using $B_g^n(x,r) \ge 0$ and Young's inequality,

$$\alpha\int_{\{u_n>0\}} |Du_n|^p g(u_n)\exp(G(u_n))\,dx\,dt$$
$$\le \exp(\|g\|_{L^1(\mathbb{R})}/\alpha)(\|\gamma\|_{L^1(Q)}+\|f\|_{L^1(Q)}+\|b(x,u_0)\|_{L^1(\Omega)})$$
$$+C_1\|g\|_\infty\exp(\|g\|_{L^1(\mathbb{R})}/\alpha)\int_Q |F|^{p'}\,dx\,dt$$
$$+\frac{\alpha}{2p}\int_Q |Du_n|^p\frac{g(u_n)}{\alpha}\exp(G(u_n))\chi_{\{u_n>0\}}\,dx\,dt$$
$$+C_2\int_0^\infty g(s)\,ds\|g\|_\infty\exp\big(\|g\|_{L^1(\mathbb{R})}/\alpha\big)\int_Q |F|^{p'}\,dx\,dt$$
$$+\frac{\alpha}{2p}\int_Q |Du_n|^p\frac{g(u_n)}{\alpha}\exp(G(u_n))\chi_{\{u_n>0\}}\,dx\,dt.$$

We obtain

$$\int_{\{u_n>0\}} g(u_n)|Du_n|^p\exp(G(u_n))\,dx\,dt \le C_3.$$

Similarly, taking $\varphi=\int_{u_n}^0 g(s)\chi_{\{s<0\}}ds$ as a test function in (3.6), we conclude that

$$\int_{\{u_n<0\}} g(u_n)|Du_n|^p\exp(G(u_n))\,dx\,dt \le C_4.$$

Consequently,

$$\int_Q g(u_n)|Du_n|^p\exp(G(u_n))\,dx\,dt \le C_5. \tag{3.10}$$

Above, $C_1, \ldots, C_5$ are constants independent of $n$. We deduce that

$$\int_Q |DT_k(u_n)^+|^p \, dx \, dt \le C_6 k. \tag{3.11}$$

Similarly to (3.11) we take $\varphi = T_k(u_n)^- \chi_{(0,\tau)}$ in (3.6) to deduce that

$$\int_Q |DT_k(u_n)^-|^p \, dx \, dt \le C_7 k. \tag{3.12}$$

Combining (3.11) and (3.12) we conclude that

$$\|T_k(u_n)\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} \le C_8 k. \tag{3.13}$$

where $C_6$, $C_7$, $C_8$ are constants independent of $n$. Thus, $T_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, independently of $n$ for any $k > 0$. We deduce from (3.7), (3.8) and (3.13) that

$$\int_\Omega B^n_{k,G}(x, u_n(\tau)) \, dx \le Ck. \tag{3.14}$$

Now we turn to proving the almost everywhere convergence of $u_n$ and $b_n(x, u_n)$. Consider a nondecreasing function $g_k \in C^2(\mathbb{R})$ such that $g_k(s) = s$ for $|s| \le k/2$ and $g_k(s) = k$ for $|s| \ge k$. Multiplying the approximate equation by $g'_k(u_n)$, we get

$$\begin{aligned}\frac{\partial B^n_k(x,u_n)}{\partial t} &- \operatorname{div}(a(x,t,u_n,Du_n) g'_k(u_n)) \\ &+ a(x,t,u_n,Du_n) g''_k(u_n) Du_n + H_n(x,t,u_n,Du_n) g'_k(u_n) \\ &= f_n g'_k(u_n) - \operatorname{div}(F g'_k(u_n)) + F g''_k(u_n) Du_n\end{aligned} \tag{3.15}$$

where $B^n_k(x,z) = \int_0^z \frac{\partial b_n(x,s)}{\partial s} g'_k(s) \, ds$. As a consequence of (3.13), we deduce that $g_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and $\partial B^n_k(x,u_n)/\partial t$ is bounded in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$. Due to the properties of $g_k$ and (2.2), we conclude that $\partial g_k(u_n)/\partial t$ is bounded in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$, which implies that $g_k(u_n)$ is compact in $L^1(Q)$.

Due to the choice of $g_k$, we conclude that for each $k$, the sequence $T_k(u_n)$ converges almost everywhere in $Q$, which implies that $u_n$ converges almost everywhere to some measurable function $v$ in $Q$. Thus by using the same argument as in [2], [3], [18], we can show the following lemma.

LEMMA 3.2. *Let $u_n$ be a solution of the approximate problem* (3.4). *Then*

$$u_n \to u \qquad a.e. \text{ in } Q, \tag{3.16}$$

$$b_n(x,u_n) \to b(x,u) \qquad a.e. \text{ in } Q. \tag{3.17}$$

*We can deduce from* (3.13) *that*

$$T_k(u_n) \rightharpoonup T_k(u) \qquad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \tag{3.18}$$

*which implies, by using* (2.3), *that for all* $k > 0$ *there exists* $\Lambda_k \in (L^{p'}(Q))^N$ *such that*

$$a(x,t,T_k(u_n),DT_k(u_n)) \rightharpoonup \Lambda_k \quad \text{weakly in } (L^{p'}(Q))^N. \tag{3.19}$$

We now establish that $b(\cdot,u)$ belongs to $L^\infty(0,T;L^1(\Omega))$. Using (3.16) and passing to lim inf in (3.14) as $n\to\infty$, we obtain $(1/k)\int_\Omega B_{k,G}(x,u(\tau))\,dx \le C$, for a.e. $\tau$ in $(0,T)$. Due to the definition of $B_{k,G}(x,s)$ and the fact that $(1/k)B_{k,G}(x,u)$ converges pointwise to

$$\int_0^u \operatorname{sgn}(s)\frac{\partial b(x,s)}{\partial s}\exp(G(s))\,ds \ge |b(x,u)|$$

as $k\to\infty$, it follows that $b(\cdot,u)$ belong to $L^\infty(0,T;L^1(\Omega))$.

Lemma 3.3. *Let* $u_n$ *be a solution of the approximate problem* (3.4). *Then*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,Du_n)Du_n\,dx\,dt = 0. \tag{3.20}$$

*Proof.* Set $\varphi = T_1(u_n - T_m(u_n))^+ = \alpha_m(u_n)$ in (3.5); this function is admissible since $\varphi\in L^p(0,T;W_0^{1,p}(\Omega))$ and $\varphi\ge 0$. Then we have

$$\begin{aligned}\int_Q \frac{\partial b_n(x,u_n)}{\partial t}&\exp(G(u_n))\alpha_m(u_n)\,dx\,dt\\ &+\int_{\{m\le u_n\le m+1\}} a(x,t,u_n,Du_n)Du_n\exp(G(u_n))\,dx\,dt\\ &\le \int_Q f_n\exp(G(u_n))\alpha_m(u_n) + \int_Q \gamma(x,t)\exp(G(u_n))\alpha_m(u_n)\,dx\,dt\\ &+\int_{\{m\le u_n\le m+1\}} FDu_n\exp(G(u_n))\,dx\,dt\\ &+\int_Q FDu_n\frac{g(u_n)}{\alpha}\exp(G(u_n))\alpha_m(u_n)\,dx\,dt.\end{aligned}$$

This gives, by setting $B^m_{n,G}(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\exp(G(s))\alpha_m(s)\,ds$, and by Young's inequality,

$$\begin{aligned}\int_\Omega B^m_{n,G}(x,u_n)(T)\,dx &+ \int_{\{m\le u_n\le m+1\}} a(x,t,u_n,Du_n)Du_n\exp(G(u_n))\,dx\,dt\\ \le \exp(\|g\|_{L^1(\mathbb{R})}/\alpha)&\Big[\int_{\{|u_n|>m\}}(|f_n|+|\gamma|)\,dx\,dt + \int_{\{|u_{n0}|>m\}}|b_n(x,u_{0n})|\,dx\Big]\\ &+C_1\int_{\{u_n\ge m\}}|F|^{p'}\,dx\,dt + \frac{\alpha}{p}\int_{\{m\le u_n\le m+1\}}|Du_n|^p\exp(G(u_n))\,dx\,dt\\ &+C_2\int_{\{u_n\ge m\}}|F|^{p'}\,dx\,dt + C_3\int_{\{u_n\ge m\}}|Du_n|^p g(u_n)\exp(G(u_n))\,dx\,dt.\end{aligned}$$

Using (2.5) and since $B^m_{n,G}(x,u_n)(T) > 0$, we obtain

$$\tag{3.21} \frac{p-1}{p}\int_{\{m\le u_n\le m+1\}} a(x,t,u_n,Du_n)Du_n\exp(G(u_n))\,dx\,dt$$
$$\le \exp(\|g\|_{L^1(\mathbb{R})}/\alpha)\Big[\int_{\{|u_n|>m\}}(|f_n|+|\gamma|)\,dx\,dt+\int_{\{|u_{n0}|>m\}}|b_n(x,u_{0n})|\,dx\Big]$$
$$+C_4\int_{\{u_n\ge m\}}|F|^{p'}\,dx\,dt+C_5\int_{\{u_n>m\}}g(u_n)\exp(G(u_n))|Du_n|^p\,dx\,dt.$$

Taking $\varphi=\rho_m(u_n)=\int_0^{u_n}g(s)\chi_{\{s>m\}}\,ds$ as a test function in (3.5), we obtain

$$\Big[\int_\Omega B^n_m(x,u_n)\,dx\Big]_0^T+\int_Q a(x,t,u_n,Du_n)Du_ng(u_n)\chi_{\{u_n>m\}}\exp(G(u_n))\,dx\,dt$$
$$\le\Big(\int_m^\infty g(s)\chi_{\{s>m\}}\,ds\Big)\exp(\|g\|_{L^1(\mathbb{R})}/\alpha)(\|\gamma\|_{L^1(Q)}+\|f_n\|_{L^1(Q)})$$
$$+\int_Q FDu_ng(u_n)\chi_{\{u_n>m\}}\exp(G(u_n))\,dx\,dt$$
$$+\Big(\int_m^\infty g(s)\chi_{\{s>m\}}\,ds\Big)\int_Q FDu_n\frac{g(u_n)}{\alpha}\exp(G(u_n))\chi_{\{u_n>m\}}\,dx\,dt,$$

where $B^n_m(x,r)=\int_0^r\frac{\partial b_n(x,s)}{\partial s}\rho_m(s)\exp(G(s))\,ds$, which implies, since $B^n_m(x,r)\ge 0$, by (2.5) and Young's inequality,

$$\tag{3.22} \frac{\alpha(p-1)}{p}\int_{\{u_n>m\}}|Du_n|^pg(u_n)\exp(G(u_n))\,dx\,dt$$
$$\le\Big(\int_m^\infty g(s)\,ds\Big)\exp(\|g\|_{L^1(\mathbb{R})}/\alpha)(\|\gamma\|_{L^1(Q)}+\|f_n\|_{L^1(Q)}$$
$$+\|b_n(x,u_{0n})\|_{L^1(\Omega)}+C\|F\|^{p'}_{(L^{p'}(Q))^N}).$$

Using (3.22) and the strong convergence of $f_n$ in $L^1(\Omega)$ and $b_n(x,u_{0n})$ in $L^1(\Omega)$, $\gamma\in L^1(\Omega)$, $g\in L^1(\mathbb{R})$ and $F\in(L^{p'}(Q))^N$, by Lebesgue's theorem, passing to the limit in (3.21), we conclude that

$$\tag{3.23} \lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le u_n\le m+1\}}a(x,t,u_n,Du_n)Du_n\,dx\,dt=0.$$

On the other hand, taking $\varphi=T_1(u_n-T_m(u_n))^-$ as a test function in (3.6) and reasoning as in the proof of (3.23) we deduce that

$$\tag{3.24} \lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{-(m+1)\le u_n\le -m\}}a(x,t,u_n,Du_n)Du_n\,dx\,dt=0.$$

Thus (3.20) follows from (3.23) and (3.24).

Step 2: *Almost everywhere convergence of the gradients.* This step is devoted to introducing for $k \geq 0$ fixed a time regularization of the function $T_k(u)$ in order to apply the monotonicity method. This kind of method was first introduced by R. Landes (see Lemma 6 and Proposition 3, p. 230, and Proposition 4, p. 231, in [11]). Let $\psi_i \in \mathcal{D}(\Omega)$ be a sequence which converges strongly to $u_0$ in $L^1(\Omega)$. Set $w^i_\mu = (T_k(u))_\mu + e^{-\mu t} T_k(\psi_i)$ where $(T_k(u))_\mu$ is the mollification of $T_k(u)$ with respect to time. Note that $w^i_\mu$ is a smooth function having the following properties:

$$\frac{\partial w^i_\mu}{\partial t} = \mu(T_k(u) - w^i_\mu), \quad w^i_\mu(0) = T_k(\psi_i), \quad |w^i_\mu| \leq k, \tag{3.25}$$

$$w^i_\mu \to T_k(u) \quad \text{in } L^p(0,T; W^{1,p}_0(\Omega)) \text{ as } \mu \to \infty. \tag{3.26}$$

We introduce the following function of one real variable $s$:

$$h_m(s) = \begin{cases} 1 & \text{if } |s| \leq m, \\ 0 & \text{if } |s| \geq m+1, \\ m+1+|s| & \text{if } m \leq |s| \leq m+1. \end{cases}$$

For $m > k$, let $\varphi = (T_k(u_n) - w^i_\mu)^+ h_m(u_n) \in L^p(0,T; W^{1,p}_0(\Omega)) \cap L^\infty(Q)$ and $\varphi \geq 0$. If we take this function in (3.5), we obtain

$$\begin{aligned} &\int_{\{T_k(u_n)-w^i_\mu \geq 0\}} \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n))(T_k(u_n) - w^i_\mu) h_m(u_n)\, dx\, dt \\ &\quad + \int_{\{T_k(u_n)-w^i_\mu \geq 0\}} a(x,t,u_n,Du_n) D(T_k(u_n) - w^i_\mu) h_m(u_n)\, dx\, dt \\ &\quad - \int_{\{m \leq u_n \leq m+1\}} \exp(G(u_n)) a(x,t,u_n,Du_n) Du_n (T_k(u_n) - w^i_\mu)^+\, dx\, dt \\ &\leq \int_Q (\gamma(x,t) + f_n) \exp(G(u_n))(T_k(u_n) - w^i_\mu)^+ h_m(u_n)\, dx\, dt \\ &\quad + \int_Q F Du_n \frac{g(u_n)}{\alpha} \exp(G(u_n))(T_k(u_n) - w^i_\mu)^+ h_m(u_n)\, dx\, dt \\ &\quad + \int_{\{T_k(u_n)-w^i_\mu \geq 0\}} \exp(G(u_n)) F D(T_k(u_n) - w^i_\mu) h_m(u_n)\, dx\, dt \\ &\quad - \int_{\{m \leq u_n \leq m+1\}} \exp(G(u_n)) F Du_n (T_k(u_n) - w^i_\mu)^+\, dx\, dt. \end{aligned} \tag{3.27}$$

Observe that

$$\Big|\int_{\{m\le u_n\le m+1\}} \exp(G(u_n))a(x,t,u_n,Du_n)Du_n(T_k(u_n)-w^i_\mu)^+\,dx\,dt\Big|$$
$$\le 2k\exp(\|g\|_{L^1(\mathbb{R})}/\alpha)\int_{\{m\le u_n\le m+1\}} a(x,t,u_n,Du_n)Du_n\,dx\,dt,$$

and

$$\Big|\int_{\{m\le u_n\le m+1\}} \exp(G(u_n))FDu_n(T_k(u_n)-w^i_\mu)^+\,dx\,dt\Big|\le$$
$$2k\exp(\|g\|_{L^1(\mathbb{R})}/\alpha)\frac{\|F\|_{(L^{p'}(Q))^N}}{\alpha^{1/p}}\Big(\int_{\{m\le u_n\le m+1\}} a(x,t,u_n,Du_n)Du_n\,dx\,dt\Big)^{1/p}.$$

Thanks to (3.20) the third and fourth integrals on the right hand side tend to zero as $n$ and $m$ tend to infinity, and by Lebesgue's theorem and $F\in(L^{p'}(Q))^N$, we deduce that the right hand side converges to zero as $n$, $m$ and $\mu$ tend to infinity. Since

$$(T_k(u_n)-w^i_\mu)^+h_m(u_n)\rightharpoonup(T_k(u)-w^i_\mu)^+h_m(u)\quad\text{weakly}^*\text{ in }L^\infty(Q)$$

as $n\to\infty$ and strongly in $L^p(0,T;W^{1,p}_0(\Omega))$, and $(T_k(u)-w^i_\mu)^+h_m(u)\rightharpoonup 0$ weakly$^*$ in $L^\infty(Q)$ and strongly in $L^p(0,T;W^{1,p}_0(\Omega))$ as $\mu\to\infty$, it follows that the first and second integrals on the right-hand side of (3.27) converge to zero as $n,m,\mu\to\infty$.

Below, we denote by $\varepsilon_l(n,m,\mu,i)$, $l=1,2,\dots$, various functions that tend to zero as $n$, $m$, $i$ and $\mu$ tend to infinity.

The very definition of the sequence $w^i_\mu$ makes it possible to establish the following lemma.

Lemma 3.4 (see [16]). *For $k\ge 0$ we have*

$$\int_{\{T_k(u_n)-w^i_\mu\ge 0\}}\frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))(T_k(u_n)-w^i_\mu)h_m(u_n)\,dx\,dt \ge \varepsilon(n,m,\mu,i). \tag{3.28}$$

On the other hand, the second term on the left hand side of (3.27) reads

$$\int_{\{T_k(u_n)-w^i_\mu\ge 0\}} a(x,t,u_n,Du_n)D(T_k(u_n)-w^i_\mu)h_m(u_n)\,dx\,dt$$
$$=\int_{\{T_k(u_n)-w^i_\mu\ge 0,\,|u_n|\le k\}} a(x,t,T_k(u_n),DT_k(u_n))D(T_k(u_n)-w^i_\mu)h_m(u_n)\,dx\,dt$$
$$-\int_{\{T_k(u_n)-w^i_\mu\ge 0,\,|u_n|\ge k\}} a(x,t,u_n,Du_n)Dw^i_\mu h_m(u_n)\,dx\,dt.$$

Since $m > k$, and $h_m(u_n) = 0$ on $\{|u_n| \geq m+1\}$, one has

$$
\begin{aligned}
(3.29)\quad & \int\limits_{\{T_k(u_n)-w_\mu^i \geq 0\}} a(x,t,u_n,Du_n)D(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt \\
&= \int\limits_{\{T_k(u_n)-w_\mu^i \geq 0\}} a(x,t,T_k(u_n),DT_k(u_n))D(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt \\
&\quad - \int\limits_{\{T_k(u_n)-w_\mu^i \geq 0,\,|u_n|\geq k\}} a(x,t,T_{m+1}(u_n),DT_{m+1}(u_n))Dw_\mu^i h_m(u_n)\,dx\,dt \\
&= J_1 + J_2.
\end{aligned}
$$

In the following we pass to the limit in (3.29): first we let $n$ tend to $\infty$, then $\mu$ and finally $m$ tend to $\infty$. Since $a(x,t,T_{m+1}(u_n),DT_{m+1}(u_n))$ is bounded in $(L^{p'}(Q))^N$ we see that $a(x,t,T_{m+1}(u_n),DT_{m+1}(u_n))h_m(u_n)\chi_{\{|u_n|>k\}} \to \Lambda_m\chi_{\{|u|>k\}}h_m(u)$ strongly in $(L^{p'}(Q))^N$ as $n \to \infty$. It follows that

$$
\begin{aligned}
J_2 &= \int\limits_{\{T_k(u)-w_\mu^i \geq 0\}} \Lambda_m Dw_\mu^i h_m(u)\chi_{\{|u|>k\}}\,dx\,dt + \varepsilon(n) \\
&= \int\limits_{\{T_k(u)-w_\mu^i \geq 0\}} \Lambda_m(DT_k(u)_\mu - e^{-\mu t}DT_k(\psi_i))h_m(u)\chi_{\{|u|>k\}}\,dx\,dt + \varepsilon(n).
\end{aligned}
$$

Letting $\mu \to \infty$ implies that $J_2 = \int_Q \Lambda_m DT_k(u)\,dx\,dt + \varepsilon(n,\mu)$. Using now the term $J_1$ of (3.29) one can easily show that

$$
\begin{aligned}
(3.30)\quad & \int\limits_{\{T_k(u_n)-w_\mu^i \geq 0\}} a(x,t,T_k(u_n),DT_k(u_n))D(T_k(u_n)-w_\mu^i)h_m(u_n)\,dx\,dt \\
&= \int\limits_{\{T_k(u_n)-w_\mu^i \geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \\
&\qquad\qquad \times [DT_k(u_n) - DT_k(u)]h_m(u_n)\,dx\,dt \\
&\quad + \int\limits_{\{T_k(u_n)-w_\mu^i \geq 0\}} a(x,t,T_k(u_n),DT_k(u))(DT_k(u_n)-DT_k(u))h_m(u_n)\,dx\,dt \\
&\quad + \int\limits_{\{T_k(u_n)-w_\mu^i \geq 0\}} a(x,t,T_k(u_n),DT_k(u_n))DT_k(u)h_m(u_n)\,dx\,dt \\
&\quad - \int\limits_{\{T_k(u_n)-w_\mu^i \geq 0\}} a(x,t,T_k(u_n),DT_k(u_n))Dw_\mu^i h_m(u_n)\,dx\,dt \\
&= K_1 + K_2 + K_3 + K_4.
\end{aligned}
$$

We shall pass to the limit as $n$, $\mu \to \infty$ in the last three integrals. Starting with $K_2$, we have, by letting $n \to \infty$,

$$
(3.31)\qquad K_2 = \varepsilon(n).
$$

For $K_3$, we have, by letting $n \to \infty$ and using (3.19),

$$K_3 = \varepsilon(n). \tag{3.32}$$

For $K_4$ we can write

$$K_4 = -\int_{\{T_k(u)-w^i_\mu \geq 0\}} \Lambda_k D w^i_\mu h_m(u)\,dx\,dt + \varepsilon(n).$$

Letting $\mu \to \infty$ implies that

$$K_4 = -\int_Q \Lambda_k DT_k(u)\,dx\,dt + \varepsilon(n,\mu). \tag{3.33}$$

We then conclude that

$$\begin{aligned}
&\int_{\{T_k(u_n)-w^i_\mu \geq 0\}} a(x,t,T_k(u_n),DT_k(u_n))\nabla(T_k(u_n)-w^i_\mu)h_m(u_n)\,dx\,dt \\
&\quad= \int_{\{T_k(u_n)-w^i_\mu \geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \\
&\qquad\qquad \times [DT_k(u_n) - DT_k(u)]\, h_m(u_n)\,dx\,dt + \varepsilon(n,\mu).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\int_{\{T_k(u_n)-w^i_\mu \geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \\
&\qquad\qquad \times [DT_k(u_n) - DT_k(u)]\,dx\,dt \\
&= \int_{\{T_k(u_n)-w^i_\mu \geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \\
&\qquad\qquad \times [DT_k(u_n) - DT_k(u)]h_m(u_n)\,dx\,dt \\
&\quad + \int_{\{T_k(u_n)-w^i_\mu \geq 0\}} a(x,t,T_k(u_n),DT_k(u_n)) \\
&\qquad\qquad \times (DT_k(u_n) - DT_k(u))(1-h_m(u_n))\,dx\,dt \\
&\quad - \int_{\{T_k(u_n)-w^i_\mu \geq 0\}} a(x,t,T_k(u_n),DT_k(u)) \\
&\qquad\qquad \times (DT_k(u_n) - DT_k(u))(1-h_m(u_n))\,dx\,dt.
\end{aligned} \tag{3.34}$$

Since $h_m(u_n) = 1$ in $\{|u_n| \leq m\}$ and $\{|u_n| \leq k\} \subset \{|u_n| \leq m\}$ for $m$ large enough, we deduce from (3.34) that

$$\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \times [DT_k(u_n)-DT_k(u)]\,dx\,dt$$

$$= \int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \times [DT_k(u_n)-DT_k(u)]h_m(u_n)\,dx\,dt$$

$$+ \int_{\{T_k(u_n)-w_\mu^i\geq 0,\,|u_n|>k\}} a(x,t,T_k(u_n),DT_k(u))DT_k(u)(1-h_m(u_n))\,dx\,dt.$$

It is easy to see that the last terms of the last equality tend to zero as $n\to\infty$, which implies that

$$\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \times [DT_k(u_n)-DT_k(u)]\,dx\,dt$$

$$= \int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \times [DT_k(u_n)-DT_k(u)]h_m(u_n)\,dx\,dt + \varepsilon(n).$$

Combining (3.28) and (3.30)–(3.34) we obtain

$$\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \times [DT_k(u_n)-DT_k(u)]\,dx\,dt \leq \varepsilon(n,\mu,m). \tag{3.35}$$

Passing to the limit in (3.35) as $n,m\to\infty$, we obtain

$$\lim_{n\to\infty}\int_{\{T_k(u_n)-w_\mu^i\geq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \times [DT_k(u_n)-DT_k(u)]\,dx\,dt = 0. \tag{3.36}$$

On the other hand, take $\varphi = (T_k(u_n)-w_\mu^i)^- h_m(u_n)$ in (3.6). Similarly, we can deduce as in (3.36) that

$$\lim_{n\to\infty}\int_{\{T_k(u_n)-w_\mu^i\leq 0\}} [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \times [DT_k(u_n)-DT_k(u)]\,dx\,dt = 0. \tag{3.37}$$

Combining (3.36) and (3.37), we conclude

$$\lim_{n\to\infty}\int_Q [a(x,t,T_k(u_n),DT_k(u_n)) - a(x,t,T_k(u_n),DT_k(u))] \times [DT_k(u_n)-DT_k(u)]\,dx\,dt = 0. \tag{3.38}$$

This implies that

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } L^p(0,T;W_0^{1,p}(\Omega)) \quad \forall k. \tag{3.39}$$

Now, observe that, for every $\sigma > 0$,

$$\begin{aligned}
\operatorname{meas}\{(x,t) \in \Omega \times [0,T] : |Du_n - Du| > \sigma\} \\
\le \operatorname{meas}\{(x,t) \in \Omega \times [0,T] : |Du_n| > k\} \\
+ \operatorname{meas}\{(x,t) \in \Omega \times [0,T] : |u| > k\} \\
+ \operatorname{meas}\{(x,t) \in \Omega \times [0,T] : |DT_k(u_n) - DT_k(u)| > \sigma\}.
\end{aligned}$$

Then as a consequence of (3.39) we also find that $Du_n$ converges to $Du$ in measure and therefore, for a subsequence,

$$Du_n \to Du \quad \text{a.e. in } Q, \tag{3.40}$$

which implies that

$$a(x,t,T_k(u_n),DT_k(u_n)) \rightharpoonup a(x,t,T_k(u),DT_k(u)) \quad \text{in } (L^{p'}(Q))^N. \tag{3.41}$$

STEP 3: *Equi-integrability of the nonlinearity sequence.* We shall now prove that $H_n(x,t,u_n,Du_n) \to H(x,t,u,Du)$ strongly in $L^1(Q)$ by using Vitali's theorem. Since $H_n(x,t,u_n,Du_n) \to H(x,t,u,Du)$ a.e. in $Q$, considering now $\varphi = \rho_h(u_n) = \int_0^{u_n} g(s)\chi_{\{s>h\}}\,ds$ as a test function in (3.5), we obtain

$$\begin{aligned}
\Big[\int_\Omega B_h^n(x,u_n)\,dx\Big]_0^T + \int_Q a(x,t,u_n,Du_n)Du_n g(u_n)\chi_{\{u_n>h\}}\exp(G(u_n))\,dx\,dt \\
\le \Big(\int_h^\infty g(s)\chi_{\{s>h\}}\,ds\Big)\exp(\|g\|_{L^1(\mathbb{R})}/\alpha)(\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)}) \\
+ \int_Q FDu_n g(u_n)\chi_{\{u_n>h\}}\exp(G(u_n))\,dx\,dt \\
+ \Big(\int_h^\infty g(s)\chi_{\{s>h\}}\,ds\Big)\int_Q |FDu_n|\frac{g(u_n)}{\alpha}\exp(G(u_n))\chi_{\{u_n>h\}}\,dx\,dt,
\end{aligned}$$

where $B_h^n(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s}\rho_h(s)\exp(G(s))\,ds$, which implies, in view of $B_h^n(x,r) \ge 0$, (2.5) and Young's inequality,

$$\begin{aligned}
\frac{\alpha(p-1)}{p}\int_{\{u_n>h\}} |Du_n|^p g(u_n)\exp(G(u_n))\,dx\,dt \\
\le \Big(\int_h^\infty g(s)\,ds\Big)\exp(\|g\|_{L^1(\mathbb{R})}/\alpha)(\|\gamma\|_{L^1(Q)} + \|f_n\|_{L^1(Q)} \\
+ \|b_n(x,u_{0n})\|_{L^1(\Omega)} + C\|F\|_{(L^{p'}(Q))^N}).
\end{aligned}$$

We conclude that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}} g(u_n)|Du_n|^p\,dx\,dt = 0.$$

Consequently,

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}} g(u_n)|Du_n|^p\,dx\,dt = 0,$$

which implies, for $h$ large enough and for a subset $E$ of $Q$,

$$\begin{aligned}\lim_{\text{meas}(E)\to 0}\int_E g(u_n)|Du_n|^p\,dx\,dt \le \|g\|_\infty \lim_{\text{meas}(E)\to 0}\int_E |DT_h(u_n)|^p\,dx\,dt \\ + \int_{\{|u_n|>h\}} g(u_n)|Du_n|^p\,dx\,dt,\end{aligned}$$

so $g(u_n)|Du_n|^p$ is equi-integrable. Thus we have shown that $g(u_n)|Du_n|^p$ converges to $g(u)|Du|^p$ strongly in $L^1(Q)$. Consequently, by using (2.6), we conclude that

$$H_n(x,t,u_n,Du_n) \to H(x,t,u,Du) \quad \text{strongly in } L^1(Q). \tag{3.42}$$

Step 4: *Proof that $u$ satisfies (2.8).* Observe that for any fixed $m \ge 0$ one has

$$\begin{aligned}\int_{\{m\le|u_n|\le m+1\}} a(u_n,Du_n)Du_n &= \int_Q a(u_n,Du_n)(DT_{m+1}(u_n) - DT_m(u_n)) \\ &= \int_Q a(T_{m+1}(u_n),DT_{m+1}(u_n))DT_{m+1}(u_n) \\ &\quad - \int_Q a(T_m(u_n),DT_m(u_n))DT_m(u_n).\end{aligned}$$

According to (3.41) and (3.39), one can pass to the limit as $n\to\infty$ for fixed $m\ge 0$ to obtain

$$\begin{aligned}\lim_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(u_n,Du_n)Du_n\,dx\,dt \\ = \int_Q a(T_{m+1}(u),DT_{m+1}(u))DT_{m+1}(u)\,dx\,dt \\ - \int_Q a(T_m(u),DT_m(u))DT_m(u)\,dx\,dt \\ = \int_{\{m\le|u|\le m+1\}} a(u,Du)Du\,dx\,dt.\end{aligned} \tag{3.43}$$

Taking the limit as $m\to\infty$ in (3.43) and using the estimate (3.20) shows that $u$ satisfies (2.8).

STEP 5: *Proof that $u$ satisfies (2.9) and (2.10).* Let $S \in W^{2,\infty}(\mathbb{R})$ be such that $S'$ has a compact support. Let $M > 0$ such that $\operatorname{supp}(S') \subset [-M, M]$. Pointwise multiplication of the approximate equation (3.4) by $S'(u_n)$ leads to

$$\begin{aligned}(3.44)\qquad &\frac{\partial B_S^n(x,u_n)}{\partial t} - \operatorname{div}[S'(u_n)a(x,t,u_n,Du_n)]\\ &+ S''(u_n)a(x,t,u_n,Du_n)Du_n + S'(u_n)H_n(x,t,u_n,Du_n\\ &= fS'(u_n) - \operatorname{div}(FS'(u)) + S''(u)FDu \quad \text{in } D'(Q).\end{aligned}$$

In what follows we pass to the limit in (3.44) as $n$ tends to $\infty$.

• *Limit of $\partial B_S^n(x,u_n)/\partial t$.* Since $S$ is bounded and continuous, $u_n \to u$ a.e. in $Q$ implies that $B_S^n(x,u_n)$ converges to $B_S(x,u)$ a.e. in $Q$ and $L^\infty$ weak*. Then $\partial B_S^n(x,u_n)/\partial t$ converges to $\partial B_S(x,u)/\partial t$ in $\mathcal{D}'(Q)$ as $n \to \infty$.

• *Limit of $-\operatorname{div}[S'(u_n)a_n(x,t,u_n,Du_n)]$.* Since $\operatorname{supp}(S') \subset [-M, M]$, we have, for $n \geq M$,

$$S'(u_n)a_n(x,t,u_n,Du_n) = S'(u_n)a(x,t,T_M(u_n),DT_M(u_n)) \quad \text{a.e. in } Q.$$

The pointwise convergence of $u_n$ to $u$ and (3.41) and the boundedness of $S'$ yield, as $n \to \infty$,

$$(3.45)\quad S'(u_n)a_n(x,t,u_n,Du_n) \rightharpoonup S'(u)a(x,t,T_M(u),DT_M(u)) \text{ in } (L^{p'}(Q))^N.$$

$S'(u)a(x,t,T_M(u),DT_M(u))$ has been denoted by $S'(u)a(x,t,u,Du)$ in equation (2.9).

• *Limit of $S''(u_n)a(x,t,u_n,Du_n)Du_n$.* Consider the "energy" term

$$S''(u_n)a(x,t,u_n,Du_n)Du_n = S''(u_n)a(x,t,T_M(u_n),DT_M(u_n))DT_M(u_n)$$

a.e. in $Q$. The pointwise convergence of $S'(u_n)$ to $S'(u)$ and (3.41) as $n \to \infty$ and the boundedness of $S''$ yield

$$(3.46)\quad S''(u_n)a_n(x,t,u_n,Du_n)Du_n \rightharpoonup S''(u)a(x,t,T_M(u),DT_M(u))DT_M(u).$$

weakly in $L^1(Q)$ Recall that

$$S''(u)a(x,t,T_M(u),DT_M(u))DT_M(u) = S''(u)a(x,t,u,Du)D$$

a.e. in $Q$.

• *Limit of $S'(u_n)H_n(x,t,u_n,Du_n)$.* From $\operatorname{supp}(S') \subset [-M, M]$ and (3.42), we have

$$(3.47)\quad S'(u_n)H_n(x,t,u_n,Du_n) \to S'(u)H(x,t,u,Du) \ \text{ strongly in } L^1(Q)$$

as $n \to \infty$.

• *Limit of $S'(u_n)f_n$.* Since $u_n \to u$ a.e in $Q$, we have $S'(u_n)f_n \to S'(u)f$ strongly in $L^1(Q)$ as $n \to \infty$.

• *Limit of* $\operatorname{div}(S'(u_n)F)$. $S'(u_n)$ is bounded and converges to $S'(u)$ a.e. in $Q$. Hence $\operatorname{div}(S'(u_n)F) \to \operatorname{div}(S'(u)F)$ strongly in $L^{p'}(0,T;W^{-1,p'}(\Omega))$ as $n \to \infty$.

• *Limit of* $S''(u_n)FDu_n$. This term is equal to $FDS'(u_n)$. Since $DS'(u_n)$ converges to $DS'(u_n)$ weakly in $(L^p(Q))^N$, we obtain $S''(u_n)FDu_n = FDS'(u_n) \rightharpoonup FDS'(u)$ weakly in $L^1(Q)$ as $n \to \infty$. The term $FDS'(u)$ identifies with $S''(u)FDu$.

As a consequence of the above convergence result, we are in a position to pass to the limit as $n \to \infty$ in equation (3.44) and to conclude that $u$ satisfies (2.9).

It remains to show that $B_S(x,u)$ satisfies the initial condition (2.10). To this end, first remark that, $S$ being bounded, $B_S^n(x,u_n)$ is bounded in $L^\infty(Q)$. Secondly, (3.44) and the above considerations on the behavior of the terms of this equation show that $\partial B_S^n(x,u_n)/\partial t$ is bounded in $L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega))$. As a consequence, an Aubin type lemma (see, e.g., [19]) implies that $B_S^n(x,u_n)$ lies in a compact set in $C^0([0,T],L^1(\Omega))$. It follows that on the one hand, $B_S^n(x,u_n)|_{t=0} = B_S^n(x,u_0^n)$ converges to $B_S(x,u)|_{t=0}$ strongly in $L^1(\Omega)$. On the other hand, the smoothness of $S$ implies that $B_S(x,u)|_{t=0} = B_S(x,u_0)$ in $\Omega$.

As a conclusion of Steps 1 to 5, the proof of Theorem 3.1 is complete. ∎

**4. Example.** Consider the following special case: $b(x,r) = Z(x)C(s)$ where $Z \in W^{1,p}(\Omega)$, $Z(x) \geq \alpha > 0$ and $C \in C^1(\mathbb{R})$ such that for all $k > 0$, $0 < \lambda_k \equiv \inf_{|s|\leq k} C'(s)$, $C(0) = 0$ and

$$0 < \lambda_k \leq \frac{\partial b(x,s)}{\partial s} \leq A_k(x) \quad \text{and} \quad \left|\nabla_x\left(\frac{\partial b(x,s)}{\partial s}\right)\right| \leq B_k(x), \tag{4.1}$$

$$H(x,t,s,\xi) = \frac{-2s}{1+s^4}|\xi|^p \quad \text{and} \quad a(x,t,s,d) = |d|^{p-2}d. \tag{4.2}$$

It is easy to show that the $a(t,x,s,d)$ are Carathéodory functions satisfying the growth condition (2.3) and the coercivity (2.5). On the other hand the monotonicity condition is satisfied. In fact, $(a(x,t,d) - a(x,t,d'))(d-d') = (|d|^{p-2}d - |d'|^{p-2}d')(d-d') > 0$ for almost all $x \in \Omega$ and for all $d,d' \in \mathbb{R}^N$ and $d \neq d'$.

The Carathéodory function $H(x,t,s,\xi)$ satisfies the condition (2.6); indeed,

$$|H(x,t,s,\xi)| \leq \frac{2|s|}{1+s^4}|\xi|^p = g(s)|\xi|^p$$

where $g(s) = \frac{2|s|}{1+s^4}$ is a bounded positive continuous function which belongs to $L^1(\mathbb{R})$. Note that $H(x,t,s,\xi)$ does not satisfy the sign condition (1.3) or the coercivity condition.

Finally, the hypotheses of Theorem 3.1 are satisfied. Therefore, the problem

$$
(4.3)\quad
\begin{cases}
b(x,u) \in L^\infty([0,T];L^1(\Omega)) \quad \text{and} \quad T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega)),\\
\lim\limits_{m\to\infty} \displaystyle\int_{\{m\le|u|\le m+1\}} a(x,t,u,Du)Du\,dx\,dt = 0,\\
\dfrac{\partial B_S(x,u)}{\partial t} - \operatorname{div}[S'(u)|Du|^{p-2}Du] + S''(u)|Du|^p\\
\qquad -\dfrac{2u}{1+u^4}|Du|^p S'(u) = fS'(u) - \operatorname{div}(S'(u)F) + FS''(u)Du,\\
B_S(x,u)|_{t=0} = B_S(x,u_0) \quad \text{in } \Omega,\\
\forall S \in W^{2,\infty}(\mathbb{R}) \text{ with } S' \text{ having a compact support in } \mathbb{R},\\
\text{and } B_S(x,r) = \displaystyle\int_0^r \frac{\partial b(x,\sigma)}{\partial\sigma} S'(\sigma)\,d\sigma,
\end{cases}
$$

has at least one renormalized solution.

## References


[1] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] D. Blanchard and F. Murat, *Renormalized solutions of nonlinear parabolic problems with $L^1$ data: existence and uniqueness*, Proc. Roy. Soc. Edinburgh Sect. A 127 (1997) 1137–1152.

[3] D. Blanchard, F. Murat and H. Redwane, *Existence and uniqueness of renormalized solution for a fairly general class of nonlinear parabolic problems*, J. Differential Equations 177 (2001), 331–374.

[4] L. Boccardo and T. Gallouët, *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal. 87 (1989), 149–169.

[5] L. Boccardo, D. Giachetti, J.-I. Diaz and F. Murat, *Existence and regularity of renormalized solutions of some elliptic problems involving derivatives of nonlinear terms*, J. Differential Equations 106 (1993), 215–237.

[6] L. Boccardo and F. Murat, *Strongly nonlinear Cauchy problems with gradient dependent lower order nonlinearity*, in: Pitman Res. Notes in Math. 208, Longman, 1989, 247–254.

[7] —, —, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Anal. 19 (1992), 581–597.

[8] A. Dall'Aglio and L. Orsina, *Nonlinear parabolic equations with natural growth conditions and $L^1$ data*, ibid. 27 (1996), 59–73.

[9] G. Dal Maso, F. Murat, L. Orsina and A. Prignet, *Definition and existence of renormalized solutions of elliptic equations with general measure data*, C. R. Acad. Sci. Paris 325 (1997), 481–486.

[10] R. J. DiPerna and P.-L. Lions, *On the Cauchy problem for Boltzmann equations: global existence and weak stability*, Ann. of Math. (2) 130 (1989), 321–366.

[11] R. Landes, *On the existence of weak solutions for quasilinear parabolic initial-boundary value problems*, Proc. Roy. Soc. Edinburgh Sect. A 89 (1981), 321–366.

[12] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.

[13] A. Porretta, *Existence results for strongly nonlinear parabolic equations via strong convergence of truncations*, Ann. Mat. Pura Appl. (4) 177 (1999), 143–172.

[14] —, *Nonlinear equations with natural growth terms and measure data*, in: Electron. J. Differ. Equ. Conf. 9 (2002), 183–202.

[15] J.-M. Rakotoson, *Uniqueness of renormalized solutions in a $T$-set for the $L^1$-data problem and the link between various formulations*, Indiana Univ. Math. J. 43 (1994), 685–702.

[16] H. Redwane, *Existence of a solution for a class of parabolic equations with three unbounded nonlinearities*, Adv. Dynam. Systems Appl. 2 (2007), 241–264.

[17] —, *Existence results for a class of parabolic equations in Orlicz spaces*, Electron. J. Qual. Theory Differential Equations 2010, no. 2, 19 pp.

[18] —, *Solutions renormalisées de problèmes paraboliques et elliptiques non linéaires*, Ph.D. thesis, Rouen, 1997.

[19] J. Simon, *Compact sets in the space $L^p(0, T, B)$*, Ann. Mat. Pura Appl. 146 (1987), 65–96.

[20] E. Zeidler, *Nonlinear Functional Analysis and its Applications*, Springer, New York, 1990.

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