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ESTIMATION OF PARAMETERS OF A SPHERICAL INVARIANT STABLE DISTRIBUTION

Abstract. This paper concerns the estimation of the parameters that describe spherical invariant stable distributions: the index $\alpha \in (0,2]$ and the scale parameter $\sigma > 0$. We present a kind of moment estimators derived from specially transformed original data.

1. Introduction. The distribution of a random vector $\xi = (\xi_1, \ldots, \xi_d)$ is called *α-stable spherically invariant* if its characteristic function is of the form

$$\widehat{s}_{d,\alpha}(t) = \exp(-\sigma^\alpha |t|^\alpha), \qquad \sigma > 0,\ \alpha \in (0,2],\ t \in \mathbb{R}^d.$$

The parameter α is called the *index* while σ is called the *scale parameter*.

Each α-stable spherically invariant distribution determines a family of so-called *elliptically contoured* distributions. The stable distribution inherits many properties of the normal distribution.

Let $\underline{X} = (x^{(1)}, \ldots, x^{(n)})$, where $x^{(i)} = (x_1^{(i)}, \ldots, x_d^{(i)})$, $i = 1, \ldots, n$, be a sample from an α-stable spherical invariant distribution $S_{d,\alpha}(\sigma)$. From [ST, Ch. 3.6] we have

Lemma 1.1. *If a sample X is drawn from $S_{d,\alpha}(\sigma)$ distribution and $t \in S^{d-1}$ is any vector then $\langle t, X\rangle$ is a symmetric one-dimensional random variable with distribution $S_{1,\alpha}(\sigma)$.*

It seems that such a conclusion eliminates the question of multidimensionality. However, it is true only in the case of normal distribution and the analysis of the scale parameter: a sample of size n from $S_{d,2}(\sigma)$ distribution can be deemed as a sample of size nd from $S_{1,2}(\sigma)$ distribution. This follows from the fact that the vector $X \sim S_{d,2}(\sigma)$ has coordinates which are inde-

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pendent, and it is not valid when $\alpha < 2$. Hence, it can be expected that scale parameter estimators will have better properties when $\alpha < 2$.

One of the techniques for the estimation of parameters of $S_{d,\alpha}(\sigma)$ distribution is the moment estimation by Zolotarev [ZM, N].

LEMMA 1.2. *Let $X \sim S_{d,\alpha}(\sigma)$. Then $\mathrm{E}\,|X|^p < \infty$ only if $-d < p < \alpha$ and*

$$\mathrm{E}\,|X|^p = (2\sigma)^p \, \frac{\Gamma(1 - p/\alpha)\Gamma((d+p)/2)}{\Gamma(1-p/2)\Gamma(d/2)}.$$

Construction of estimators based on Lemma 1.2 by the moment method results in the technical issue of selection of p and analytical determination of estimation of α.

Hence, let us consider a random variable

$$Y = h(X),$$

where $h : \mathbb{R}^d \to \mathbb{R}$ is such that all moments of Y are finite. Examples of such functions are arctan and ln. We shall consider the function $Y = \ln|X|$.

The paper is organized as follows. In Section 2 we present one-dimensional estimators. In Section 3 the above mentioned method is extended to higher dimensions. Some remarks concerning Monte Carlo simulation are given in Section 4.

2. One-dimensional case. Consider $t = (1, \ldots, 0) \in S^{d-1}$. Denote $x_i = x_1^{(i)}$, $i = 1, \ldots, n$. Let $C_E = -\psi(1) = 0.577\ldots$, where ψ is the Euler function and $\overline{y} = n^{-1}\sum_{i=1}^n y_i$, $s_y^2 = (n-1)^{-1}\sum_{i=1}^n (y_i - \overline{y})^2$, where $y_1 = \ln|x_1|, \ldots, y_n = \ln|x_n|$.

From [Z, Ch. 3.6] we have

LEMMA 2.1. *Let $S_\alpha(\sigma, \beta, a)$ be the one-dimensional α-stable distribution with scale parameter $\sigma > 0$, shape parameter $\beta \in [-1, 1]$ and location parameter $a \in \mathbb{R}$. Consider*

$$X \sim S_\alpha\biggl(\biggl(\cos\frac{\pi\kappa\alpha}{2}\biggr)^{1/\alpha}, \beta, 0\biggr), \quad \textit{where} \quad \kappa = \frac{2}{\pi\alpha}\arctan\biggl(\beta\tan\frac{\pi\alpha}{2}\biggr).$$

The regular moments of order $s \in \mathbb{N}$ of the random variable $Y = \ln|X|$ are

$$\mathrm{E}\,Y^s = C_s(q_1, \ldots, q_s) + s\ln\kappa,$$

where

$$q_1 = C_E\biggl(\frac{1}{\alpha} - 1\biggr),$$

$$q_j = (2^j - 1)(1 - \kappa^j)\frac{\pi^j|B_j|}{j} + \biggl(\frac{1}{\alpha^j} - 1\biggr)\Gamma(j)\zeta(j), \qquad j = 2, \ldots, s,$$

and B_j are the Bernoulli numbers, ζ is the Riemann zeta function and

$$C_s(q_1,\dots,q_s)=\sum\frac{s!}{k_1!\dots k_s!}\left(\frac{q_1}{1!}\right)^{k_1}\dots\left(\frac{q_s}{s!}\right)^{k_s}$$

where the sum extends over $\{(k_1,\dots,k_s) : \sum_{j=1}^{s} jk_j = s,\ k_j \in \mathbb{N},\ j = 1,\dots,s\}$.

COROLLARY 2.2. *Let $X \sim S_\alpha(1,0,0) = S_{1,\alpha}(1)$. The first four regular moments of the random variable $Y = \ln|X|$ are*

$$\begin{aligned}\mathrm{E}\,Y &= q_1,\\ \mathrm{E}\,Y^2 &= q_1^2+q_2,\\ \mathrm{E}\,Y^3 &= q_1^3+3q_1q_2+q_3,\\ \mathrm{E}\,Y^4 &= q_1^4+3q_2^2+6q_1^2q_2+4q_1q_3+q_4,\end{aligned}$$

where

$$\begin{aligned}q_1 &= C_E\left(\frac{1}{\alpha}-1\right),\\ q_2 &= \frac{\pi^2}{12}\left(1+\frac{2}{\alpha^2}\right),\\ q_3 &= 2\zeta(3)\left(\frac{1}{\alpha^3}-1\right),\\ q_4 &= \frac{\pi^4}{120}\left(7+\frac{8}{\alpha^4}\right).\end{aligned}$$

COROLLARY 2.3. *Let $X \sim S_{1,\alpha}(\sigma)$. The expected value, variance, and fourth central moment of the random variable $Y = \ln|X|$ are*

$$\begin{aligned}\mu_1 &= \mathrm{E}\,Y = q_1 = C_E\left(\frac{1}{\alpha}-1\right)+\ln\sigma,\\ \mu_2 &= \mathrm{Var}\,Y = q_2 = \frac{\pi^2(1+2/\alpha^2)}{12},\\ \mu_4 &= \mathrm{E}\,(Y-\mu_1)^4 = 3q_2^2+q_4 = \frac{\pi^4(19\alpha^4+20\alpha^2+36)}{240\alpha^4}.\end{aligned}$$

In view of Corollary 2.3, Zolotarev [Z] proposed the moment type estimators

$$\sigma_Z = \exp(\bar{y}-C_E(\alpha^{-1}-1)) \tag{2.1}$$

and

$$\alpha_Z = \frac{1}{\sqrt{\max\left\{\frac{1}{4},\frac{6}{\pi^2}s_y^2-\frac{1}{2}\right\}}}, \tag{2.2}$$

which is the solution of the system of equations

$$\begin{cases} \mu_1 = \overline{y}, \\ \mu_2 = s_y^2. \end{cases}$$

Since $\alpha \le 2$, the estimator α_Z was altered by addition of the max function and $1/4$. Sometimes we use the notation $\sigma_Z(\underline{X})$ and $\alpha_Z(\underline{X})$ instead of σ_Z and α_Z.

The estimator of the scale parameter. First we consider the function

$$f : (2\alpha_0^{-1}, \infty) \times [\alpha_0, 2] \to \mathbb{R},$$

where $\alpha_0 \in (0, 2)$ is a constant and

$$f(z, \alpha) = \frac{4}{\exp(2C_E(1/\alpha - 1))} \left(\frac{\Gamma(1 - 2/(z\alpha))\Gamma(1/2 + 1/z)}{\Gamma(1 - 1/z)\sqrt{\pi}} \right)^z.$$

LEMMA 2.4. *The function $f(z, \alpha)$ has the properties:*

(i) $\dfrac{\partial f}{\partial \alpha}(z, \alpha) < 0$,

(ii) $\lim_{z \to \infty} f(z, \alpha) = 1$,

(iii) $\dfrac{\partial f}{\partial z}(z, 2) < 0$,

(iv) $f(z, \alpha) > 1$.

Proof. (i) Calculating directly we get

$$\frac{\partial f}{\partial \alpha}(z, \alpha) = 2(\psi(1 - 2/(z\alpha)) - \psi(1))f(z, \alpha).$$

Since the function $\psi(y) = \frac{\partial \ln \Gamma}{\partial y}(y)$ is increasing we have $\frac{\partial f}{\partial \alpha}(z, \alpha) < 0$.

(ii) Use the fact that $\Gamma(y + \Delta) \sim \Gamma(y)(1 + \psi(y)\Delta)$ as $\Delta \to 0$, where $\sim$ denotes asymptotic equivalence.

(iii) Note that

$$\frac{\partial^2 f}{\partial z^2}(z, 2) = \frac{f(z, 2)}{z^2} \left[\frac{\psi'(1/2 + 1/z)}{z} + \left(z \ln \frac{\Gamma(1/2 + 1/z)}{\sqrt{\pi}} - \psi(1/2 + 1/z) \right)^2 \right] > 0.$$

This means that the function $f(z, 2)$ is convex. Using (ii) implies that it must also be decreasing.

(iv) Note that

$$f(z, \alpha) \overset{\text{(i)}}{\ge} f(z, 2) \overset{\text{(iii)}}{>} \lim_{z \to \infty} f(z, 2) \overset{\text{(ii)}}{=} 1. \quad \blacksquare$$

Further, for simplicity, we will denote the mean square risk by R with an appropriate subscript representing the estimator's name, and with an estimated parameter in parentheses. For instance $\mathrm{R}_Z(\sigma)$ is the mean square risk of the parameter σ for the estimator σ_Z.

THEOREM 2.5. *The estimator σ_Z has the properties:*

(i) *for each $n > [1/\alpha]$,*

$$\mathrm{E}\,\sigma_Z = \sigma\sqrt{f(2n,\alpha)} > \sigma,$$

(ii) *for each $n > [2/\alpha]$,*

$$\begin{aligned}\operatorname{Var}\sigma_Z &= \sigma^2[f(n,\alpha) - f(2n,\alpha)],\\ \mathrm{R}_Z(\sigma) &= \sigma^2[f(n,\alpha) - 2\sqrt{f(2n,\alpha)} + 1],\end{aligned}$$

(iii) *for each $\lambda > 0$,*

$$\sigma_Z(\lambda \underline{X}) = \lambda\sigma_Z(\underline{X}),$$

(iv) *the distribution of $n^{1/2}(\sigma_Z - \sigma)$ is asymptotic to $N(0,b^2)$, where*

$$b^2 = \sigma^2\frac{\pi^2(1+2/\alpha^2)}{12}.$$

Proof. (i) We have

$$\mathrm{E}\,\sigma_Z = \frac{1}{\exp(C_E(1/\alpha - 1))}\mathrm{E}\exp(\overline{y}).$$

Note that

$$\mathrm{E}\exp(\overline{y}) = \mathrm{E}\prod_{i=1}^{n}|x_i|^{1/n}.$$

Because the random variables x_i, $i = 1,\ldots,n$, are independent,

$$\mathrm{E}\exp(\overline{y}) = (\mathrm{E}\,|x_1|^{1/n})^n.$$

From Lemma 1.2, for $d = 1$ and $p = 1/n < \alpha$ we get

$$\mathrm{E}\exp(\overline{y}) = 2\sigma\left(\frac{\Gamma(1-1/(n\alpha))\Gamma(1/2+1/(2n))}{\Gamma(1-1/(2n))\sqrt{\pi}}\right)^n.$$

Hence, we have the conclusion.

(ii) We have

$$\operatorname{Var}\sigma_Z \overset{\text{(i)}}{=} \mathrm{E}\,\sigma_Z^2 - \sigma^2 f(2n,\alpha)$$

and

$$\mathrm{E}\,\sigma_Z^2 = \frac{1}{\exp(2C_E(1/\alpha - 1))}\,\mathrm{E}\exp(2\overline{y}).$$

By analogy to (i) we show $\mathrm{E}\,\sigma_Z^2 = \sigma^2 f(n,\alpha)$. After substitution we get the conclusion. Similarly we prove the equality for the risk.

(iii) Note that

$$\sigma_Z(\underline{X}) = \frac{\sqrt[n]{\prod_{i=1}^n |x_i|}}{\exp(C_E(1/\alpha - 1))}.$$

Then for $\lambda > 0$ we have

$$\sigma_Z(\lambda\underline{X}) = \frac{\sqrt[n]{\prod_{i=1}^n |\lambda x_i|}}{\exp(C_E(1/\alpha - 1))} = \frac{\sqrt[n]{\lambda^n \prod_{i=1}^n |x_i|}}{\exp(C_E(1/\alpha - 1))} = \lambda\sigma_Z(\underline{X}).$$

(iv) From the Lindeberg–Lévy theorem [S, Ch. 1.9], $\sqrt{n}(\overline{y} - \mu_1) \to_d N(0, \mu_2)$. Let $g(z) = \exp(z - C_E(\alpha^{-1} - 1)$. Then $g'(z) = g(z)$. From the convergence theorem (see also [R]) we have $\sqrt{n}(g(\overline{y}) - g(\mu_1)) \to_d N(0, (g'(\mu_1))^2\mu_2)$. Therefore

$$b^2 = \sigma^2\mu_2 = \sigma_2 q_2 = \sigma^2\frac{\pi^2(1 + 2/\alpha^2)}{12}. \ \blacksquare$$

Corollary 2.6. *The estimator σ_Z is biased and*

$$\mathrm{E}\,\sigma_Z - \sigma = \sigma(\sqrt{f(2n,\alpha)} - 1) > 0.$$

From Theorem 2.5(i) we can consider the new unbiased estimator

$$\sigma_{NZ} = \frac{\sigma_Z}{\sqrt{f(2n,\alpha)}}.$$

We will call it the *new Zolotarev estimator.*

Theorem 2.7. *Let $\alpha \in (0, 2]$ be a constant. The estimator σ_{NZ} has the properties:*

(i) *for $n > [1/\alpha]$,*

$$\mathrm{E}\,\sigma_{NZ} = \sigma,$$

(ii) *for $n > [2/\alpha]$,*

$$\operatorname{Var}\sigma_{NZ} = \mathrm{R}_{NZ}(\sigma) = \sigma^2\left[\frac{f(n,\alpha)}{f(2n,\alpha)} - 1\right],$$

(iii) *for all $\lambda > 0$,*

$$\sigma_{NZ}(\lambda X) = \lambda\sigma_{NZ}(X),$$

(iv) *the distribution of $n^{1/2}(\sigma_{NZ} - \sigma)$ is asymptotic to $N(0, b^2)$, where*

$$b^2 = \sigma^2\frac{\pi^2(1 + 2/\alpha^2)}{12}.$$

Proof. (i) This follows directly from the definition of σ_{NZ}.

(ii) The estimator σ_{NZ} is unbiased. Moreover

$$\operatorname{Var}\sigma_{NZ} = \frac{\operatorname{Var}\sigma_Z}{f(2n,\alpha)}.$$

(iii) The proof is analogous to that for σ_Z.

(iv) The estimators σ_Z and σ_{NZ} are asymptotically equivalent. Hence

$$\sigma_{NZ} \sim \sigma_Z, \qquad n \to \infty.$$

We have

$$\frac{\sigma_Z}{\sigma_{NZ}} = \sqrt{f(2n, \alpha)}.$$

Next, we use Lemma 2.4(ii). ∎

COROLLARY 2.8. *For $n > [2/\alpha]$,*

$$\mathrm{R}_Z(\sigma) > \mathrm{R}_{NZ}(\sigma) \quad \textit{and} \quad \mathrm{Var}(\sigma_Z) > \mathrm{Var}(\sigma_{NZ}).$$

The inequalities in Corollary 2.8 mean that the estimator σ_Z is inadmissible.

The estimator for the index. Let us consider the Zolotarev estimator α_Z of the form (2.1).

THEOREM 2.9. *The estimator α_Z has the properties:*

(i) *for each $\lambda > 0$, $\alpha_Z(\lambda \underline{X}) = \alpha_Z(\underline{X})$,*
(ii) *for $n > 1$ we have*

$$\mathrm{R}_Z(\alpha) \le \frac{16}{5}\left(\frac{22}{\alpha^4} + \frac{10}{\alpha^2} + 13\right)\frac{1}{n} + 8\left(\frac{2}{\alpha^2} + 1\right)^2 \frac{1}{n(n-1)},$$

(iii) *the distribution $n^{1/2}(\alpha_Z - \alpha)$ is asymptotic to $N(0, m^2)$, where*

$$m^2 = \frac{\alpha^2(13\alpha^4 + 10\alpha^2 + 22)}{20}.$$

Proof. (i) Note that $s_y^2(\lambda \underline{X}) = s_y^2(\underline{X})$. This yields the given equality.

(ii) In [Z, Ch. 4.3] there is a proof. Regretfully, the fourth central moment is incorrectly calculated, so the estimation contains an error. Therefore we give a detailed argument.

Let $v = 1/\alpha^2$ and $\tilde{v} = \max\{1/4, \widehat{v}\}$, where $\widehat{v} = (6/\pi^2)s_y^2 - 1/2$. We have

$$|\alpha_Z - \alpha| = |\tilde{v}^{-1/2} - v^{-1/2}| = \frac{|\tilde{v} - v|}{v\tilde{v}^{1/2} + \tilde{v}v^{1/2}}.$$

Because $v, \tilde{v} \ge 1/4$ we have

$$|\alpha_Z - \alpha| \le 4|\tilde{v} - v|.$$

Furthermore

$$|\tilde{v} - v| = |\max\{1/4, \widehat{v}\} - \max\{1/4, v\}| \le |\widehat{v} - v|.$$

Hence

$$\mathrm{E}\,(\alpha_Z - \alpha)^2 \le 16\,\mathrm{E}\,(\widehat{v} - v)^2 = \frac{576}{\pi^4}\,\mathrm{Var}\, s_y^2. \tag{2.3}$$

To find $\mathrm{Var}\, s_y^2$ we use formula (4.1.18) of [Z]. We have

$$\mathrm{Var}\, s_y^2 = (\mu_4 - \mu_2^2)\frac{1}{n} + 2\mu_2^2 \frac{1}{n(n-1)},$$

where μ_2, μ_4 were determined in Corollary 2.3. After simplification we obtain

$$\operatorname{Var} s_y^2 = \frac{\pi^4}{180\alpha^4}(13\alpha^4 + 10\alpha^2 + 22)\frac{1}{n} + \frac{\pi^4}{72}(1 + 2/\alpha^2)^2\frac{1}{n(n-1)},$$

which yields the desired result.

(iii) Note that $\max\{1/4, (6/\pi^2)S_y^2 - 1/2\}$ is asymptotic to $(6/\pi^2)S_y^2 - 1/2$. From the Lindeberg–Lévy theorem we have

$$\sqrt{n}(s_y^2 - \mu_2) \to_d N(0, \mu_4 - \mu_2^2).$$

Let $g(z) = \frac{1}{\sqrt{(6/\pi^2)z - 1/2}}$. Hence $g'(z) = -(3/\pi^2)g(z)^3$. Next we have

$$\sqrt{n}(g(s_y^2) - g(\mu_2)) \to_d N(0, (g'(\mu_2))^2(\mu_4 - \mu_2^2)).$$

Then

$$m^2 = (g'(\mu_2))^2(\mu_4 - \mu_2^2) = \frac{72\pi^2(2q_2^2 + q_4)}{(12q_2 - \pi^2)^3} = \frac{\alpha^2(13\alpha^4 + 10\alpha^2 + 22)}{20}. \blacksquare$$

3. Multidimensional case. We will denote by $\overline{y} = n^{-1}\sum_{i=1}^n y_i$ the sample mean, where $y_1 = \ln|x^{(1)}|, \ldots, y_n = \ln|x^{(n)}|$, and by $s_y^2 = (n-1)^{-1} \cdot \sum_{i=1}^n (y_i - \overline{y})^2$ the variance from the sample.

Lemma 3.1. *Let $X \sim S_{d,\alpha}(\sigma)$. Then the expected value, variance and fourth central moment of the random variable $Y = \ln|X|$ are*

$$\mu_1 = \mathrm{E}\,Y = \ln(2\sigma) + C_E\left(\frac{1}{\alpha} - \frac{1}{2}\right) + \frac{1}{2}\psi\left(\frac{d}{2}\right), \tag{3.1}$$

$$\mu_2 = \operatorname{Var} Y = \frac{\pi^2}{6}\left(\frac{1}{\alpha^2} - \frac{1}{4}\right) + \frac{1}{4}\psi'\left(\frac{d}{2}\right), \tag{3.2}$$

$$\begin{aligned} \mu_4 = \mathrm{E}\,(Y - \mathrm{E}\,Y)^4 = {} & \frac{\pi^4(\alpha^4 - 40\alpha^2 + 144)}{960\alpha^4} + \frac{\pi^2(4 - \alpha^2)}{16\alpha^2}\psi'\left(\frac{d}{2}\right) \\ & + \frac{3}{16}[\psi'(d/2)]^2 + \frac{1}{16}\psi'''(d/2). \end{aligned} \tag{3.3}$$

Proof. Let $A \sim S_{\alpha/2}((\cos(\pi\alpha/4))^{2/\alpha}, 1, 0)$ and $Z = \sqrt{2\sigma^2}(Z_1, \ldots, Z_d)$, where $Z_i \sim N(0,1)$, $i = 1, \ldots, d$, are independent random variables. Then from Cor. 2.5.5(3) of [ST] we have

$$X = A^{1/2}Z = \sqrt{2\sigma^2}A^{1/2}(Z_1, \ldots, Z_d).$$

Thus $|X|^2 = 2\sigma^2|A|T$ where $T = Z_1^2 + \cdots + Z_d^2 \sim \chi^2(d)$. Hence

$$Y = \frac{1}{2}\ln 2 + \ln\sigma + \frac{1}{2}\ln|A| + \frac{1}{2}\ln T.$$

From Corollary 2.3 we have

$$\mathrm{E}\ln|A| = C_E\left(\frac{2}{\alpha} - 1\right),$$

and from [P, formulas 2.6.21]

$$\mathrm{E}\ln T = \ln 2 + \psi\left(\frac{d}{2}\right)$$

we have

$$\mathrm{E}\,Y = \ln(2\sigma) + C_E\left(\frac{1}{\alpha} - \frac{1}{2}\right) + \frac{1}{2}\psi\left(\frac{d}{2}\right).$$

We have proven (3.1). To prove (3.2) we shall use the fact that (see also Corollary 2.3)

$$\mathrm{Var}\ln|A| = \frac{\pi^2}{6}\left(\frac{4}{\alpha^2} - 1\right)$$

and (see [P, formulas 2.6.21])

$$\mathrm{Var}\ln T = \psi'\left(\frac{d}{2}\right).$$

Since $\mathrm{Var}\,Y = \frac{1}{4}\mathrm{Var}\ln|A| + \frac{1}{4}\mathrm{Var}\ln T$, after substitution we have

$$\mathrm{Var}\,Y = \frac{\pi^2}{6}\left(\frac{1}{\alpha^2} - \frac{1}{4}\right) + \frac{1}{4}\psi'\left(\frac{d}{2}\right).$$

Property (3.3) is proved analogously. We have

$$\mathrm{E}\,(\ln A - \mathrm{E}\ln A)^4 = \frac{\pi^4(\alpha^4 - 40\alpha^2 + 144)}{60\alpha^4}$$

and (see [P, formulas 2.6.21])

$$\mathrm{E}\,(\ln T - \mathrm{E}\ln T)^4 = 3\left[\psi'\left(\frac{d}{2}\right)\right]^2 + \psi'''\left(\frac{d}{2}\right).$$

Note that

$$\mathrm{E}\,(Y - \mathrm{E}\,Y)^4 = \frac{1}{16}\,\mathrm{E}\,[(\ln|A| - \mathrm{E}\ln|A|) + (\ln T - \mathrm{E}\ln T)]^4.$$

Taking advantage of independence of the random variables A and T and the formula $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ we have after simplification

$$\mathrm{E}\,(Y - \mathrm{E}\,Y)^4 = \frac{\pi^4(\alpha^4 - 40\alpha^2 + 144)}{960\alpha^4} + \frac{\pi^2(4-\alpha^2)}{16\alpha^2}\psi'\left(\frac{d}{2}\right)$$
$$+ \frac{3}{16}\left[\psi'\left(\frac{d}{2}\right)\right]^2 + \frac{1}{16}\psi'''\left(\frac{d}{2}\right). \ \blacksquare$$

Estimators (see also [N, U, ZM]) based on (3.1) and (3.2) for the parameters α and σ, analogously to the one-dimensional case, have the form

$$\sigma_Z^d = \exp(\overline{y} - A_d - C_E(\alpha^{-1} - 1)), \tag{3.4}$$

$$\alpha_Z^d = \frac{1}{\sqrt{\max\{1/4, (6/\pi^2)s_y^2 + B_d\}}}, \tag{3.5}$$

where

$$A_d = \ln 2 + \frac{1}{2}(C_E + \psi(d/2)) = \begin{cases} \ln 2 + \sum_{j=1}^{m-1} \frac{1}{2j}, & d = 2m, \\ \sum_{j=1}^{m} \frac{1}{2j-1}, & d = 2m+1, \end{cases}$$

and

$$B_d = \frac{1}{4} - \frac{3}{2\pi^2}\psi'(d/2) = \begin{cases} \frac{6}{\pi^2} \sum_{j=1}^{m-1} \frac{1}{(2j)^2}, & d = 2m, \\ -\frac{1}{2} + \frac{6}{\pi^2} \sum_{j=1}^{m} \frac{1}{(2j-1)^2}, & d = 2m+1. \end{cases}$$

The estimator for the scale parameter. Let $\alpha_0 \in (0,2)$ be a constant number. To analyse scale parameter estimators we will use the function

$$g : [2^{-1}, \infty) \times (0, \alpha_0 2^{-1}) \times [1, 2\alpha_0^{-1}] \to \mathbb{R}$$

defined as

$$g(p,z,a) = \frac{1}{\exp(C_E(a-1) + \psi(p))} \left(\frac{\Gamma(1-az)\Gamma(p+z)}{\Gamma(1-z)\Gamma(p)} \right)^{1/z}.$$

LEMMA 3.2. *The function $g(p,z,a)$ has the following properties:*

(i) $g(2^{-1}, z^{-1}, 2\alpha^{-1}) = f(z,\alpha)$, *where f was defined in Lemma 2.4,*

(ii) $\frac{\partial g}{\partial a}(p,z,a) > 0$,

(iii) $\frac{\partial g}{\partial p}(p,z,a) < 0$,

(iv) $g(p,z,a) > 1$.

Proof. (i) Obvious.

(ii) Calculating the derivative directly we have

$$\frac{\partial g}{\partial a}(p,z,a) = -g(p,z,a)(\psi(1-az) - \psi(1)).$$

The function $\psi(y)$ is increasing. Therefore $\psi(1-az) - \psi(1) < 0$.

(iii) Calculating the derivative directly we have

$$\frac{\partial g}{\partial p}(p,z,a) = g(p,z,a)\left(\frac{\psi(p+z) - \psi(p)}{z} - \psi'(p) \right).$$

Since $\psi''(y) < 0$, $\psi(y)$ is concave, which means that each difference quotient is less than the derivative at p.

(iv) Note that

$$g(p,z,a) \overset{\text{(ii)}}{\geq} g(p,z,1) \overset{\text{(iii)}}{\geq} g(2^{-1},z,1) \overset{\text{(i)}}{=} f(z^{-1},2) > 1.$$

The latter inequality results from Lemma 2.4(iv). ■

THEOREM 3.3. *Fix $\alpha \in (0,2]$. The estimator σ_Z^d has the following properties:*

(i) *for $n > [1/\alpha]$,*

$$\mathrm{E}\,\sigma_Z^d = \sigma\sqrt{g(d/2, 1/(2n), 2/\alpha)} > \sigma,$$

(ii) *for $n > [2/\alpha]$,*

$$\begin{aligned} \operatorname{Var}\sigma_Z^d &= \sigma^2[g(d/2,1/n,2/\alpha) - g(d/2,1/(2n),2/\alpha)],\\ \mathrm{R}_Z^d(\sigma) &= \sigma^2[g(d/2,1/n,2/\alpha) - 2\sqrt{g(d/2,1/(2n),2/\alpha)} + 1], \end{aligned}$$

(iii) *for each $\lambda > 0$,*

$$\sigma_Z^d(\lambda\underline{X}) = \lambda\sigma_Z^d(\underline{X}),$$

(iv) *the distribution of $n^{1/2}(\sigma_Z^d - \sigma)$ is asymptotic to $N(0,b^2)$, where*

$$b^2 = \sigma^2\left[\frac{\pi^2}{6}\left(\frac{1}{\alpha^2} - \frac{1}{4}\right) + \frac{1}{4}\psi'\left(\frac{d}{2}\right)\right].$$

Proof. (i) We argue as in the one-dimensional case. The strict inequality results from Lemma 3.2(iv).

Items (ii), (iii) and (iv) are proved as in one-dimensional case. For (iv) we additionally take advantage of the central moment data. ■

COROLLARY 3.4. *The estimator σ_Z^d is biased and its bias equals*

$$\mathrm{E}\,\sigma_Z^d - \sigma = \sigma(\sqrt{g(d/2,1/(2n),2/\alpha)} - 1) > 0.$$

To get rid of the bias in σ_Z^d we propose a new estimator

$$\sigma_{NZ}^d = \frac{\sigma_Z^d}{\sqrt{g(d/2,1/(2n),2/\alpha)}}.$$

THEOREM 3.5. *Let $\alpha \in (0,2]$. The estimator σ_{NZ}^d has the following properties:*

(i) *for $n > [1/\alpha]$,*

$$\mathrm{E}\,\sigma_{NZ}^d = \sigma,$$

(ii) *for $n > [2/\alpha]$,*

$$\operatorname{Var}\sigma_{NZ}^d = \mathrm{R}_{NZ}^d(\sigma) = \sigma^2\left[\frac{g(d/2,1/n,2/\alpha)}{g(d/2,1/(2n),2/\alpha)} - 1\right],$$

(iii) *for each $\lambda > 0$,*

$$\sigma_{NZ}^d(\lambda\underline{X}) = \lambda\sigma_{NZ}^d(\underline{X}),$$

(iv) *the distribution of* $n^{1/2}(\sigma_{NZ}^d - \sigma)$ *is asymptotic to* $N(0, b^2)$, *where*

$$b^2 = b^2(\alpha, \sigma) = \sigma^2 \left[\frac{\pi^2}{6} \left(\frac{1}{\alpha^2} - \frac{1}{4} \right) + \frac{1}{4} \psi' \left(\frac{d}{2} \right) \right].$$

Proof. (i) results directly from the definiton of σ_{NZ}^d. The other statements are proved as in the one-dimensional case. ∎

With the increase of the dimension or of α the estimator's bias decreases. The minimum value is reached for $\alpha = 2$.

Analogously to $d = 1$ the estimator σ_{NZ}^d is unbiased and has variance as well as mean squared risk lower than those of σ_Z^d (see also Corollary 2.8). This means that the estimator σ_Z^d is unacceptable.

THEOREM 3.6. *For each* $d \in \mathbb{N}$,

$$\mathrm{R}_{NZ}^{d+1}(\sigma) < \mathrm{R}_{NZ}^d(\sigma).$$

Proof. It is sufficient to note that the risk is decreasing as a function of d, which is shown as in the proof of Lemma 3.2(iii). ∎

It follows from the above statement that the NZ-estimator in the one-dimensional case has larger risk than its multidimensional counterpart.

The estimator for the index. Let us consider the Zolotarev estimator α_Z of the form (3.5).

REMARK 3.7. The estimator α_Z^d is biased. Just as in the one-dimensional case, we shall not prove this fact.

We shall not include bias-related numerical results either since the estimator α_Z^d behaves analogously to its one-dimensional equivalent. The estimator either underestimates or overestimates the index α.

REMARK 3.8. For $\alpha = 2$ and a finite sample we have $\mathrm{E}\, \alpha_Z^d < 2$.

THEOREM 3.9. *The estimator* α_{NZ}^d *has the following properties:*

(i) *for* $\lambda > 0$, $\alpha_Z^d(\lambda \underline{X}) = \alpha_Z^d(\underline{X})$,
(ii) *for* $n > 1$ *we have* $\mathrm{R}_Z(\alpha) \le \frac{T_1}{n} + \frac{T_2}{n(n-1)}$ *where*

$$\begin{aligned} T_1 = {} & -\frac{2}{5} \left(1 + \frac{40}{\alpha^2} - \frac{176}{\alpha^4} \right) + \frac{24}{\pi^2} \psi' \left(\frac{d}{2} \right) \left(\frac{4}{\alpha^2} - 1 \right) \\ & + \frac{36}{\pi^2} \left(2\psi' \left(\frac{d}{2} \right) + \psi''' \left(\frac{d}{2} \right) \right), \\ T_2 = {} & \frac{1152}{\pi^4} \left[\frac{\pi^2}{6} \left(\frac{1}{\alpha^2} - \frac{1}{4} \right) + \frac{1}{4} \psi' \left(\frac{d}{2} \right) \right]^2, \end{aligned}$$

(iii) *the distribution of* $n^{1/2}(\alpha_Z^d - \alpha)$ *is asymptotic to* $N(0, m^2)$, *where*

$$m^2 = \frac{\alpha^2}{160\pi^4}[\alpha^4(90\psi'''(d/2) + 180\psi'(d/2)^2 - 60\pi^2\psi'(d/2) - \pi^4) + \alpha^2 40\pi^2(6\psi'(d/2) - \pi^2) + 176\pi^4].$$

Proof. The proof is analogous to the one-dimensional case. For (ii) and (iii) we additionally take advantage of (3.2) and (3.3). ■

COROLLARY 3.10. *The bias and the square root of the risk of the estimator* α_Z^d *are* $O(n^{-1/2})$ *as* $n \to \infty$. *In particular, the estimator is asymptotically unbiased.*

With the increase of the sample size the risk decreases, whereas with the increase of α it initially increases and then decreases. The estimation of the risk $\mathrm{R}_Z^d(\alpha)$ suggested in Theorem 3.9 can be useful only in asymptotic applications.

4. Monte-Carlo simulations. The estimation of parameters of multidimensional spherical invariant α-stable distributions takes place according to the following sequence:

(i) estimation of α by means of α_Z^d,

(ii) estimation of σ by means of σ_{NZ}^d and substitution of α by α_Z^d.

Each vector $\zeta \sim S_{d,\alpha}(\sigma)$ when $\alpha < 2$ can be presented in a way that facilitates simulation (see also [ST, Cor. 2.5.5]).

LEMMA 4.1. *Let* $\alpha \in (0, 2)$, $\sigma > 0$, *suppose a random variable* A *has distribution* $S_{\alpha/2}((\cos(\pi\alpha/4))^{2/\alpha}, 1, 0)$ *and* $Y_1, \ldots, Y_d$ *are independent random variables of equal distribution* $N(0, \sqrt{2})$. *Then* $\zeta = A^{1/2}Y$ *has distribution* $S_{d,\alpha}(\sigma)$.

The estimators α_Z^d and σ_{NZ}^d have been evaluated by means of Monte-Carlo simulation. Sample sizes taken were $n = 50, 100, 500$. $N = 10^4$ simulations were executed for each sample. The Chambers algorithm (see also [C, W]) was used to simulate random variables with α-stable distributions.

Tables 1–3 include the results of estimation of the index α and scale parameter σ. For each sample size we give an estimation of the mean value (first row) and mean square risk (second row).

In the case of α_Z^d (see Table 1) it is clearly visible that the bias and mean square risk decrease with the increase of sample size and vector dimension. This is also the case for σ_{NZ}^d (see Tables 2 and 3). Comparison of Tables 2 and 3 shows that for $\alpha < 1$ the mean square risk when α does not require estimation is larger than when the index has to be estimated. This apparent contradiction is an outcome of a very "heavy tail" and biasness of α_Z^d. For $\alpha > 1$ the situation raises no doubts.

Table 1. Expected value and mean square risk of the estimator α_Z^d

$\alpha = 0.5$					$\alpha = 1.5$				
n	$d=1$	$d=2$	$d=3$	$d=5$	n	$d=1$	$d=2$	$d=3$	$d=5$
50	0.519	0.517	0.516	0.516	50	1.581	1.542	1.533	1.523
	0.007	0.005	0.005	0.005		0.131	0.047	0.031	0.024
100	0.509	0.508	0.508	0.507	100	1.571	1.525	1.515	1.513
	0.003	0.003	0.002	0.002		0.095	0.025	0.016	0.013
500	0.502	0.502	0.502	0.502	500	1.526	1.505	1.503	1.503
	0.001	0.001	0.000	0.000		0.028	0.005	0.003	0.002

Table 2. Expected value and mean square risk of the estimator σ_{NZ}^d (index α known)

$\alpha = 0.5$					$\alpha = 1.5$				
n	$d=1$	$d=2$	$d=3$	$d=5$	n	$d=1$	$d=2$	$d=3$	$d=5$
50	1.008	1.003	1.004	1.002	50	0.999	1.000	0.997	1.000
	0.175	0.152	0.146	0.149		0.032	0.015	0.011	0.009
100	0.999	0.997	1.003	0.997	100	1.002	1.000	1.000	1.000
	0.081	0.070	0.069	0.066		0.015	0.007	0.005	0.004
500	0.999	1.001	1.001	1.000	500	1.000	1.000	1.000	1.000
	0.015	0.013	0.013	0.013		0.003	0.001	0.001	0.001

Table 3. Expected value and mean square risk of the estimator σ_{NZ}^d (α estimated by α_Z^d)

$\alpha = 0.5$					$\alpha = 1.5$				
n	$d=1$	$d=2$	$d=3$	$d=5$	n	$d=1$	$d=2$	$d=3$	$d=5$
50	0.994	0.998	0.995	0.993	50	1.008	1.003	1.003	1.003
	0.128	0.104	0.096	0.092		0.053	0.016	0.010	0.007
100	0.997	0.996	0.997	0.998	100	1.009	1.002	1.001	1.000
	0.064	0.049	0.048	0.045		0.029	0.008	0.005	0.003
500	0.998	0.999	0.997	1.000	500	1.004	1.000	1.001	1.000
	0.012	0.010	0.009	0.009		0.006	0.002	0.001	0.001

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