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A. Raouf Chouikha (Paris)

MONOTONICITY OF THE PERIOD FUNCTION FOR SOME PLANAR DIFFERENTIAL SYSTEMS. PART II: LIÉNARD AND RELATED SYSTEMS

Abstract. We are interested in conditions under which the two-dimensional autonomous system

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y,$$

has a local center with monotonic period function. When f and g are (non-odd) analytic functions, Christopher and Devlin [C-D] gave a simple necessary and sufficient condition for the period to be constant. We propose a simple proof of their result. Moreover, in the case when f and g are of class C^3, the Liénard systems can have a monotonic period function in a neighborhood of 0 under certain conditions. Necessary conditions are also given. Furthermore, Raleigh systems having a monotonic (or non-monotonic) period are considered.

1. Introduction. The Liénard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

or its equivalent two-dimensional form

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y, \tag{2}$$

holds an important place in the theory of dynamical systems. Several problems have been considered by authors studying this equation, including existence, boundedness, uniqueness, and multiplicity of periodic solutions and related questions.

We assume $g(0) = 0$, so that the origin is a critical point for the equivalent two-dimensional problem (2).

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One of the goals of this paper is to study the conditions under which system (2) has a center of constant period (or, alternatively, for which (1) has a non-isolated periodic solution with locally constant period).

Recently, significant results on the isochronicity problem have been presented, [S], [C-D]. Our results on the Liénard system are an attempt to classify the isochronous systems of a significant infinite-dimensional family of planar polynomial systems.

The notations $' = d/dx$ and $'' = d^2/dx^2$ will be used throughout.

The next result has been proved by Christopher, Devlin, Lloyd and Sabatini in the analytic case.

Proposition 1. *Let f and g be analytic odd functions of x with $xg(x)>0$ in a neighborhood of the origin. Then system* (2) *has an isochronous center at the origin if and only if $g'(0) > 0$ and*

$$(*)\qquad g(x) = g'(0)x + \frac{1}{x^3}\Big(\int_0^x \xi f(\xi)\,d\xi\Big)^2.$$

If $f(x)$ and $g(x)$ are odd functions of class C^1, then $(*)$ is a sufficient condition for the origin to be isochronous.

In [A-F-G] several cases of non-odd polynomial Liénard systems (2) are considered. It is proved, in particular, that 0 is not an isochronous center when f and g do not satisfy $(*)$.

Moreover, it is conjectured that the assumption that f and g are odd functions is superfluous.

In this direction, Proposition 1 has been generalized by Christopher and Devlin [C-D]:

Proposition 2. *System* (2) *with f and g analytic such that $f(0) = g(0) = 0$ and $g'(0) = 1$ has an isochronous center at the origin if and only if*

$$(3)\qquad g = ss'\bigg(1 + \frac{1}{s^4}\Big(\int_0^x s(\xi)f(\xi)\,d\xi\Big)^2\bigg),$$

where $s(x)$ solves the functional equation

$$(4)\qquad F(x - 2s(x)) = F(x), \qquad s(0) = 0,\ s'(0) = 1.$$

where $F(x) = \int_0^x f(t)\,dt$. In particular, if $f(x)$ or $g(x)$ is odd, then (2) *has an isochronous center at the origin if and only if $f(x)$ is odd and*

$$(*)\qquad g(x) = x + \frac{1}{x^3}\Big(\int_0^x \xi f(\xi)\,d\xi\Big)^2.$$

This latter result was first proved by Sabatini [S] for Liénard systems which are not necessarily analytic.

Results of this paper can be viewed as a contribution to the proof of the above conjecture.

Let us formulate our main results:

1) We propose (in Section 2) an alternative simpler proof of the above Proposition 2.

2) Moreover, we study (in Section 3) the monotonicity of the period function T for system (2) when f and g are of class C^3. The inequality

$$g'(0)g^{(3)}(x) - \frac{5}{3}\,g''(x)^2 - \frac{2}{3}\,f'(x)^2 g'(0) \neq 0$$

implies the monotonicity of the period function T in a neighborhood of 0.

We deduce in particular that:

(a) If f and g are of class C^3 and $g^{(3)}(0) > 0$, then $g'(0)g^{(3)}(0) - \frac{5}{3}\,g''(0)^2 - \frac{2}{3}g'(0)f'(0)^2 = 0$ and $f'(0)g''(0) - g'(0)f''(0) = 0$ are necessary conditions for the center 0 to be isochronous.

(b) If $g(x) = x$ and $f'(0) \neq 0$ then the period function is increasing in a neighborhood of 0.

(c) When f and g are of class C^4 and $g'(0)g^{(3)}(0) - \frac{5}{3}g''(0)^2 - \frac{2}{3}g'(0)f'(0)^2 = 0$ we establish other necessary conditions for the center 0 of the Liénard system (2) to be isochronous.

2. A generalization of Christopher and Devlin. As mentioned in the introduction, Christopher and Devlin [C-D] generalized Proposition 1 and proved a classification theorem for isochronous centers for Liénard systems of the form

$$\dot{x} = y, \qquad \dot{y} = -g(x) - f(x)y, \tag{2}$$

The proof of Proposition 2 consisted in rewriting system (2) in a normal form using the function derived from the complex separatrices of the system at the origin.

Below we give another proof of that result, by reducing the system to the type considered by Sabatini. For completeness, we present parts of Sabatini's result needed for the proof.

Proof of Proposition 2. Let us recall briefly some basic results on isochronous Liénard systems. Any isochronous family of periodic orbits surrounds a unique non-degenerate critical point of center type. We shall assume that (2) linearizes to a non-degenerate center at the origin. A translation of this critical point does not modify the form of system (2). Scaling the x-axis preserves the isochronicity of the center. Therefore, without loss of generality, we may choose f and g such that

$$g(0) = 0, \qquad g'(0) = 1, \qquad f(0) = 0.$$

It can be shown that if (2) has an isochronous center, then necessarily $g(x)$ has $x=0$ as its only zero. Moreover, if we suppose in addition $g''(0)=0$ then necessarily $f''(0)=0$.

Concerning the existence of a center, a result of Cherkas [C] shows that system (2) has a center if and only if there exist polynomials A and B such that the integrals

$$F(x)=\int_0^x f(t)\,dt, \qquad G(x)=\int_0^x g(t)\,dt$$

satisfy

$$F(x)=A(M(x)), \qquad G(x)=B(M(x)),$$

for a polynomial $M(x)=x^2+\cdots$. In particular, if $g(x)$ (and $f(x)$) are odd then $M=x^2$.

Let us define $s=s(x)$ such that

$$F(x-2s(x))=F(x), \qquad s(0)=0, \qquad s'(0)=1.$$

The case $M=x^2$ holds if and only if $s(x)=x$. The same holds true if we replace F by any analytic function $M=x^2+\cdots$ for which $F=\chi(M)$ for some analytic function χ. If g is odd then $M=x^2$.

Following [C-D], if the non-constant term of the lowest degree of the power series of $F(x)$ is even then the function $s(x)$ defined in (4) is clearly unique and analytic in a neighborhood of $x=0$. So, we assume $F''(0)\neq 0$.

From the defining equation for s, we find that

$$F(x)=F(x-2s(x))=F(x-2s(x)-2s(x-2s(x)))$$

so that, by uniqueness of the solution of $F(x)=F(y(x))$ with $y'(0)=1$,

$$x=x-2s(x)-2s(x-2s(x)),$$

and hence

$$x(-s)=x(s)-2s. \tag{5}$$

We thus have $x=s+\phi(s)$, where ϕ is even in s.

Now, (4) and (5) imply that $F(x)=\widetilde{F}(s)$ for some even analytic function $\widetilde{F}$. We can also take $G(x)=\widetilde{G}(s)$ for some analytic function $\widetilde{G}$, and scale system (2) by dx/ds to get

$$ds/dt=y, \qquad dy/dt=-\widetilde{g}(s)-\widetilde{f}(s)y, \tag{6}$$

where

$$\widetilde{f}=\frac{d\widetilde{F}}{ds}, \qquad \widetilde{g}=\frac{d\widetilde{G}}{ds}.$$

Then (6) is a Liénard system with $\widetilde{f}(s)$ an odd analytic function. When $\widetilde{g}$ is odd, then the origin of (6) is a center by symmetry in the y-axis. According to the argument of Cherkas [C], if $\widetilde{g}$ is not odd then (6) cannot have a center.

Thus, systems (2) and (6) have a center at the origin if and only if $\widetilde{g}$ is an odd function of s. Moreover,

$$\widetilde{g}(s) = g(x)(1+\phi'(s)), \qquad \widetilde{f}(s) = f(x)(1+\phi'(s)).$$

By Proposition 1, since $\widetilde{f}(s)$ and $\widetilde{g}(s)$ are odd, system (6) has an isochronous center at 0 if and only if

(7) $$\widetilde{g}(s) = \widetilde{g}'(0)s + \frac{1}{s^3}\Big(\int_0^s \sigma \widetilde{f}(\sigma)\,d\sigma\Big)^2.$$

An easy calculation yields $\widetilde{g}'(0) = g'(0) = 1$ and after a change of variables

$$\widetilde{g}(s) = (1+\phi(s))g(x) = s + \frac{1}{s^3}\Big(\int_0^s \sigma f(\xi)(1+\phi'(\sigma))\,d\sigma\Big)^2.$$

Here $\sigma = s(\xi)$, $(1+\phi'(\sigma))\,d\sigma = d\xi$, and hence

$$(1+\phi(s))g(x) = s + \frac{1}{s^3}\Big(\int_0^x s(\xi)f(\xi)\,d\xi\Big)^2.$$

So, by Proposition 1, system (6) has an isochronous center at 0 if and only if

(3) $$g = ss'\bigg(1 + \frac{1}{s^4}\Big(\int_0^x s(\xi)f(\xi)\,d\xi\Big)^2\bigg).$$

Now consider the transformation of (6) given by $Y = y + sN(s)$ with

$$N = \frac{1}{s^2}\int_0^s \sigma\widetilde{f}(\sigma)\,d\sigma,$$

an odd polynomial in s, which brings system (6) to the form

$$\dot{s} = Y - sN(s), \qquad \dot{Y} = -s - YN(s) + K(s),$$

where $K(s) = s + s^{-3}(\int_0^s \sigma\widetilde{f}(\sigma)\,d\sigma)^2 - \widetilde{g}$, which is odd in s. In polar coordinates $\dot{\theta} = -1$, which means that $K(s) = 0$ if and only if the center of (6) (and hence of (2)) is isochronous (the converse holds by (7) and Proposition 1). However, this is just condition $(*)$ above. Changing back to the original coordinates, we get condition (3). The proposition is thus proved.

3. Monotonicity of the period function for Liénard equations

3.1. *Preliminary results.* Consider

(1) $$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

with $g(0) = 0$. Thus, $x \equiv 0$ is a trivial solution, and the origin is a singular point of the equivalent system

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = -g(x) - f(x)y, \end{cases} \tag{2}$$

where f, g are functions of class C^k, $k \geq 1$.

Suppose 0 is a center of (2), and let γ_0 be the *central region*, the open connected set covered by cycles surrounding the center. These periodic trajectories may be parametrized for example by choosing their initial values in the segment $(0, \pi)$ on the x-axis, $x(0) = c$.

Recall that $T : \gamma_0 \to \mathbb{R}$ is the function defined by associating to every point $(x, 0) \in \gamma_0$ the minimum period of the trajectory starting at $(x, 0)$, and reaching the negative x-axis. T is the *period function* and is constant on cycles. We say T is *increasing* (resp. strictly increasing) if, for every couple of cycles γ_1 and γ_2 with γ_1 included in γ_2, we have $T(\gamma_1) \leq T(\gamma_2)$ (resp. $T(\gamma_1) < T(\gamma_2)$).

We say 0 is an *isochronous center* if T is constant in a neighborhood of 0.

Functions f and g are assumed to be C^3 on an open interval J containing 0 and to satisfy $f(0) = g(0) = 0$ and $g'(0) > 0$. These assumptions ensure that the origin is a center, so that the period function T is defined.

Multiplication of (1) by $\alpha^{-1/2}$ ($\alpha > 0$) does not change the nature (of monotonicity) of the period but only changes each period by a constant multiple. More precisely, for any positive real number α equation (1) is equivalent to

$$\ddot{x} + \frac{1}{\sqrt{\alpha}} f(X)\dot{x} + \frac{1}{\alpha} g(X) = 0$$

by the scaling $x(t) = X(\sqrt{\alpha}\, t)$.

We are led to a system of the form

$$\begin{cases} \dot{x} = -\dfrac{1}{\alpha^{1/2}}\, y, \\ \dot{y} = \dfrac{1}{\alpha^{1/2}}\, g(x) - \dfrac{1}{\alpha^{1/2}}\, f(x)y. \end{cases} \tag{8}$$

3.2. *Main theorem..* The following result specifies the behavior of the period function for the Liénard system in the neighborhood of the center 0. We need the hypothesis that f and g are of class C^3. It allows us to deduce several interesting corollaries. In particular, we obtain simple conditions for the monotonicity of T or for the isochronicity of the center.

THEOREM 3. *Let $f, g \in C^3(J)$, where J is an open interval containing* 0, *and suppose that $f(0) = g(0) = 0$, $g'(0) > 0$ and the origin* 0 *is a center of*

$$\ddot{x} + f(x)\dot{x} + g(x) = 0. \tag{1}$$

If

$$g'(0)g^{(3)}(0) - \frac{5}{3}\,g''(0)^2 - \frac{2}{3}\,f'(0)^2 g'(0) \neq 0 \tag{9}$$

then the period function T *of equation* (1) *is monotonic in a neighborhood of* 0. *More precisely, if*

$$g'(0)g^{(3)}(0) - \frac{5}{3}\,g''(0)^2 - \frac{2}{3}\,f'(0)^2 g'(0) < 0 \tag{9+}$$

then T *is increasing in a neighborhood of* 0, *and if*

$$g'(0)g^{(3)}(0) - \frac{5}{3}\,g''(0)^2 - \frac{2}{3}\,f'(0)^2 g'(0) > 0 \tag{9−}$$

then T *is decreasing in a neighborhood of* 0.

Proof. To establish the asserted conditions, we will give an expansion of the period function near the center. We will use implicit function techniques.

Since the origin is a center, orbits of solutions starting on the positive x-axis must be closed. Let $(x(t,c), y(t,c))$ be a solution other than the origin of

$$\begin{cases} \dot{x} = -\sqrt{g'(0)}\,y, \\ \dot{y} = \dfrac{1}{\sqrt{g'(0)}}\,g(x) - \dfrac{1}{\sqrt{g'(0)}}\,f(x)y, \end{cases} \tag{10}$$

with $x(0,c) = c$ and $y(0,c) = 0$. Suppose c is a positive constant close to 0. After a certain time close to $2\pi/\sqrt{g'(0)}$ this solution will go around the origin and will again intersect the positive x-axis at $x(T,c)$. Consider the following functions depending on c:

$$\phi(T,c) = x(T,c) - c, \qquad \psi(T,c) = y(T,c).$$

We will solve $\psi(T,c) = 0$ for $T = T(c)$, a function of c small. Thus, ϕ is a function of c. Let $\Phi(c) = \phi(T(c),c)$. We find that the position of return is $x = c + \Phi(c)$. Thus, the orbit is closed if and only if

$$\Phi(c) = 0.$$

We will find the behavior of $\Phi(c)$ when c tends to 0 by calculating its first derivatives at 0. First, we have $\Phi(0) = 0$, $T_0 = T(0) = 2\pi/\sqrt{g'(0)}$ and the partial derivatives of ϕ and ψ are

$$\phi_T(T_0,0) = \dot{x}\left(\frac{2\pi}{\sqrt{g'(0)}}, 0\right) = 0, \qquad \psi_T(T_0,0) = \dot{y}\left(\frac{2\pi}{\sqrt{g'(0)}}, 0\right) = 0$$
$$\phi_c(T_0,0) = x_c\left(\frac{2\pi}{\sqrt{g'(0)}}, 0\right) - 1, \qquad \psi_c(T_0,0) = y_c\left(\frac{2\pi}{\sqrt{g'(0)}}, 0\right).$$

Here the subscript c or T denotes differentiation with respect to c or T.

The derivatives of $x_c(t,c)$ and $y_c(t,c)$ with respect to t satisfy the Liénard system

$$(11)\qquad \begin{cases} \dot{x}_c = -\sqrt{g'(0)}\, y_c, \\ \dot{y}_c = \dfrac{1}{\sqrt{g'(0)}}\, g'(x)x_c - \dfrac{1}{\sqrt{g'(0)}}\, f(x)y_c - \dfrac{1}{\sqrt{g'(0)}}\, f'(x)x_c y, \end{cases}$$

with initial conditions $x_c(T_0,c) = 1$, $y_c(T_0,c) = 0$. According to our hypotheses, $g(0) = f(0) = 0$. If we set $c = 0$ the system becomes

$$(12)\qquad \begin{cases} \dot{x}_c(t,0) = -\sqrt{g'(0)}\, y_c, \\ \dot{y}_c(t,0) = \sqrt{g'(0)}\, x_c. \end{cases}$$

This implies in particular $x_c(t,0)=\cos(\sqrt{g'(0)}\,t)$ and $y_c(t,0)=\sin(\sqrt{g'(0)}\,t)$. Thus, $\phi(T_0,0) = 0$ and $\psi(T_0,0) = 0$.

We now calculate the second derivatives:

$$\phi_{TT}(T_0,0) = \ddot{x}\left(\frac{2\pi}{\sqrt{g'(0)}},0\right) = 0, \qquad \psi_{TT}(T_0,0) = \dot{y}'\left(\frac{2\pi}{\sqrt{g'(0)}},0\right) = 0,$$

$$\phi_{Tc}(T_0,0) = \dot{x}_c\left(\frac{2\pi}{\sqrt{g'(0)}},0\right) = 0, \qquad \psi_{Tc}(T_0,0) = \dot{y}_c\left(\frac{2\pi}{\sqrt{g'(0)}},0\right) = \sqrt{g'(0)},$$

$$\phi_{cc}(T_0,0) = x_{cc}\left(\frac{2\pi}{\sqrt{g'(0)}},0\right), \qquad \psi_{cc}(T_0,0) = y_{cc}\left(\frac{2\pi}{\sqrt{g'(0)}},0\right).$$

The derivatives satisfy the system

$$\begin{cases} \dot{x}_{cc} = -\sqrt{g'(0)}\, y_{cc}, \\ \dot{y}_{cc} = \dfrac{1}{\sqrt{g'(0)}}\, [[g'(x) - f'(x)y]x_{cc} \\ \qquad + [g''(x) - f''(x)y](x_c)^2 - f(x)y_{cc} - 2f'(x)x_c y_c]. \end{cases}$$

Setting now $c = 0$, according to the above initial conditions we get

$$\begin{cases} \dot{x}_{cc} = -\sqrt{g'(0)}\, y_{cc}, \\ \dot{y}_{cc} = \dfrac{1}{\sqrt{g'(0)}}\, [g'(0)x_{cc} + g''(0)(\cos(\sqrt{g'(0)}\,t))^2 \\ \qquad - 2f'(0)\cos(\sqrt{g'(0)}\,t)\sin(\sqrt{g'(0)}\,t)]. \end{cases}$$

The solution of the latter system is

$$\begin{cases} \dot{x}_{cc} = \dfrac{1}{6\sqrt{g'(0)}}\, [-3g''(0) + g''(0)\cos(\sqrt{g'(0)}\,t) + 4f'(0)\sin(\sqrt{g'(0)}\,t) \\ \qquad + g''(0)\cos(2\sqrt{g'(0)}\,t) - 2f'(0)\sin(2\sqrt{g'(0)}\,t)], \\ \dot{y}_{cc} = \dfrac{1}{3\sqrt{g'(0)}}\, [g''(0)\sin(\sqrt{g'(0)}\,t) - 2f'(0)\cos(\sqrt{g'(0)}\,t) \\ \qquad + g''(0)\sin(2\sqrt{g'(0)}\,t) + 2f'(0)\cos(\sqrt{g'(0)}\,t)]. \end{cases}$$

We deduce from this the values

$$\phi_{cc}(T_0,0)=\psi_{cc}(T_0,0)=0.$$

By a similar method we establish the value of the third derivatives. We find in particular

$$\begin{aligned}
\phi_{Tcc}(T_0,0) &= \dot{x}_{cc}\left(\frac{2\pi}{\sqrt{g'(0)}},0\right)=0,\\
\psi_{Tcc}(T_0,0) &= \dot{y}_{cc}\left(\frac{2\pi}{\sqrt{g'(0)}},0\right)=\frac{g''(0)}{2\sqrt{g'(0)}},\\
\phi_{TTc}(T_0,0) &= \ddot{x}_{c}\left(\frac{2\pi}{\sqrt{g'(0)}},0\right)=-g'(0),\\
\psi_{TTc}(T_0,0) &= \dot{y}'_{c}\left(\frac{2\pi}{\sqrt{g'(0)}},0\right)=0,\\
\phi_{ccc}(T_0,0) &= x_{ccc}\left(\frac{2\pi}{\sqrt{g'(0)}},0\right)=\frac{3\pi}{2\sqrt{g'(0)}}\left[\frac{g''(0)}{g'(0)}\,\frac{f'(0)}{2\sqrt{g'(0)}}-\frac{f''(0)}{2\sqrt{g'(0)}}\right],\\
\psi_{ccc}(T_0,0) &= y_{ccc}\left(\frac{2\pi}{\sqrt{g'(0)}},0\right)\\
&= \frac{\pi}{2\sqrt{g'(0)}}\left[-\frac{f'(0)^2}{g'(0)}-\frac{10g''(0)^2}{4g'(0)^2}+9\,\frac{g^{(3)}(0)}{6g'(0)}\right].
\end{aligned}$$

From the above calculation, we may write the Taylor expansions of $\psi(T,c)$ and $\phi(T,c)$ near $(T_0,0)$:

$$\begin{aligned}
\phi(T,c) &= -\frac{g'(0)}{2}\left(T-\frac{2\pi}{\sqrt{g'(0)}}\right)^2 c+\frac{1}{6}\,\phi_{ccc}(T_0+\theta(T-T_0),\theta c)c^3,\\
\psi(T,c) &= \left(T-\frac{2\pi}{\sqrt{g'(0)}}\right)c+\frac{g''(0)}{2\sqrt{g'(0)}}\left(T-\frac{2\pi}{\sqrt{g'(0)}}\right)c^2\\
&\quad+\frac{1}{6}\,\psi_{ccc}(T_0+\theta(T-T_0),\theta c)c^3,
\end{aligned}$$

for some θ such that $0<\theta<1$.

Finally, we return to our problem. Let us solve $\psi(T(c),c)=0$ for T as an implicit function of c. Notice that $T(c)\to T_0$ as $c\to 0$ and by hypothesis ψ is of class C^3. Hence its third derivative at $(T_0,0)$ is such that $\phi_{ccc}(T_0+\theta(T-T_0))\to\psi_{ccc}(T_0,0)$ as $c\to 0$. Thus,

$$T(c)=\frac{2\pi}{\sqrt{g'(0)}}-\frac{1}{6}\,\psi_{ccc}(T_0,0)c^2+o(c^3).$$

Recall the orbit is closed if $\Phi(c)=\phi(T(c),c)=0$. Substituting $T=T(c)$ we get

$$\Phi(c) = \frac{1}{6}\,\phi_{ccc}(T_0,0)c^3 + o(c^3).$$

Thus, we get the expansion of the period function for c small:

$$T(c) = \frac{2\pi}{\sqrt{g'(0)}} - \frac{\pi}{12\sqrt{g'(0)}}\left[-\frac{f'(0)^2}{g'(0)} - \frac{10g''(0)^2}{4g'(0)^2} + 9\,\frac{g^{(3)}(0)}{6g'(0)}\right]c^2 + o(c^3),$$

which leads to the stated condition and proves the theorem.

First, we deduce the following useful consequence:

LEMMA 1. *Let $f \in C^2(J)$ and $g \in C^3(J)$ satisfy $f(0) = g(0) = 0$, $g'(0) > 0$ and the Liénard equation* (1). *Then a necessary condition for the origin to be a center of* (2) *is*

$$f'(0)g''(0) - g'(0)f''(0) = 0.$$

Proof. In the proof of Theorem 1, we have given the behavior of the period near the origin. $\Phi(c) \neq 0$ for small c means there is no closed orbit in a neighborhood of 0. Indeed, $c + \Phi(c)$ corresponds to the position of return to the x-axis. Since we have seen

$$\Phi(c) = \phi(T(c),c) = -\frac{g'(0)}{2}\left(T - \frac{2\pi}{\sqrt{g'(0)}}\right)^2 c + \frac{1}{6}\,\phi_{ccc}(T_0,0)c^3 + o(c^3)$$

and according to the above expansion of $T(c)$, a necessary condition for the origin to be a center is

$$\phi_{ccc}(T_0,0) = \frac{3\pi}{2\sqrt{g'(0)}}\left[\frac{g''(0)}{g'(0)}\,\frac{f'(0)}{2\sqrt{g'(0)}} - \frac{f''(0)}{2\sqrt{g'(0)}}\right] = 0.$$

This yields the assertion.

3.3. *Remarks.* (i) The problem of the monotonicity of the period was already considered in special cases for a subfamily of Liénard systems, notably in [C2] and [F-G-G], which considered the system

$$\begin{cases} \dot{x} = -y + A(x), \\ \dot{y} = A'(x), \end{cases} \tag{13}$$

where A is a smooth function such that $A(0) = A'(0) = 0$. So, $f = -A'$ and $g = A$ in our notations. We may verify that for any $A(x) = kI_i(x)$ for $i = 1,\ldots,9$ and $k > 0$ (see the notations of [F-G-G, §4.3]) one gets

$$A''(0)A^{(4)}(0) - \frac{5}{3}\,(A^{(3)})^2(0) - \frac{2}{3}\,A''^3(0) < 0 \text{ (resp. } > 0).$$

Thus, this system has an increasing (resp. decreasing) period in the neighborhood of the origin.

(ii) Consider the Liénard equation with linear restoring term

$$(L_l) \quad \ddot{x} + f(x)\dot{x} + x = 0.$$

In [C-G-M] the monotonicity of its period is proved by computing the period constants in the case where f is analytic.

As a corollary of Theorem 3 one proves the following

COROLLARY 4. *Let $f \in C^1(J)$ be such that $f'(0) \neq 0$ and let the origin be a center for equation (L_l). Then the period function is increasing in a neighborhood of* 0.

Indeed, if $g(x) = x$ we get

$$-\frac{f'(0)^2}{g'(0)} - \frac{5g''(0)^2}{2g'(0)^2} + 3\,\frac{g^{(3)}(0)}{2g'(0)} = -f'(0)^2 < 0,$$

so T is increasing.

3.4. *Some corollaries.* We deduce from Theorem 3 other consequences, in particular

COROLLARY 5. *Let $f, g \in C^3(J)$ satisfy $f(0) = g(0) = 0$, $g'(0) > 0$ and let the origin be a center of* (2). *If $g'''(0) < 0$ then the period function T of* (2) *is increasing in a neighborhood of* 0.

Indeed, $g'''(0) < 0$ implies

$$-\frac{f'(0)^2}{g'(0)} - \frac{5g''(0)^2}{2g'(0)^2} + 3\,\frac{g^{(3)}(0)}{2g'(0)} < 0.$$

COROLLARY 6. *Consider the conservative equation $\ddot{x} + g(x) = 0$ and the Liénard equations $\ddot{x} \pm f(x)\dot{x} + g(x) = 0$. Let $f, g \in C^3(J)$ satisfy $f(0) = g(0) = 0$, $g'(0) > 0$ and let the origin be a center of these equations. If the period function of $\ddot{x} + g(x) = 0$ is increasing in a neighborhood of 0, then so is the period function of $\ddot{x} \pm f(x)\dot{x} + g(x) = 0$.*

Indeed, according to Schaaf's criterion (see [Ch1]), T increases if $5g''(x)^2 - 3g'(x)g^3(x) > 0$ in a neighborhood of 0. This implies in particular $5g''(0)^2 - 3g'(0)g^3(0) > 0$. Adding $f'(0)^2/g'(0)$ we deduce that

$$-\frac{f'(0)^2}{g'(0)} - \frac{5g''(0)^2}{2g'(0)^2} + 3\frac{g^{(3)}(0)}{2g'(0)} \quad \text{and} \quad -5g''(0)^2 + 3g'(0)g^3(0)$$

have the same sign.

The following is another interesting consequence:

COROLLARY 7. *Consider the conservative equation $\ddot{x} + g(x) = 0$ and the Liénard equations $\ddot{x} \pm f(x)\dot{x} + g(x) = 0$. Let $f, g \in C^3(J)$ satisfy $f(0) = g(0) = 0$, $g'(0) > 0$ and let the origin be a center of these equations. If $g''(0) \neq 0$ and the center 0 of $\ddot{x} \pm f(x)\dot{x} + g(x) = 0$ is isochronous then the period function T of $\ddot{x} + g(x) = 0$ is strictly decreasing in a neighborhood of 0.*

This follows from Corollary 6, on account of the fact that g is not a odd function.

To establish the existence of an isochronous center for the Liénard system, it is necessary first to make sure that the period of the associated conservative system is decreasing at least in a neighborhood of 0. For this, one will be able to use the different criteria of monotonicity of T.

Corollary 8. *Under the same hypotheses on f and g, suppose in addition $g''(0) \neq 0$ and the origin is a center of* (1). *Then the following conditions on the functions f, g are necessary for equation* (1) *to have T constant in a neighborhood of* 0:

(i) $g^{(3)}(0) > 0$,

(ii) $$f'(0) = \pm \frac{1}{\sqrt{2}} \sqrt{3g^{(3)}(0) - \frac{5}{g'(0)}\, g''(0)^2},$$

(iii) $$f''(0) = \pm \frac{g''(0)}{\sqrt{2}\, g'(0)} \sqrt{3g^{(3)}(0) - \frac{5}{g'(0)}\, g''(0)^2}.$$

Indeed, it is sufficient to remark that by Corollary 7 the period function of the equation $\ddot{x} + g(x) = 0$ has to be decreasing. Then, according to the criterion of Schaaf it is necessary that $3g^{(3)}(x) - (5/g'(x))g''(x)^2 > 0$ in a neighborhood of 0. Moreover, by Lemma 1, $f''(0) = (g''(0)/g'(0))f'(0)$ and we get the second expression.

This result is of interest because we do not need to suppose $g(x)$ odd.

3.5. *Other consequences.* In fact the Liénard system is equivalent to another system more convenient to study. We can prove the following, which agrees with Lemma 2 of [S] for $g'(0) = 1$:

Lemma 2. *Suppose f, g are continuous functions of class C^k, $k \geq 1$, on an interval J containing* 0 *and $f(0) = 0$. Let*

$$C(x) = \frac{1}{g'(0)}\, g(x) - \frac{1}{g'(0)x^3} \Big[\int_0^x s f(s)\, ds \Big]^2.$$

Then the system

$$\text{(14)} \qquad \begin{cases} \dot{x} = y - \dfrac{1}{x\sqrt{g'(0)}} \displaystyle\int_0^x s f(s)\, ds, \\ \dot{y} = -g(x) - \dfrac{1}{g'(0)x^3} \Big[\displaystyle\int_0^x s f(s)\, ds \Big]^2 - \dfrac{y}{x^2\sqrt{g'(0)}} \displaystyle\int_0^x s f(s)\, ds, \end{cases}$$

is of class C^k in a neighborhood of 0 *and is equivalent to* (2).

Indeed, to see this, define

$$\psi(x) = \frac{1}{\sqrt{g'(0)}} \int_0^x s f(s)\, ds.$$

By the de l'Hospital rule we get

$$\lim_{x\to 0} \frac{\psi(x)}{x^2} = \frac{f(0)}{2\sqrt{g'(0)}}.$$

One also proves that the function $\psi(x)/x^2$ is differentiable at 0 with derivative $f'(0)/3\sqrt{g'(0)}$.

Moreover, the function $C(x)$ is obviously differentiable and $C'(0) = 1$.

A calculation gives

$$C''(0) = \frac{g''(0)}{g'(0)} - \frac{2}{3g'(0)} f(0)f'(0), \qquad C'''(0) = \frac{g'''(0)}{g'(0)} - \frac{2}{3g'(0)} f'(0)^2.$$

This proves the regularity of system (4).

Furthermore, let $(x(t), y(t))$ be a solution of (2); then $x(t)$ is a solution of (1). Note that for $x \neq 0$, we get

$$x\left[\frac{\psi(x)}{x^2}\right]' = x^3 f(x) - \frac{2x\psi(x)}{x^3} = \frac{f(x)}{\sqrt{g'(0)}} - 2\,\frac{\psi(x)}{x^2}.$$

Moreover, by differentiating $y = \dot{x} + x\frac{\psi(x)}{x^2}$, we get

$$\ddot{x} = -\frac{1}{g'(0)}\, g(x) - \frac{1}{\sqrt{g'(0)}}\, f(x)\dot{x}.$$

This is equivalent to (1) by scaling the time, here $\alpha = g'(0)$. This yields

$$x\left(\frac{\tau}{\sqrt{g'(0)}}\right) = X(\tau).$$

Since f and g are independent of t, we have $f(x) = f(X)$ and $g(x) = g(X)$. We then obtain, by differentiating with respect to τ,

$$\ddot{x} = -g(X) - f(X)\dot{x}.$$

The following result is a version of Theorem 1 of [S] and may be proved in the same manner.

Proposition 9. *Let $f \in C^2(J)$ and $g \in C^3(J)$ satisfy $f(0) = g(0) = 0$, $g'(0) > 0$, let the origin be a center of* (2), *and in addition suppose that $g''(0) = f''(0) = 0$. Let $C(x)$ be as defined in Lemma* 2. *If*

$$(15) \qquad C(x) \text{ is } \begin{cases} \text{strictly convex for } x \in J \text{ and } x < 0, \\ \text{strictly concave for } x \in J \text{ and } x > 0, \end{cases}$$

then T is increasing in a neighborhood of 0; *if*

$$(16)\qquad C(x) \text{ is } \begin{cases} \textit{strictly convex for } x\in J \textit{ and } x>0, \\ \textit{strictly concave for } x\in J \textit{ and } x<0, \end{cases}$$

then T is decreasing in a neighborhood of 0; *if*

$$(17)\qquad \frac{d^2}{dx^2}C(x)\equiv 0 \quad \textit{for } x\in J,$$

then T is constant in a neighborhood of 0.

Notice that after a change to polar coordinates (r,θ), Sabatini [S] obtains the following system equivalent to (2):

$$(18)\qquad \begin{cases} \dot r = -\sqrt{g'(0)}\, r\cos\theta\sin\theta - r\beta(r\cos\theta) - \sin\theta\, C(r\cos\theta), \\ \dot\theta = -\dfrac{1}{\sqrt{g'(0)}}\cos^2\theta - \sin^2\theta - \dfrac{\cos\theta\, C(r\cos\theta) - C'(0)(r\cos\theta)}{r} \\ \quad = \omega(r,\theta), \end{cases}$$

where

$$\beta(r\cos\theta) = \frac{1}{r^2\cos^2\theta}\int_0^{r\cos\theta} sf(s)\,ds.$$

We also observe that from system (2) we get

$$r^2\dot\theta = r^2\omega(r,\theta) = -xC(x) - y^2,$$

where $\omega = d\theta/dt$ so $T(r) = \int_{[0,2\pi]} d\theta/\omega$.

By Theorem 1 in [S], it is sufficient to prove for example that hypothesis (9−) implies that $\partial\omega(r,\theta)/\partial r \le 0$ for almost all values $\theta\in[0,2\pi]$.

A calculation gives

$$\frac{\partial\omega(r,\theta)}{\partial r} = \frac{r\cos^2\theta C'(r\cos\theta) - \cos\theta C(r\cos\theta)}{r^2}.$$

Hence

$$\partial\omega(r,\theta)\partial r = \frac{-xC(x)+x^2C'(x)}{(x^2+y^2)^{3/2}} = x\,\frac{-C(x)+xC'(x)}{(x^2+y^2)^{3/2}}.$$

Note that $(xC'(x)-C(x))' = xC''(x)$. Then according to hypotheses, $C\in C^3(J)$, and the condition $xC''(x)\le 0$ for $x\in J$, which is equivalent to hypothesis (9−), implies $\partial\omega(r,\theta)/\partial r\le 0$.

In the same way, we prove that the condition $xC''(x)\ge 0$ for $x\in J$, which is equivalent to hypothesis (9+), implies $\partial\omega(r,\theta)/\partial r > 0$.

We have thus proved that the functions $\partial\omega/\partial r$ and $xC''(x)$ have the same sign.

In fact, we can see this by another method. A calculation gives

$$C''(0) = \frac{g''(0)}{g'(0)} - \frac{2}{3g'(0)}f(0)f'(0), \qquad C'''(0) = \frac{g'''(0)}{g'(0)} - \frac{2}{3g'(0)}f'(0)^2.$$

Then necessarily we have $C''(0) = 0$, since $xC''(x) \neq 0$ (if $x \neq 0$) implies the monotonicity of the period. Thus, $xC''(x)$ and $g'''(x) - \frac{2}{3} f'(x)^2$ have the same sign in a neighborhood of 0. This determines the monotonicity of T.

Also, the condition $f''(0) = 0$ turns out to be necessary by Lemma 1.

REMARK. Notice that by definition of $C(x)$, $f(x) \equiv 0$ implies $C(x) \equiv g(x)/g'(0)$. Corollary 5 slightly improves Theorem 3 in [S] since $g^3(0) < 0$ (implying $C^3(0) < 0$) is a sufficient condition for the monotonicity of T without need to suppose $g''(0) = 0$.

Another remark is that the condition $g''(0) = 0$ turns out to be necessary to study various monotonicity conditions of the period function as we have seen above.

Moreover, when system (2) is isochrone and $g''(0) = 0$ then $\frac{d^2}{dx^2} C(x) \equiv 0$ implies

$$C'''(0) = \frac{g^{(3)}(0)}{g'(0)} - \frac{2}{3g'(0)} f'^2(0) = 0,$$

which is a particular case of Corollary 8.

On the other hand, consider the following function introduced in [S]:

$$\sigma(x) = 2x^2 \frac{1}{g'(0)} f(x) \int_0^x s f(s)\, ds - 4 \frac{1}{g'(0)} \left[\int_0^x s f(s)\, ds \right]^2 + \frac{x^3}{g'(0)} [g(x) - x g'(x)].$$

The following properties are proved in [S, Theorem A, Theorem 2 and Corollary 1]:

PROPOSITION 10. *Let $f, g \in C^3(a,b)$ with $f(0) = g(0) = 0$ and $g''(0) = f''(0) = 0$, the origin being a center of* (2). *If $xC(x) > 0$ in a punctured neighborhood J of* 0, *then we have*:

(1) *if $\sigma(x) \leq 0$ for $x \in J$, then T is decreasing in a neighborhood of* 0;
(2) *if $\sigma(x) \geq 0$ for $x \in J$, then T is increasing in a neighborhood of* 0;
(3) *if $\sigma(x) \equiv 0$ for $x \in J$, then T is constant in a neighborhood of* 0.

Notice that while considering the assumption $xC(x) > 0$ in the case where g and f are C^3, we have $g'(0) > 0$ and $C(x)$ is C^3. Furthermore, $\sigma(x)$ and $C(x)$ are related by

$$\sigma(x) = -x^5 \frac{d}{dx} \left(\frac{C(x)}{x} \right).$$

This function may be written in a neighborhood of 0 as

$$\sigma(x) = -2x^6 \left[g^{(3)}(x) - \frac{2}{3} f'(x)^2 \right] + \cdots .$$

This imposes an additional condition $g''(0) = 0$ which turns out to be necessary since we have seen $C''(0) = g''(0) = f''(0)$ by Lemma 1.

If $g''(0) \neq 0$ then $d^5\sigma/dx^5 = -C''(0)/2 \neq 0$ and the proposition cannot be applied. Thus, Theorem 3 gives an improvement of that proposition.

3.6. *Another necessary condition.* In the case where the expression

$$g'(0)g^{(3)}(0) - \frac{5}{3}\, g''(0)^2 - \frac{2}{3}\, f'(0)^2 g'(0)$$

vanishes it is still possible to establish necessary conditions for the period function T of the Liénard system to be monotonic.

Indeed, if f and g are at least of class C^4 we can re-iterate the procedure used in the proof of Theorem 3 in order to find the next term of the expansion of the period $T = T(c)$. Take again the functions ϕ and ψ and their Taylor expansions,

$$\begin{aligned} \phi(T,c) &= -\frac{g'(0)}{2}\left(T - \frac{2\pi}{\sqrt{g'(0)}}\right)^2 c + \frac{1}{6}\,\phi_{ccc}(T_0,0)c^3 + \frac{1}{24}\,\phi_{c^4} + o(c^4), \\ \psi(T,c) &= \left(T - \frac{2\pi}{\sqrt{g'(0)}}\right)c + \frac{g''(0)}{2\sqrt{g'(0)}}\left(T - \frac{2\pi}{\sqrt{g'(0)}}\right)c^2 \\ &\qquad + \frac{1}{6}\,\psi_{ccc}(T_0,0)c^3 + \frac{1}{24}\,\psi_{c^4} + o(c^4). \end{aligned}$$

We then establish the values of their fourth derivatives. By using implicit techniques we are able to calculate the expansion of $T(c)$ near the center 0. Solving $\psi(T,c) = 0$ for $T = T(c)$ and knowing that T is an even function of c we may write

$$T(c) = \frac{2\pi}{\sqrt{g'(0)}} - \frac{\pi}{12\sqrt{g'(0)}}\left[-\frac{f'(0)^2}{g'(0)} - \frac{10g''(0)^2}{4g'(0)^2} + 9\frac{g^{(3)}(0)}{6g'(0)}\right]c^2 + \alpha c^4 + o(c^4).$$

For brevity, we will omit the details (which actually require very long calculations). The coefficient α of c^4 in the expansion of $T(c)$ is

$$\begin{aligned} \alpha = \frac{\pi}{4^3 3^2 g'^{7/2}} \Big[& -g^{(4)}g'^2 + 5f'^2 g''^2 g' + \frac{225}{8}\, g''^2 g^{(3)} g' \\ & + 7f'^2 g^{(3)} g'^2 - \frac{235}{16}\, g''^4 - \frac{13}{4}\, f'^4 g'^2 - \frac{51}{16}\, g^{(3)} g'^2 \Big] \end{aligned}$$

where all the derivatives are evaluated at zero. In particular, when g and f are odd functions one gets

$$\alpha = \frac{\pi}{4^3 3^2 g'^{7/2}}\left[7f'^2 g^{(3)} g'^2 - \frac{235}{16}\, g''^4 - \frac{13}{4}\, f'^4 g'^2 - \frac{51}{16}\, g^{(3)} g'^2\right].$$

This term is of interest only if the preceding one in the expansion of T vanishes, i.e.

$$(L_c) \quad g^{(3)} g' = \frac{5}{3}\, g''^2 + \frac{2}{3}\, f'^2 g'.$$

In the latter case we may assert the following:

THEOREM 11. *Suppose f and g are C^4 functions and 0 is a center of Liénard system* (2) *such that $g^{(3)}g' - \frac{5}{3}g''^2 + \frac{2}{3}f'^2g' = 0$. Then:*

(i) *If $-\frac{3}{5}g^{(4)}g'^2 + 17f'^2g'g''^2 + g''^4 > 0$ then the period function $T(c)$ is increasing in a neighborhood of 0.*
(ii) *If $-\frac{3}{5}g^{(4)}g'^2 + 17f'^2g'g''^2 + g''^4 < 0$ then the period function $T(c)$ is decreasing in a neighborhood of 0.*
(iii) *$g^{(3)}g' - \frac{5}{3}g''^2 + \frac{2}{3}f'^2g' = 0$ and $-\frac{3}{5}g^{(4)}g'^2 + 17f'^2g'g''^2 + g''^4 = 0$ are necessary conditions for the center 0 to be isochronous.*

To verify the monotonicity of T, the following is a practical result when (L_c) is assumed:

COROLLARY 12. *Under the hypotheses of Theorem* 11 *suppose in addition $g^{(3)}g' - \frac{5}{3}g''^2 + \frac{2}{3}f'^2g' = 0$. Then:*

(i) *If $g^{(4)}(0) \leq 0$ then the period function T of* (2) *is increasing in a neighborhood of* 0.
(ii) *If g is a non-odd function then $g^{(4)}(0) > 0$ is a necessary condition for* (2) *to have an isochronous center at* 0.

4. The Raleigh systems. In [C2], Chicone considers the differential equation of the form

$$\ddot{x} + F(\dot{x}) + G(x) = 0$$

with Dirichlet or Neumann boundary values and $F(0) = G(0) = 0$. It is equivalent to the system

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x - x\widetilde{g}(x) - y\widetilde{f}(y). \end{cases} \tag{19}$$

In the standard Neumann situation the functions $\widetilde{f}$ and $\widetilde{g}$ are such that $\widetilde{f}, \widetilde{g} \in C^2([-a,a])$ and satisfy conditions **(C)** of Chicone's paper [C2],

$$(\mathbf{C}) \quad \begin{cases} \text{(i) } \widetilde{f}(-s) = -\widetilde{f}(s), \quad \widetilde{g}(-s) = -\widetilde{g}(s), \quad \text{for } s \in [-a,a] \\ \text{(ii) } \widetilde{f}'(s) \geq 0, \quad \widetilde{g}'(s) \geq 0, \quad \widetilde{f}''(s) \geq 0, \quad \widetilde{g}''(s) \geq 0, \quad \text{for } s \in [0,a]. \end{cases}$$

The trajectories of this system are symmetric with respect to the x-axis. So, it has obviously a center at the origin of the phase plane. Chicone proved that under the above conditions, the period function is increasing.

The following class of Raleigh equations with linear restoring term may have an increasing period in a neighborhood of the center 0:

$$(R) \quad \ddot{x} + F(\dot{x}) + x = 0.$$

Without supposing the hypotheses above, we may prove an analogous result which also improves Corollary 10 of [S]. We only need to suppose that $F(x)$ is an even function with $F''(0) \neq 0$, instead of conditions (**C**). We then deduce from Theorem 3 the same result as in [C2] but with a weaker hypothesis.

THEOREM 13. *Let F be an analytic even function such that $F(0) = 0$, $F''(0) \neq 0$ and the origin 0 is a center of (R). Then the period function T of equation (R) is increasing in a neighborhood of 0.*

Proof. The system

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x + F(y), \end{cases} \tag{20}$$

which is equivalent to (R), has a unique singular point at the origin. By exchanging the variables and multiplying by -1 we obtain the equivalent system

$$\begin{cases} \dot{x} = -y - F(x), \\ \dot{y} = x, \end{cases} \tag{21}$$

which is a Liénard system

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

with $g(x) = x$ and $f(x) = \pm F'(x)$. Moreover, the origin is clearly a center since $f(x)$ is odd. The conclusion holds by Theorem 3 since $-\frac{2}{3}f'(0)^2g'(0) < 0$.

5. General remarks. Our result (Theorem 3) has the advantage of being natural and easily applicable. The monotonicity and isochronicity conditions can be verified in an easier way. In particular, the function $C(x)$ can be reduced to $g(x)$ when $f(x) \equiv 0$ in a neighborhood of 0. This shows the link between Theorem 3 and Opial's monotonicity condition for the period of conservative systems. More precisely, Opial's condition requires $g''(0) = 0$. Recall

$$C''(0) = g''(0) - \frac{2}{3}f(0)f'(0) = 0, \quad C'''(0) = g'''(0) - \frac{2}{3}f'(0)^2.$$

The condition $C'''(x) < 0$ for $x \in (a, b) \subset J$ implies (9+), and $C'''(x) > 0$ implies (9−) when g is C^3 and $g'''(0) < 0$ (this requires necessarily $C'''(0) < 0$).

It naturally seems that the function $C(x)$ plays the same role for system (2) as does $g(x)$ for the conservative system. Indeed, if we take $f(x) \equiv 0$, conditions (15) and (16) reduce to $xg''(x) < 0$ and $xg''(x) > 0$ respectively (see Proposition 1 of [Ch1]). Notice that the Rothe condition for the monotonicity of the period function,

$$R(g) = x\left[3g'(x)^2 - g(x)g''(x) - 3\frac{g'(0)^2}{g''(0)}g''(x)\right] \geq 0 \ (\leq 0),$$

is more general than $xg''(x) < 0$ (and $xg''(x) > 0$ respectively).

We may expect that Theorem 3 can be generalized. That is, $C(x)$ might be replaced by another more general appropriate function, say $D(x)$, which itself can be reduced to the Rothe function

$$3\,\frac{g'(0)^2}{g''(0)}\,g''(x)-3g'(x)^2+g(x)g''(x)$$

for the conservative case $f(x)\equiv 0$. We know its first derivatives. In particular

$$D^{(3)}(0)=g^{(3)}(0)-\frac{5}{3g'(0)}\,g''(0)^2-\frac{2}{3}\,f'^2(0).$$

Therefore the strong condition $g''(0)=0$ will not be required.

We may also expect that the sign of the function

$$F(x)=g'(x)g^{(3)}(x)-\frac{5}{3}\,g''(x)^2-\frac{2}{3}\,f'(x)^2g'(x)$$

determines the global monotonicity of the period function T of the Liénard system

$$\begin{cases}\dot{x}=-y,\\ \dot{y}=g(x)-f(x)y.\end{cases}$$

In the conservative case, the function $F(x)$ reduces to $g'(x)g^{(3)}(x)-\frac{5}{3}g''(x)^2$ which intervenes in the Schaaf condition (see [Ch1, Section 3]). Recall that the last one is weaker than the Rothe condition.

Moreover, according to Corollary 8 in order to determine the isochronous centers at the origin for the Liénard system (other than those determined by [S]) we have to ensure that the associated conservative system has a decreasing period function.

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LAGA
Université Paris 13
93430 Villetaneuse, France
E-mail: chouikha@math.univ-paris13.fr

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