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A PIEZOELECTRIC CONTACT PROBLEM WITH NORMAL COMPLIANCE

Abstract. We consider a mathematical model which describes the static frictional contact between a piezoelectric body and an insulator foundation. We use a nonlinear electroelastic constitutive law to model the piezoelectric material and the normal compliance condition associated to a version of Coulomb's friction law to model the contact. We derive a variational formulation for the model which is in the form of a coupled system involving the displacement and the electric potential fields. Then we provide the existence of a weak solution to the problem and, under a smallness assumption, its uniqueness. We also study the dependence of the solution on the contact conditions and derive a convergence result.

1. Introduction. The piezoelectric phenomenon represents the coupling between the mechanical and electrical behavior of a class of materials, called piezoelectric materials. In simplest terms, when a piezoelectric material is squeezed, an electric charge collects on its surface; conversely, when a piezoelectric material is subjected to a voltage drop, it mechanically deforms. Many crystalline materials exhibit piezoelectric behavior. A few materials exhibit the phenomenon strongly enough to be used in applications that take advantage of their properties. These include quartz, Rochelle salt, lead titanate zirconate ceramics, barium titanate, and polyvinylidene flouride (a polymer film).

On a nanoscopic scale, the piezoelectric phenomenon arises from a nonuniform charge distribution within a crystal's unit cells. When such a crystal is mechanically deformed, the positive and negative charge centers displace by differing amounts. So while the overall crystal remains electrically neu-

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tral, the difference in charge center displacements results in an electric polarization within the crystal. Electric polarization due to mechanical input is perceived as piezoelectricity.

Piezoelectric materials are used extensively as switches and actuary in many engineering systems, in radioelectronics, electroacoustics and measuring equipment. General models for elastic materials with piezoelectric effects can be found in [11, 12, 13, 20] and more recently in [1, 5, 19]. Currently, there is a considerable interest in frictional contact problems involving piezoelectric materials (see for instance [2, 9] and the references therein). However, there exists virtually no mathematical results about contact problems for such materials and there is a need to expand the emerging Mathematical Theory of Contact Mechanics to include the coupling between the mechanical and electrical properties.

The aim of this paper is to study the process of frictional contact between a piezoelectric body which is acted upon by volume forces and surface tractions, and an obstacle, the so-called foundation. We assume the process is static and the properties of the body are electroelastic; we model the contact with normal compliance associated to a general version of Coulomb's law of dry friction. The normal compliance contact condition was introduced in [10] and used in a large number of papers (see for instance [3, 6, 7, 16] and the references therein). We derive a variational formulation of the model which consists in a system coupling a variational inequality for the displacement field and a variational equation for the electric field. Then we provide the existence of a unique weak solution to the model and we study its continuous dependence on the contact conditions. The results obtained in this paper extend part of the results obtained in [14] where the analysis of a frictional contact problem with normal compliance for nonlinear elastic materials was provided. Indeed, in comparison with the problem in [14], the novelty of this paper consists in the fact that here we take into account the piezoelectric properties of the material, which leads to a new and interesting mathematical model.

Following this introduction, the rest of the paper is structured as follows. The model of the contact process of the piezoelectric body is presented in Section 2. In Section 3 we list the assumptions on the problem data, derive the variational formulation of the problem and state our main existence and uniqueness result, Theorem 3.1. The proof of the theorem is presented in Section 5. It is carried out in several steps, based on an abstract result for quasivariational inequalities that we recall in Section 4. Finally, in Section 6 we study the dependence of the solution on the contact conditions and derive a convergence result, Theorem 6.1.

2. Problem statement. We consider the following physical setting. An electroelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with a smooth boundary $\partial\Omega = \Gamma$. The body is submitted to the action of body forces of density $\boldsymbol{f}_0$ and volume electric charges of density q_0. It is also submitted to mechanical and electric constraints on the boundary. To describe them, we consider a partition of Γ into three measurable parts Γ_1, Γ_2, Γ_3, on the one hand, and on two measurable parts Γ_a and Γ_b, on the other hand, such that $\operatorname{meas} \Gamma_1 > 0$, $\operatorname{meas} \Gamma_a > 0$, and $\Gamma_3 \subseteq \Gamma_b$. We assume that the body is clamped on Γ_1 and surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3 the body is in frictional contact with an insulator obstacle, the so-called foundation. We model the contact with normal compliance and a static version of Coulomb's law of dry friction. We also assume that the electrical potential vanishes on Γ_a and a surface electric charge of density q_2 is prescribed on Γ_b. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. Also, below $\boldsymbol{\nu}$ represents the unit outward normal on Γ while "$\cdot$" and $\|\cdot\|$ denote the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively.

With the assumptions above, the problem of equilibrium of the electroelastic body in frictional contact with a foundation is the following.

PROBLEM P. *Find a displacement field* $\boldsymbol{u} : \Omega \to \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$, *an electric potential* $\varphi : \Omega \to \mathbb{R}$ *and an electric displacement field* $\boldsymbol{D} : \Omega \to \mathbb{R}^d$ *such that*

$$\boldsymbol{\sigma} = \mathcal{F}\varepsilon(\boldsymbol{u}) - \mathcal{E}^T \boldsymbol{E}(\varphi) \quad \text{in } \Omega, \tag{2.1}$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \boldsymbol{\beta}\boldsymbol{E}(\varphi) \quad \text{in } \Omega, \tag{2.2}$$

$$\operatorname{Div} \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in } \Omega, \tag{2.3}$$

$$\operatorname{div} \boldsymbol{D} = q_0 \quad \text{in } \Omega, \tag{2.4}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1, \tag{2.5}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on } \Gamma_2, \tag{2.6}$$

$$\sigma_\nu = -p_\nu(u_\nu - g) \quad \text{on } \Gamma_3, \tag{2.7}$$

$$\begin{cases} \|\boldsymbol{\sigma}_\tau\| \le p_\tau(u_\nu - g), \\ \boldsymbol{\sigma}_\tau = -p_\tau(u_\nu - g)\dfrac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \quad \text{if } \boldsymbol{u}_\tau \neq \boldsymbol{0} \end{cases} \quad \text{on } \Gamma_3, \tag{2.8}$$

$$\varphi = 0 \quad \text{on } \Gamma_a, \tag{2.9}$$

$$\boldsymbol{D} \cdot \boldsymbol{\nu} = q_2 \quad \text{on } \Gamma_b. \tag{2.10}$$

In (2.1)–(2.10) and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$. Equations (2.1) and (2.2) represent the electroelastic constitutive law of the material in which $\mathcal{F}$ is a given nonlinear function, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the small strain tensor, $\boldsymbol{E}(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E}$ represents the third order piezoelectric tensor, $\mathcal{E}^T$ is its transposed and $\boldsymbol{\beta}$ denotes the electric permittivity tensor. Details of the linear version of the constitutive relations (2.1) and (2.2) can be found in [1, 2]. Equations (2.3) and (2.4) represent the equilibrium equations for the stress and electric displacement fields, respectively, (2.5) and (2.6) are the displacement and traction boundary conditions, respectively, and (2.9), (2.10) represent the electric boundary conditions.

We now provide some comments on the frictional contact conditions (2.7) and (2.8) in which we are specially interested. Condition (2.7) represents the normal compliance contact condition in which σ_ν and u_ν are the normal stress and the normal displacement, respectively, p_ν is a given function, and g represents the initial gap between the potential contact surface Γ_3 and the foundation, measured along the direction of the outward normal $\boldsymbol{\nu}$. When positive, $u_\nu - g$ represents the penetration of the surface asperities into those of the foundation. Condition (2.8) represents the associated Coulomb's law of dry friction in which $\boldsymbol{\sigma}_\tau$ is the tangential stress, $\boldsymbol{u}_\tau$ denotes the tangential displacement and p_τ is a given function. According to (2.8) the tangential shear cannot exceed the maximum frictional resistance $p_\tau(u_\nu - g)$, the so-called friction bound. Moreover, when sliding commences, the tangential shear reaches the friction bound and the shear opposes it. Frictional contact conditions of the form (2.7), (2.8) have been used in various papers (see for instance [4, 16] and the references therein).

3. Variational formulations and main result. In this section we list the assumptions on the data, derive a variational formulation for the contact problem (2.1)–(2.10) and state our main existence and uniqueness result, Theorem 3.1. To this end we need to introduce some notation and preliminary material.

We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \qquad \|\boldsymbol{v}\| = (\boldsymbol{v}\cdot\boldsymbol{v})^{1/2} \qquad \forall \boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^d,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2} \qquad \forall \boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^d.$$

Here and everywhere in this paper i, j, k, l run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \partial u_i/\partial x_j$.

Everywhere below we use the classical notation for L^p and Sobolev spaces associated to Ω and Γ. Moreover, we use the notation $L^2(\Omega)^d$, $H^1(\Omega)^d$,

$\mathcal{H}$ and $\mathcal{H}_1$ for the following spaces:

$$L^2(\Omega)^d = \{\boldsymbol{v} = (v_i) \mid v_i \in L^2(\Omega)\},$$
$$H^1(\Omega)^d = \{\boldsymbol{v} = (v_i) \mid v_i \in H^1(\Omega)\},$$
$$\mathcal{H} = \{\boldsymbol{\tau} = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\},$$
$$\mathcal{H}_1 = \{\boldsymbol{\tau} \in \mathcal{H} \mid \tau_{ij,j} \in L^2(\Omega)\}.$$

The spaces $L^2(\Omega)^d$, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u},\boldsymbol{v})_{L^2(\Omega)^d} = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,dx, \qquad (\boldsymbol{u},\boldsymbol{v})_{H^1(\Omega)^d} = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}\,dx + \int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\,dx,$$
$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx, \qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} = \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,dx + \int_\Omega \operatorname{Div}\boldsymbol{\sigma}\cdot\operatorname{Div}\boldsymbol{\tau}\,dx,$$

and the associated norms $\|\cdot\|_{L^2(\Omega)^d}$, $\|\cdot\|_{H^1(\Omega)^d}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. Here and below we use the notation

$$\nabla\boldsymbol{v} = (v_{i,j}), \quad \boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v})), \quad \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}(v_{i,j}+v_{j,i}) \quad \forall \boldsymbol{v}\in H^1(\Omega)^d,$$
$$\operatorname{Div}\boldsymbol{\tau} = (\tau_{ij,j}) \quad \forall\boldsymbol{\tau}\in\mathcal{H}_1.$$

For every element $\boldsymbol{v} \in H^1(\Omega)^d$ we also write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ given by $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$, $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$.

Let now consider the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \{\boldsymbol{v}\in H^1(\Omega)^d \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\}.$$

Since meas$(\Gamma_1) > 0$, the following Korn inequality holds:

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \ge c_K\|\boldsymbol{v}\|_{H^1(\Omega)^d} \quad \forall\boldsymbol{v}\in V, \tag{3.1}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. On the space V we consider the inner product given by

$$(\boldsymbol{u},\boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \tag{3.2}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (3.1) that $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent norms on V. Therefore $(V,\|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, and (3.1) and (3.2), there exists a constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \le c_0\|\boldsymbol{v}\|_V \quad \forall\boldsymbol{v}\in V. \tag{3.3}$$

We also introduce the spaces

$$W = \{\psi\in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_a\},$$
$$\mathcal{W}_1 = \{\boldsymbol{D} = (D_i) \mid D_i\in L^2(\Omega),\ D_{i,i}\in L^2(\Omega)\}.$$

Since $\mathrm{meas}(\Gamma_a) > 0$, the following Friedrichs–Poincaré inequality holds:

$$\|\nabla\psi\|_{L^2(\Omega)^d} \ge c_F\,\|\psi\|_{H^1(\Omega)} \qquad \forall \psi \in W, \tag{3.4}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. On the space W we consider the inner product given by

$$(\varphi, \psi)_W = \int_\Omega \nabla\varphi \cdot \nabla\psi \, dx$$

and let $\|\cdot\|_W$ be the associated norm. It follows from (3.4) that $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_W$ are equivalent norms on W and therefore $(W, \|\cdot\|_W)$ is a real Hilbert space. Moreover, the space $\mathcal{W}_1$ is a real Hilbert space with the inner product

$$(\boldsymbol{D}, \boldsymbol{E})_{\mathcal{W}_1} = \int_\Omega \boldsymbol{D} \cdot \boldsymbol{E} \, dx + \int_\Omega \mathrm{div}\,\boldsymbol{D} \cdot \mathrm{div}\,\boldsymbol{E} \, dx,$$

where $\mathrm{div}\,\boldsymbol{D} = (D_{i,i})$, and with the associated norm $\|\cdot\|_{\mathcal{W}_1}$.

In the study of the mechanical problem (2.1)–(2.10) we assume that

$$\left\{\begin{array}{ll} \text{(a)} & \mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b)} & \text{There exists } M_{\mathcal{F}} > 0 \text{ such that} \\ & \|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\xi}_2)\| \le M_{\mathcal{F}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\| \\ & \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c)} & \text{There exists } m_{\mathcal{F}} > 0 \text{ such that} \\ & (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\xi}_1)) - (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge m_{\mathcal{F}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2 \\ & \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(d)} & \text{The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \xi) \text{ is Lebesgue measurable on } \Omega, \\ & \text{for any } \boldsymbol{\xi} \in \mathbb{S}^d. \\ \text{(e)} & \text{The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } \mathcal{H}. \end{array}\right. \tag{3.5}$$

$$\left\{\begin{array}{ll} \text{(a)} & \mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d. \\ \text{(b)} & \mathcal{E}(\boldsymbol{x}, \boldsymbol{\tau}) = (e_{ijk}(\boldsymbol{x}) \tau_{jk}) \quad \forall \boldsymbol{\tau} = (\boldsymbol{\tau}_{ij}) \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c)} & e_{ijk} = e_{ikj} \in L^\infty(\Omega). \end{array}\right. \tag{3.6}$$

$$\left\{\begin{array}{ll} \text{(a)} & \boldsymbol{\beta} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d. \\ \text{(b)} & \boldsymbol{\beta}(\boldsymbol{x}, \boldsymbol{E}) = (\beta_{ij}(\boldsymbol{x}) E_j) \quad \forall \boldsymbol{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c)} & \beta_{ij} = \beta_{ji} \in L^\infty(\Omega). \\ \text{(d)} & \text{There exists } m_\beta > 0 \text{ such that } \beta_{ij}(\boldsymbol{x}) E_i E_j \ge m_\beta \|\boldsymbol{E}\|^2 \\ & \forall \boldsymbol{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \end{array}\right. \tag{3.7}$$

$$(3.8) \quad \begin{cases} \text{(a) } p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+ \text{ for } r = \nu, \tau. \\ \text{(b) There exists } L_r > 0 \text{ such that} \\ \quad |p_r(\boldsymbol{x}, u_1) - p_r(\boldsymbol{x}, u_2)| \le L_r |u_1 - u_2| \\ \quad \forall u_1,\, u_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto p_r(\boldsymbol{x}, u) \text{ is measurable on } \Gamma_3, \\ \quad \text{for any } u \in \mathbb{R}. \\ \text{(d) } p_r(\boldsymbol{x}, u) = 0 \quad \forall u \le 0,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases}$$

$$(3.9) \qquad \boldsymbol{f}_0 \in L^2(\Omega)^d, \qquad \boldsymbol{f}_2 \in L^2(\Gamma_3)^d.$$

$$(3.10) \qquad q_0 \in L^2(\Omega), \qquad q_2 \in L^2(\Gamma_b), \qquad q_2 = 0 \quad \text{a.e. on } \Gamma_3.$$

$$(3.11) \qquad g \in L^2(\Gamma_3), \quad g \ge 0 \quad \text{a.e. on } \Gamma_3.$$

We make some comments on the assumptions (3.5)–(3.11).

First, we note that the condition (3.5) is satisfied in the case of the linear elastic constitutive law $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$ in which

$$(3.12) \qquad \mathcal{F}\boldsymbol{\xi} = (f_{ijkl}\xi_{kl}),$$

provided that $f_{ijkl} \in L^\infty(\Omega)$ and there exists $\alpha > 0$ such that

$$f_{ijkl}(\boldsymbol{x})\xi_k\xi_l \ge \alpha\|\boldsymbol{\xi}\|^2 \quad \forall \boldsymbol{\xi} = (\xi_i) \in \mathbb{S}^d,\ \text{a.e. } \boldsymbol{x} \in \Omega.$$

To provide examples of nonlinear constitutive laws which satisfy (3.5), for every tensor $\boldsymbol{\xi} \in \mathbb{S}^d$ we denote by $\operatorname{tr}\boldsymbol{\xi}$ the trace of $\boldsymbol{\xi}$ and be $\boldsymbol{\xi}^D$ the deviatoric part of $\boldsymbol{\xi}$ given by

$$\operatorname{tr}\boldsymbol{\xi} = \xi_{ii}, \qquad \boldsymbol{\xi}^D = \boldsymbol{\xi} - \frac{1}{d}(\operatorname{tr}\boldsymbol{\xi})\boldsymbol{I}_d,$$

where $\boldsymbol{I}_d \in \mathbb{S}^d$ represents the identity tensor. Let K denote a nonempty closed convex set in $\mathbb{S}^d$ and let P_K represent the projection mapping on K. We also consider a fourth order symmetric and positive definite tensor $\mathcal{A} : \mathbb{S}^d \to \mathbb{S}^d$ and take

$$(3.13) \qquad \mathcal{F}(\boldsymbol{\xi}) = \mathcal{A}\boldsymbol{\xi} + \frac{1}{\lambda}(\boldsymbol{\xi} - P_K\boldsymbol{\xi}) \quad \forall \boldsymbol{\xi} \in \mathbb{S}^d,$$

where λ is a strictly positive constant. Using the properties of the projection mapping it is straightforward to see that the elasticity operator $\mathcal{F}$ defined by (3.13) satisfies condition (3.5). Constitutive laws of the form $\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})$ with $\mathcal{F}$ given by (3.13) have been considered in many papers (see e.g. [8], [15, p. 97] and [18, p. 68]). In most of them the convex set K is defined by $K = \{\boldsymbol{\xi} \in \mathbb{S}^d \mid G(\boldsymbol{\xi}) \le k\}$ where $G : \mathbb{S}^d \to \mathbb{R}$ is a convex continuous function such that $G(\boldsymbol{0}) = 0$ and $k > 0$. Another example of nonlinear elastic equations which satisfies condition (3.5) is provided by nonlinear Hencky materials (see [4] for details).

Next, as shown in (3.6) and (3.7), the piezoelectric operator $\mathcal{E}$ as well as the electric permittivity operator $\boldsymbol{\beta}$ are assumed to be linear, have measurable bounded components denoted e_{ijk} and β_{ij}, respectively, and, moreover, $\boldsymbol{\beta}$ is symmetric and positive definite. Recall also that the transposed tensor $\mathcal{E}^T$ is given by $\mathcal{E}^T = (e^T_{ijk})$ where $e^T_{ijk} = e_{kij}$, and the following equality holds:

$$\mathcal{E}\boldsymbol{\sigma} \cdot \boldsymbol{v} = \boldsymbol{\sigma} \cdot \mathcal{E}^T \boldsymbol{v} \quad \forall \boldsymbol{\sigma} \in \mathbb{S}^d,\ \boldsymbol{v} \in \mathbb{R}^d. \tag{3.14}$$

Conditions (3.8) are assumed to be valid for both p_ν and p_τ, i.e. for $r = \nu, \tau$. An example of a normal compliance function p_ν which satisfies conditions (3.8) is $p_\nu(u) = c_\nu u_+$ where $c_\nu \in L^\infty(\Gamma_3)$ is a positive function, the stiffness coefficient, and $u_+ = \max\{0, u\}$. The choices $p_\tau = \mu p_\nu$ and $p_\tau = \mu p_\nu (1 - \delta p_\nu)_+$ in (2.8), where $\mu \in L^\infty(\Gamma_3)$ and $\delta \in L^\infty(\Gamma_3)$ are positive functions, lead to the usual or modified Coulomb's law of dry friction, respectively (see [4, 16, 17] for details). Here μ represents the coefficient of friction and δ is a small positive material constant related to the wear and hardness of the contact surface. We note that if the function p_ν satisfies condition (3.8) then it follows that the function p_τ also satisfies this condition, in both the examples above. Therefore, we conclude that our results below are valid for the corresponding piezoelectric frictional contact models.

Assumptions (3.9) represent regularity assumptions on the densities of volume forces and surface tractions while (3.10) are regularity assumptions on the densities of volume and surface electric charges; the last part of this assumption, $q_2 = 0$ on Γ_3, is a compatibility condition which is needed because the foundation is supposed to be insulator. Finally, assumptions (3.11) describe the properties of the gap function g.

We now turn to the variational formulation of Problem P and, to this end, we introduce further notation. Let $h : V \times V \to \mathbb{R}$ be the functional

$$h(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} p_\nu(u_\nu - g) v_\nu \, da + \int_{\Gamma_3} p_\tau(u_\nu - g) \|\boldsymbol{v}_\tau\| \, da \quad \forall \boldsymbol{u}, \boldsymbol{v} \in V \tag{3.15}$$

and consider the elements $\boldsymbol{f} \in V$ and $q \in W$ given by

$$(\boldsymbol{f}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v} \, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v} \, da \quad \forall \boldsymbol{v} \in V, \tag{3.16}$$

$$(q, \psi)_W = \int_\Omega q_0 \psi \, dx - \int_{\Gamma_b} q_2 \psi \, da \quad \forall \psi \in W. \tag{3.17}$$

Note that the definitions of $\boldsymbol{f}$ and q follow by using Riesz's representation theorem, since the linear functionals

$$\boldsymbol{v} \mapsto \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v} \, dx + \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v} \, da, \qquad \psi \mapsto \int_\Omega q_0 \psi \, dx - \int_{\Gamma_b} q_2 \psi \, da$$

are continuous on the spaces V and W, respectively. Also, note that by assumptions (3.8)–(3.11) it follows that the integrals in (3.15)–(3.17) are well defined.

Using integration by parts, it is straightforward to see that if $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D})$ are sufficiently regular functions which satisfy (2.3)–(2.10) then

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} + h(\boldsymbol{u}, \boldsymbol{v}) - h(\boldsymbol{u}, \boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \forall \boldsymbol{v} \in V, \tag{3.18}$$

$$(\boldsymbol{D}, \nabla\psi)_{L^2(\Omega)^d} + (q, \psi)_W = 0 \quad \forall \psi \in W. \tag{3.19}$$

We plug (2.1) in (3.18), (2.2), in (3.19) and use the notation $\boldsymbol{E} = -\nabla\varphi$ to obtain the following variational formulation of Problem P, in terms of the displacement and electric potential fields.

PROBLEM P_V. *Find a displacement field $\boldsymbol{u} \in V$ and an electric potential field $\varphi \in W$ such that*

$$(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} + (\mathcal{E}^T\nabla\varphi, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_{L^2(\Omega)^d} + h(\boldsymbol{u}, \boldsymbol{v}) - h(\boldsymbol{u}, \boldsymbol{u}) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V \quad \forall \boldsymbol{v} \in V, \tag{3.20}$$

$$(\boldsymbol{\beta}\nabla\varphi, \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\psi)_{L^2(\Omega)^d} = (q, \psi)_W \quad \forall \psi \in W. \tag{3.21}$$

Our main existence and uniqueness result which we establish in Section 5 is the following.

THEOREM 3.1. *Assume* (3.5)–(3.11) *hold. Then*:

1) *Problem P_V has a solution $(\boldsymbol{u}, \varphi) \in V \times W$.*
2) *There exists L_0 which depends only on Ω, Γ_1, Γ_3, $\mathcal{F}$, $\boldsymbol{\beta}$ such that if $L_\tau + L_\nu < L_0$ then Problem P_V has a unique solution $(\boldsymbol{u}, \varphi) \in V \times W$ which depends Lipschitz continuously on $(\boldsymbol{f}, q) \in V \times W$.*

A quadruple $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D})$ of functions which satisfy (2.1), (2.2), (3.20) and (3.21) is called a *weak solution* of the piezoelectric contact problem P. We conclude by Theorem 3.1 that, under the assumptions (3.5)–(3.11), the piezoelectric contact problem (2.2)–(2.10) has a weak solution $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D})$ such that $\boldsymbol{u} \in V$ and $\varphi \in W$. Moreover, it is easy to see that $\boldsymbol{\sigma} \in \mathcal{H}_1$ and $\boldsymbol{D} \in \mathcal{W}_1$. The solution is unique and depends Lipschitz continuously on the data $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, q_0 and q_2, when $L_\nu + L_\tau$ is sufficiently small.

4. An abstract existence and uniqueness result. To prove Theorem 3.1 we shall use an abstract existence and uniqueness result on quasi-variational inequalities that we recall in what follows, for the convenience of the reader.

Everywhere in this section X will be a real Hilbert space endowed with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$. We denote by "$\rightharpoonup$" the weak convergence on X. Let $A : X \to X$ be a monotone operator, $j : X \times X \to \mathbb{R}$ and $f \in X$. With these data we consider the following quasivariational inequality: find $x \in X$ such that

$$(Ax, y - x)_X + j(x, y) - j(x, x) \geq (f, y - x)_X \quad \forall y \in X. \tag{4.1}$$

In order to solve (4.1) we assume that A is strongly monotone and Lipschitz continuous, i.e.

$$\begin{cases} \text{(a) There exists } m > 0 \text{ such that} \\ \quad (Ax_1 - Ax_2, x_1 - x_2)_X \geq m\|x_1 - x_2\|_X^2 \quad \forall x_1, x_2 \in X. \\ \text{(b) There exists } M > 0 \text{ such that} \\ \quad \|Ax_1 - Ax_2\|_X \leq M\|x_1 - x_2\|_X \quad \forall x_1, x_2 \in X. \end{cases} \tag{4.2}$$

The functional $j : X \times X \to \mathbb{R}$ satisfies

$$j(\eta, \cdot) : X \to \mathbb{R} \quad \text{is a convex functional on } X, \text{ for all } \eta \in X. \tag{4.3}$$

Keeping in mind (4.3) it is well known that there exists the directional derivative of j with respect to the second argument given by

$$j_2'(\eta, x; y) = \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \left[j(\eta, x + \lambda y) - j(\eta, x) \right] \quad \forall \eta, x, y \in X. \tag{4.4}$$

We now formulate some conditions on j and we recall that the m below represents the positive constant defined in (4.2).

$$\begin{cases} \text{For every sequence } \{x_n\} \subset X \text{ with } \|x_n\|_X \to \infty \\ \text{and every sequence } \{t_n\} \subset [0, 1] \text{ one has} \\ \liminf_{n \to \infty} \left[\dfrac{1}{\|x_n\|_X^2} \, j_2'(t_n x_n, x_n; -x_n) \right] < m. \end{cases} \tag{4.5}$$

$$\begin{cases} \text{For every sequence } \{x_n\} \subset X \text{ with } \|x_n\|_X \to \infty \\ \text{and every bounded sequence } \{\eta_n\} \subset X \text{ one has} \\ \liminf_{n \to \infty} \left[\dfrac{1}{\|x_n\|_X^2} \, j_2'(\eta_n, x_n; -x_n) \right] < m. \end{cases} \tag{4.6}$$

$$\begin{cases} \text{For any sequences } \{x_n\} \subset X \text{ and } \{\eta_n\} \subset X \text{ such that} \\ x_n \rightharpoonup x \in X, \ \eta_n \rightharpoonup \eta \in X \text{ and for every } y \in X \text{ one has} \\ \limsup_{n \to \infty} [j(\eta_n, y) - j(\eta_n, x_n)] \leq j(\eta, y) - j(\eta, x). \end{cases} \tag{4.7}$$

$$\begin{cases} \text{There exists } \alpha < m \text{ such that} \\ j(x, y) - j(x, x) + j(y, x) - j(y, y) \leq \alpha \|x - y\|_X^2 \quad \forall x, y \in X. \end{cases} \tag{4.8}$$

For the quasivariational inequality (4.1) we have the following result.

THEOREM 4.1. *Let* (4.2)–(4.3) *hold. Then*:

1) *Under the assumptions* (4.5)–(4.7) *there exists at least one element* $x \in X$ *which solves* (4.1).
2) *Under the assumptions* (4.5)–(4.8), *problem* (4.1) *has unique solution* $x = x_f$ *which depends Lipschitz continuously on* f *with the Lipschitz constant* $(m - \alpha)^{-1}$.

Theorem 4.1 has been obtained in [14] and therefore we do not provide the details of the proof here. We just specify that the proof was obtained in several steps and it is based on standard arguments of elliptic variational inequalities and topological degree theory.

5. Proof of Theorem 3.1. The proof of Theorem 3.1 will be carried out in several steps. To present it we consider the product space $X = V \times W$ together with the inner product

$$(x, y)_X = (\boldsymbol{u}, \boldsymbol{v})_V + (\varphi, \psi)_W \quad \forall x = (\boldsymbol{u}, \varphi),\ y = (\boldsymbol{v}, \psi) \in X \tag{5.1}$$

and the associated norm $\|\cdot\|_X$. Everywhere below we assume that (3.5)–(3.11) hold.

We use again Riesz's representation theorem to define the operator $A : X \to X$ by the formula

$$\begin{aligned}(Ax, y)_X = (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\boldsymbol{\beta}\nabla\varphi, \nabla\psi)_{L^2(\Omega)^d} \\ + (\mathcal{E}^T\nabla\varphi, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\psi)_{L^2(\Omega)^d} \\ \forall x = (\boldsymbol{u}, \varphi),\ y = (\boldsymbol{v}, \psi) \in X,\end{aligned} \tag{5.2}$$

and we extend the functional (3.15) to a functional j defined on $X \times X$, that is,

$$j(x, y) = h(\boldsymbol{u}, \boldsymbol{v}) \quad \forall x = (\boldsymbol{u}, \varphi),\ y = (\boldsymbol{v}, \psi) \in X. \tag{5.3}$$

Finally, we consider the element $f \in X$ given by

$$f = (\boldsymbol{f}, q) \in X. \tag{5.4}$$

We start with the following equivalence result.

LEMMA 5.1. *The couple* $x = (\boldsymbol{u}, \varphi)$ *is a solution to Problem* P_V *if and only if*

$$(Ax, y - x)_X + j(x, y) - j(x, x) \geq (f, y - x)_X \quad \forall y \in X. \tag{5.5}$$

Proof. Let $x = (\boldsymbol{u}, \varphi) \in X$ be a solution to Problem P_V and let $y = (\boldsymbol{v}, \psi) \in X$. We use the test function $\psi - \varphi$ in (3.21), add the corresponding equality to (3.20) and use (5.1)–(5.4) to obtain (5.5). Conversely, let $x = (\boldsymbol{u}, \varphi) \in X$ be a solution to the quasivariational inequality (5.5). We take $y = (\boldsymbol{v}, \varphi)$ in (5.5) where $\boldsymbol{v}$ is an arbitrary element of V and obtain (3.20); then we take successively $y = (\boldsymbol{v}, \varphi + \psi)$ and $y = (\boldsymbol{v}, \varphi - \psi)$ in (5.5), where ψ

is an arbitrary element of W; as a result we obtain (3.21), which concludes the proof. ∎

Notice that the quasivariational inequality (5.5) derived in Lemma 5.1 is of the form (4.1). Therefore, in order to apply the abstract result provided by Theorem 4.1, we start with the study of the the properties of the operator A given by (5.2).

Lemma 5.2. *The operator $A : X \to X$ is strongly monotone and Lipschitz continuous.*

Proof. Consider two elements $x_1 = (\boldsymbol{u}_1, \varphi_1)$, $x_2 = (\boldsymbol{u}_2, \varphi_2) \in X$. Using (5.2) we have

$$(5.6)\quad \begin{aligned}(Ax_1 - Ax_2, x_1 - x_2)_X &= (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_2), \boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2))_{\mathcal{H}} \\ &\quad + (\boldsymbol{\beta}\nabla\varphi_1 - \boldsymbol{\beta}\nabla\varphi_2, \nabla\varphi_1 - \nabla\varphi_2)_{L^2(\Omega)^d} \\ &\quad + (\mathcal{E}^T\nabla\varphi_1 - \mathcal{E}^T\nabla\varphi_2, \boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2))_{\mathcal{H}} \\ &\quad - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_1), \nabla\varphi_1 - \nabla\varphi_2)_{L^2(\Omega)^d}\end{aligned}$$

and, since it follows by (3.14) that $(\mathcal{E}^T\nabla\varphi, \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} = (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\varphi)_{L^2(\Omega)^d}$ for all $x = (\boldsymbol{u}, \varphi) \in X$, we find

$$\begin{aligned}&(Ax_1 - Ax_2, x_1 - x_2)_X \\ &= (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_2), \boldsymbol{\varepsilon}(\boldsymbol{u}_1) - \boldsymbol{\varepsilon}(\boldsymbol{u}_2))_{\mathcal{H}} + (\boldsymbol{\beta}\nabla\varphi_1 - \boldsymbol{\beta}\nabla\varphi_2, \nabla\varphi_1 - \nabla\varphi_2)_{L^2(\Omega)^d}.\end{aligned}$$

We now use (3.5) and (3.7) to see that there exists $c_1 > 0$ which depends only on $\mathcal{F}$, $\boldsymbol{\beta}$ and Ω such that

$$(5.7)\quad (Ax_1 - Ax_2, x_1 - x_2)_X \geq c_1(\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V^2 + \|\varphi_1 - \varphi_2\|_W^2)$$

and, keeping in mind (5.1), we obtain

$$(5.8)\quad (Ax_1 - Ax_2, x_1 - x_2)_X \geq c_1\|x_1 - x_2\|_X^2.$$

In the same way, using (3.5)–(3.7), after some algebra it follows that there exists $c_2 > 0$ which depends only on $\mathcal{F}$, $\boldsymbol{\beta}$ and $\mathcal{E}$ such that

$$\begin{aligned}(Ax_1 - Ax_2, y)_X \leq c_2(&\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V\|\boldsymbol{v}\|_V + \|\varphi_1 - \varphi_2\|_W\|\psi\|_W \\ &+ \|\varphi_1 - \varphi_2\|_W\|\boldsymbol{v}\|_V + \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V\|\psi\|_W)\end{aligned}$$

for all $y = (\boldsymbol{v}, \psi) \in X$. We use (5.1) and the previous inequality to obtain

$$(Ax_1 - Ax_2, y)_X \leq 4c_2\|x_1 - x_2\|_X\|y\|_X \qquad \forall y \in X$$

and, taking $y = Ax_1 - Ax_2 \in X$, we find

$$(5.9)\quad \|Ax_1 - Ax_2\|_X \leq 4c_2\|x_1 - x_2\|_X.$$

Lemma 5.2 is now a consequence of inequalities (5.8) and (5.9). ∎

Next we investigate the properties of the functional j given by (5.3), (3.15). We first remark that j satisfies condition (4.3). Moreover, we have the following results.

LEMMA 5.3. *The functional j satisfies conditions* (4.5)–(4.7).

Proof. Let $\eta = (\boldsymbol{w}, \xi)$, $x = (\boldsymbol{u}, \varphi) \in X$ and let $\lambda \in]0, 1]$. Using (5.3) and (3.15) it results that

$$j(\eta, x - \lambda x) - j(\eta, x) = -\lambda \int_{\Gamma_3} p_\nu(w_\nu - g)u_\nu \, da - \lambda \int_{\Gamma_3} p_\tau(w_\nu - g)\|\boldsymbol{u}_\tau\| \, da$$

and, since $p_\tau \geq 0$ a.e. on Γ_3, we deduce that

$$j(\eta, x - \lambda x) - j(\eta, x) \leq -\lambda \int_{\Gamma_3} p_\nu(w_\nu - g)u_\nu \, da.$$

Therefore, by (4.4) we obtain

$$j'_2(\eta, x; -x) \leq -\int_{\Gamma_3} p_\nu(w_\nu - g)u_\nu \, da \quad \forall\, \eta = (\boldsymbol{w}, \xi),\ x = (\boldsymbol{u}, \varphi) \in X. \tag{5.10}$$

Now consider sequences $\{x_n\} = \{(\boldsymbol{u}_n, \varphi_n)\} \subset X$ and $\{t_n\} \subset [0, 1]$ such that $\|x_n\|_X \to \infty$. From (3.8) and (3.11) it follows that $p_\nu(t_n u_{n\nu} - g)(u_{n\nu} - g) \geq 0$ a.e. on Γ_3 and therefore (5.10) yields

$$j'_2(t_n x_n, x_n; -x_n) \leq -\int_{\Gamma_3} g p_\nu(t_n w_{n\nu} - g) \, da \quad \forall n \in \mathbb{N}.$$

Thus, since $g \geq 0$ and $p_\nu \geq 0$ a.e. on Γ_3, we deduce that

$$j'_2(t_n x_n, x_n; -x_n) \leq 0 \quad \forall n \in \mathbb{N}$$

and we conclude that j satisfies the assumption (4.5).

Now consider sequences $\{x_n\} = \{(\boldsymbol{u}_n, \varphi_n)\} \subset X$ and $\{\eta_n\} = \{(\boldsymbol{w}_n, \xi_n)\} \subset X$ such that

$$\|\eta_n\|_X \leq c \quad \forall n \in \mathbb{N}, \tag{5.11}$$

$$\|x_n\|_X \to \infty, \tag{5.12}$$

where $c > 0$. Using (5.10) and (3.8) we obtain

$$\begin{aligned} j'_2(\eta_n, x_n; -x_n) &\leq \int_{\Gamma_3} p_\nu(w_{n\nu} - g)|u_{n\nu}| \, da \leq L_\nu \int_{\Gamma_3} |w_{n\nu} - g|\, |u_{n\nu}| \, da \\ &\leq L_\nu(\|w_{n\nu}\|_{L^2(\Gamma_3)} + \|g\|_{L^2(\Gamma_3)})\|u_n\|_{L^2(\Gamma_3)} \end{aligned}$$

for all $n \in \mathbb{N}$. Using now (3.3) and (5.11) in the previous inequality yields

$$j'_2(\eta_n, x_n; -x_n) \leq L_\nu c_0(c_0 c + \|g\|_{L^2(\Gamma_3)})\|x_n\|_X \quad \forall n \in \mathbb{N}. \tag{5.13}$$

Thus, from (5.12) and (5.13) we deduce that j satisfies the assumption (4.6).

Finally, let $\{x_n\} = \{(\boldsymbol{u}_n, \varphi_n)\} \subset X$ and $\{\eta_n\} = \{(\boldsymbol{w}_n, \xi_n)\} \subset X$ be such that $x_n \rightharpoonup x = (\boldsymbol{u}, \varphi) \in X$ and $\eta_n \rightharpoonup \eta = (\boldsymbol{w}, \xi) \in X$. Using the compactness property of the trace map and (3.8) it follows that

$$\boldsymbol{u}_{n\nu} \to \boldsymbol{u}_\nu, \quad \|\boldsymbol{u}_{n\tau}\| \to \|\boldsymbol{u}_\tau\| \quad \text{in } L^2(\Gamma_3),$$
$$p_r(w_{n\nu} - g) \to p_r(w_\nu - g) \quad \text{in } L^2(\Gamma_3)\ (r = \nu, \tau).$$

Therefore, we deduce that

$$j(\eta_n, y) \to j(\eta, y) \quad \forall y \in X \quad \text{and} \quad j(\eta_n, x_n) \to j(\eta, x), \qquad \text{as } n \to \infty,$$

which shows that the functional j satisfies the condition (4.7). ■

LEMMA 5.4. *The functional j satisfies the inequality*

$$j(x,y) - j(x,x) + j(y,x) - j(y,y) \leq c_0^2 (L_\nu + L_\tau) \|x - y\|_X^2 \quad \forall x, y \in X. \tag{5.14}$$

Proof. Let $x = (\boldsymbol{u}, \varphi)$, $y = (\boldsymbol{v}, \psi) \in X$. From (5.3), (3.15) and (3.8) it follows that

$$\begin{aligned} j(x,y) - j(x,x) + j(y,x) - j(y,y) &= \int_{\Gamma_3} (p_\nu(u_\nu - g) - p_\nu(v_\nu - g))(v_\nu - u_\nu)\, da \\ &\quad + \int_{\Gamma_3} (p_\tau(u_\nu - g) - p_\tau(v_\nu - g))(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau\|)\, da \\ &\leq \int_{\Gamma_3} |p_\nu(u_\nu - g) - p_\nu(v_\nu - g)|\, |v_\nu - u_\nu|\, da \\ &\quad + \int_{\Gamma_3} |p_\tau(u_\nu - g) - p_\tau(v_\nu - g)|\, |\, \|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau\|\, |\, da \\ &\leq (L_\nu + L_\tau) \|\boldsymbol{u} - \boldsymbol{v}\|_{L^2(\Gamma_3)^d}^2. \end{aligned}$$

Using now (3.3) and (5.1) in the previous inequality we deduce (5.14). ■

We now have all the ingredients to prove the theorem.

Proof of Theorem 3.1. 1) Lemmas 5.2 and 5.3 allow us to use the abstract results provided by the first part of Theorem 4.1. We find that the quasivariational inequality (5.5) has a solution $x = (\boldsymbol{u}, \varphi) \in X$ and, using Lemma 5.1, we deduce that $(\boldsymbol{u}, \varphi)$ is a solution to Problem P_V which satisfies $(\boldsymbol{u}, \varphi) \in V \times W$.

2) Let $L_0 = c_1/c_0^2$ where c_1 and c_0 are defined by (5.8) and (3.3), respectively. Clearly L_0 depends only on Ω, Γ_1, Γ_3, $\mathcal{F}$, $\boldsymbol{\beta}$. Now assume that $L_\nu + L_\tau < L_0$. Then there exists $\alpha \in \mathbb{R}$ such that $c_0^2(L_\nu + L_\tau) < \alpha < c_1$. Using (5.14) and (5.8) we see that the functional j satisfies condition (4.8). Therefore, by the second part of Theorem 4.1, Lemma 5.1 and (5.4), problem P_V has a unique solution $(\boldsymbol{u}, \varphi) \in V \times W$ which depends Lipschitz continuously on $(\boldsymbol{f}, q) \in V \times W$. ■

6. A continuous dependence result. In this section we study the dependence of the solution to Problem P_V on perturbations of the normal compliance functions p_ν and p_τ. To this end we suppose in what follows that the assumptions (3.5)–(3.11) hold. For every $\alpha > 0$, let p_r^α be a perturbation of p_r which satisfies (3.8) with the Lipschitz constant L_r^α, $r = \nu, \tau$. Also, we assume that

$$\text{(6.1)}\qquad \text{there exists } L_* < L_0 \text{ such that } L_\nu + L_\tau \le L_*,\ L_\nu^\alpha + L_\tau^\alpha \le L_*\ \forall \alpha > 0,$$

where L_0 is defined in the second part of Theorem 3.1, i.e. $L_0 = c_1/c_0^2$. We introduce the functional h^α obtained from h by replacing p_ν and p_τ with p_ν^α and p_τ^α, respectively, and we consider the following variational problem.

PROBLEM P_V^α. *Find a displacement field* $\boldsymbol{u}^\alpha \in V$ *and an electric potential field* $\varphi^\alpha \in W$ *such that*

$$\text{(6.2)}\qquad (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha))_{\mathcal{H}} + (\mathcal{E}^T\nabla\varphi^\alpha, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha))_{L^2(\Omega)^d} + h^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{v}) - h^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}^\alpha) \ge (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}^\alpha)_V \quad \forall \boldsymbol{v} \in V,$$

$$\text{(6.3)}\qquad (\boldsymbol{\beta}\nabla\varphi^\alpha, \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha), \nabla\psi)_{L^2(\Omega)^d} = (q, \psi)_W \quad \forall \psi \in W.$$

Clearly Problem P_V^α represents the variational formulation of the piezoelectric contact problem P^α obtained from Problem P when the normal compliance functions p_ν and p_τ are replaced by the perturbed normal compliance functions p_ν^α and p_τ^α, respectively. Using (6.1) we deduce from Theorem 3.1 that for each $\alpha > 0$, Problem P_V^α has a unique solution $(\boldsymbol{u}^\alpha, \varphi^\alpha) \in V \times W$; moreover, Problem P_V has a unique solution $(\boldsymbol{u}, \varphi) \in V \times W$.

Suppose now that the normal compliance functions satisfy the following assumptions for $r = \nu, \tau$:

$$\text{(6.4)}\qquad \begin{cases} \text{There exist } a_r : \mathbb{R}_+ \to \mathbb{R} \text{ and } b_r : \mathbb{R}_+ \to \mathbb{R} \text{ such that:} \\ \text{(a) } |p_r^\alpha(\boldsymbol{x}, u) - p_r(\boldsymbol{x}, u)| \le a_r(\alpha)\,|u| + b_r(\alpha) \quad \forall u \in \mathbb{R}, \\ \qquad \text{a.e. } \boldsymbol{x} \in \Gamma_3, \text{ for all } \alpha > 0. \\ \text{(b) } \lim\limits_{\alpha\to 0} a_r(\alpha) = 0, \qquad \lim\limits_{\alpha\to 0} b_r(\alpha) = 0. \end{cases}$$

Under these assumptions, we have the following convergence result.

THEOREM 6.1. *The solution* $(\boldsymbol{u}^\alpha, \varphi^\alpha)$ *of Problem* P_V^α *converges to the solution* $(\boldsymbol{u}, \varphi)$ *of Problem* P_V, *i.e.*

$$\text{(6.5)}\qquad \boldsymbol{u}^\alpha \to \boldsymbol{u} \quad \text{in } V \text{ as } \alpha \to 0,$$

$$\text{(6.6)}\qquad \varphi^\alpha \to \varphi \quad \text{in } W \text{ as } \alpha \to 0.$$

Proof. Let $\alpha > 0$. Everywhere below c will represent a positive constant which may depend on the data and on the solution $\boldsymbol{u}$ but is independent of α and whose value may change from place to place. From (3.20), (3.21), (6.2) and (6.3), after some computation, we find that

$$\text{(6.7)}\qquad (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} + (\mathcal{E}^T\nabla\varphi^\alpha - \mathcal{E}^T\nabla\varphi, \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha) - \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{H}} \le h(\boldsymbol{u}, \boldsymbol{u}^\alpha) - h(\boldsymbol{u}, \boldsymbol{u}) + h^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}) - h^\alpha(\boldsymbol{u}^\alpha, \boldsymbol{u}^\alpha),$$

$$\text{(6.8)}\qquad (\boldsymbol{\beta}\nabla\varphi^\alpha - \boldsymbol{\beta}\nabla\varphi, \nabla\varphi^\alpha - \nabla\varphi)_{L^2(\Omega)^d} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}), \nabla\varphi^\alpha - \nabla\varphi)_{L^2(\Omega)^d} = 0.$$

We add (6.7) and (6.8), then use (5.6) and (5.7) to obtain

$$\text{(6.9)}\qquad c_1(\|\boldsymbol{u}^\alpha-\boldsymbol{u}\|_V^2+\|\varphi^\alpha-\varphi\|_W^2)\\ \le h(\boldsymbol{u},\boldsymbol{u}^\alpha)-h(\boldsymbol{u},\boldsymbol{u})+h^\alpha(\boldsymbol{u}^\alpha,\boldsymbol{u})-h^\alpha(\boldsymbol{u}^\alpha,\boldsymbol{u}^\alpha).$$

Note that

$$h(\boldsymbol{u},\boldsymbol{u}^\alpha)-h(\boldsymbol{u},\boldsymbol{u})+h^\alpha(\boldsymbol{u}^\alpha,\boldsymbol{u})-h^\alpha(\boldsymbol{u}^\alpha,\boldsymbol{u}^\alpha)\\ =\int_{\Gamma_3}(p_\nu(u_\nu-g)-p_\nu^\alpha(u_\nu^\alpha-g))(u_\nu^\alpha-u_\nu)\,da\\ +\int_{\Gamma_3}(p_\tau(u_\nu-g)-p_\tau^\alpha(u_\nu^\alpha-g))(\|\boldsymbol{u}_\tau^\alpha\|-\|\boldsymbol{u}_\tau\|)\,da,$$

which implies that

$$\text{(6.10)}\qquad h(\boldsymbol{u},\boldsymbol{u}^\alpha)-h(\boldsymbol{u},\boldsymbol{u})+h^\alpha(\boldsymbol{u}^\alpha,\boldsymbol{u})-h^\alpha(\boldsymbol{u}^\alpha,\boldsymbol{u}^\alpha)\\ \le\int_{\Gamma_3}\big[\,|p_\nu(u_\nu-g)-p_\nu^\alpha(u_\nu^\alpha-g)|+|p_\tau(u_\nu-g)-p_\tau^\alpha(u_\nu^\alpha-g)|\big]\,\|\boldsymbol{u}^\alpha-\boldsymbol{u}\|\,da.$$

For $r=\nu$ or τ we use the triangle inequality to obtain

$$|p_r(u_\nu-g)-p_r^\alpha(u_\nu^\alpha-g)|\le|p_r(u_\nu-g)-p_r^\alpha(u_\nu-g)|\\ +|p_r^\alpha(u_\nu-g)-p_r^\alpha(u_\nu^\alpha-g)|$$

and, taking into account (3.8) and (6.4)(a), we find

$$|p_r(u_\nu-g)-p_r^\alpha(u_\nu^\alpha-g)|\le a_r(\alpha)|u_\nu-g|+b_r(\alpha)+L_r^\alpha\,|u_\nu^\alpha-u_\nu|$$

a.e. on Γ_3. We plug the last inequality in (6.10), use (3.3) and, after some computations, we deduce that

$$\text{(6.11)}\qquad h(\boldsymbol{u},\boldsymbol{u}^\alpha)-h(\boldsymbol{u},\boldsymbol{u})+h^\alpha(\boldsymbol{u}^\alpha,\boldsymbol{u})-h^\alpha(\boldsymbol{u}^\alpha,\boldsymbol{u}^\alpha)\\ \le c[a_\nu(\alpha)+a_\tau(\alpha)+b_\nu(\alpha)+b_\tau(\alpha)]\|\boldsymbol{u}^\alpha-\boldsymbol{u}\|_V\\ +c_0^2(L_\nu^\alpha+L_\tau^\alpha)\|\boldsymbol{u}^\alpha-\boldsymbol{u}\|_V^2.$$

Now, it follows from (6.1) that $c_0^2(L_\nu^\alpha+L_\tau^\alpha)\le c_0^2L_*$ and, therefore, combining (6.9) and (6.11) we find that

$$\text{(6.12)}\qquad (c_1-c_0^2L_*)\|\boldsymbol{u}^\alpha-\boldsymbol{u}\|_V^2+c_1\|\varphi^\alpha-\varphi\|_W^2\\ \le c\,\big[a_\nu(\alpha)+a_\tau(\alpha)+b_\nu(\alpha)+b_\tau(\alpha)\big]\,\|\boldsymbol{u}^\alpha-\boldsymbol{u}\|_V.$$

On the other hand, the inequality $L_*<L_0$ and equality $L_0=c_1/c_0^2$ yield $c_0^2L_*<c_1$ and therefore it follows from (6.12) that

$$\text{(6.13)}\qquad \|\boldsymbol{u}^\alpha-\boldsymbol{u}\|_V^2+\|\varphi^\alpha-\varphi\|_W^2\\ \le c[a_\nu(\alpha)+a_\tau(\alpha)+b_\nu(\alpha)+b_\tau(\alpha)]\|\boldsymbol{u}^\alpha-\boldsymbol{u}\|_V,$$

which implies that

$$\text{(6.14)}\qquad \|\boldsymbol{u}^\alpha-\boldsymbol{u}\|_V\le c[a_\nu(\alpha)+a_\tau(\alpha)+b_\nu(\alpha)+b_\tau(\alpha)].$$

Theorem 6.1 is now a consequence of (6.14), (6.13) and (6.4)(b). ■

We now extend the convergence result of Theorem 6.1 to the weak solution of the piezoelectric contact problem. To this end we denote by $\boldsymbol{\sigma}^\alpha$ and $\boldsymbol{\sigma}$ the stress fields defined by

$$\boldsymbol{\sigma}^\alpha = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha) - \mathcal{E}^T\boldsymbol{E}(\varphi^\alpha), \qquad \boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{E}^T\boldsymbol{E}(\varphi), \tag{6.15}$$

and let the electric displacement fields $\boldsymbol{D}^\alpha$ and $\boldsymbol{D}$ be given by

$$\boldsymbol{D}^\alpha = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha) + \boldsymbol{\beta}\boldsymbol{E}(\varphi^\alpha), \qquad \boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\beta}\boldsymbol{E}(\varphi). \tag{6.16}$$

It can be shown that $\boldsymbol{\sigma}^\alpha, \boldsymbol{\sigma} \in \mathcal{H}_1$ and $\boldsymbol{D}^\alpha, \boldsymbol{D} \in \mathcal{W}_1$. Moreover,

$$\operatorname{Div} \boldsymbol{\sigma}^\alpha = \operatorname{Div} \boldsymbol{\sigma} = -\boldsymbol{f}_0 \quad \text{in } \Omega, \tag{6.17}$$

$$\operatorname{div} \boldsymbol{D}^\alpha = \operatorname{div} \boldsymbol{D} = q_0 \quad \text{in } \Omega. \tag{6.18}$$

Therefore, from (6.15)–(6.18) and the assumptions (3.5)–(3.7) on the operators $\mathcal{F}$, $\mathcal{E}$ and $\boldsymbol{\beta}$, we deduce that

$$\|\boldsymbol{\sigma}^\alpha - \boldsymbol{\sigma}\|_{\mathcal{H}_1} \le c(\|\boldsymbol{u}^\alpha - \boldsymbol{u}\|_V + \|\varphi^\alpha - \varphi\|_W),$$
$$\|\boldsymbol{D}^\alpha - \boldsymbol{D}\|_{\mathcal{W}_1} \le c(\|\boldsymbol{u}^\alpha - \boldsymbol{u}\|_V + \|\varphi^\alpha - \varphi\|_W).$$

It now follows from (6.5), (6.6) that

$$\boldsymbol{\sigma}^\alpha \to \boldsymbol{\sigma} \quad \text{in } \mathcal{H}_1 \text{ as } \alpha \to 0, \tag{6.19}$$

$$\boldsymbol{D}^\alpha \to \boldsymbol{D} \quad \text{in } \mathcal{W}_1 \text{ as } \alpha \to 0. \tag{6.20}$$

In addition to the mathematical interest in the convergence result (6.5), (6.6), (6.19) and (6.20), it is of importance in applications since it indicates that small inaccuracies in the contact conditions lead to small inaccuracies in the weak solution of the piezoelectric contact problem.

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