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NUMERICAL EXPERIMENTS FOR MATHEMATICAL MODELS OF RAILWAY TRACK OSCILLATIONS

Abstract. Two mathematical models of railway track oscillations are compared on the basis of numerical experiments.

1. Introduction. Good mathematical models describing vibrations of railway track and trains are important and helpful in engineering studies concerning comfort and safety of travel (see [1]–[5]). The main goal of this work is a mathematical description of the dynamical behaviour of the system train-track. We present a numerical experiment for two different mathematical models of railway track oscillations. We consider only one-dimensional models. The first is based on Mathews' problem (see [2]), which treated a Bernoulli–Euler beam on an elastic foundation as an infinite rail track under the action of a moving (with velocity v) and harmonically oscillating force. The force of frequency $\varpi/2\pi$ is vertical to the beam and at the point x and time $t > 0$ has the value $F_0 \cos \varpi t \cdot \delta(x - vt)$, where F_0 is the amplitude. In this model we do not take into account vibrations of the particular sleepers, but we introduce an elasticity term for the whole beam. The second model (see [4]) consists of an equation of a continuous Euler–Bernoulli beam and a differential equation describing vibrations of the sleepers. The track is considered as a system consisting of an infinite continuous elastic beam connected with a finite number of discrete masses modelling the sleepers. We imitate the motion of the train by the vertical force acting at one fixed point $x = l$ and having the form $F_0 e^{-\widetilde{\omega} t} \cos \varpi t \cdot \delta(x - l)$. The positive parameter $\widetilde{\omega}$ is a substitute for the velocity of the train. Using realistic data we compare results of numerical experiments for both models.

2. Presentation of two mathematical models. We briefly describe the relevant models. In both, we shall consider only *asymptotic solutions*, i.e. solutions for which the effect of the initial condition has disappeared.

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2.1. *Mathematical model of infinite railway track oscillations.* This model is based on Mathews' problem (see [2]), which treated a Bernoulli–Euler beam on an elastic foundation as an infinite rail track under the action of a moving and harmonically oscillating force. The equation of the track is

$$EI\,\frac{\partial^4 y}{\partial x^4} + \varrho A\,\frac{\partial^2 y}{\partial t^2} + sy = F_0 \cos \varpi t \cdot \delta(x - vt), \tag{1}$$

where

$y(x,t)$ — the deflection of the beam at the point x and time t,
E — Young's modulus of the material of the beam,
I — moment of inertia of the section of the beam, with respect to the horizontal axis,
ϱ — density of the beam,
A — area of the cross-section of the beam,
s — coefficient of elasticity,
F_0 — amplitude of the force,
δ — Dirac delta function,
v — velocity of the train.

Note that the frequency of the force is $\varpi/2\pi$. We also assume that $y(x,t)$ satisfies the boundary conditions

$$\lim_{|x|\to\infty} y^{(i)}(x,t) = 0 \qquad (i = 0,\dots,4). \tag{2}$$

THEOREM 1. *The asymptotic solution $y(x,t)$ of equation* (1) *is given by*

$$\begin{aligned} y(x,t) &= \frac{F_0}{2EI}\cos\varpi t\bigg(\frac{1}{b_1}\,\frac{(4a^2-b_1^2+b_2^2)e^{-b_1|r|}\cos a|r| + 4ab_1e^{-b_1|r|}\sin a|r|}{(4a^2-b_1^2+b_2^2)^2+16a^2b_1^2} \\ &\quad + \frac{1}{b_2}\,\frac{(4a^2-b_2^2+b_1^2)e^{-b_2|r|}\cos a|r| + 4ab_2e^{-b_2|r|}\sin a|r|}{(4a^2-b_2^2+b_1^2)^2+16a^2b_2^2}\bigg) \\ &\quad \pm \frac{F_0}{2EI}\sin\varpi t\bigg(\frac{1}{b_1}\,\frac{(4a^2-b_1^2+b_2^2)e^{-b_1|r|}\sin a|r| - 4ab_1e^{-b_1|r|}\cos a|r|}{(4a^2-b_1^2+b_2^2)^2+16a^2b_1^2} \\ &\quad - \frac{1}{b_2}\,\frac{(4a^2-b_2^2+b_1^2)e^{-b_2|r|}\sin a|r| - 4ab_2e^{-b_2|r|}\cos a|r|}{(4a^2-b_2^2+b_1^2)^2+16a^2b_2^2}\bigg), \end{aligned} \tag{3}$$

where $r = x - vt$ and the negative sign holds when $r > 0$ and the positive sign for $r \le 0$. Moreover, the real and positive values of a, b_1, b_2 are obtained by *solving the system of equations*

$$
\left\{
\begin{aligned}
&2a^2 - b_1^2 - b_2^2 = \frac{\varrho A v^2}{EI},\\
&2a(b_1^2 - b_2^2) = \frac{2v\varpi\varrho A}{EI},\\
&(a^2 + b_1^2)(a^2 + b_2^2) = \frac{s - \varrho A\varpi^2}{EI}.
\end{aligned}
\right. \tag{4}
$$

REMARK 2. Since the solution $Y(x,t)$ of the homogeneous partial differential equation

$$
\alpha\frac{\partial^4 Y}{\partial x^4} + \beta\frac{\partial^2 Y}{\partial t^2} + \gamma Y = 0, \qquad \text{where } \alpha, \beta, \gamma > 0,
$$

satisfying null initial conditions is zero, the asymptotic solution $y(x,t)$ given by (3) is uniquely determined.

Sketch of proof of Theorem 1. We are only looking for the asymptotic solution of equation (1). It is easy to notice that the deflection of the beam at point $r = x - vt$ has the same value at times t and $t + 2\pi/\varpi$. In other words, $y(r,t)$ is periodic with period $2\pi/\varpi$. Suppose an asymptotic solution has the form

$$
y(x,t) = y_1(r)\cos\varpi t + y_2(r)\sin\varpi t, \tag{5}
$$

where

$$
r = x - vt.
$$

Substituting into (1) we obtain

$$
\begin{aligned}
\left(EI\,\frac{\partial^4 y_1}{\partial r^4} + \varrho A v^2\,\frac{\partial^2 y_1}{\partial r^2} - 2v\varrho A\varpi\,\frac{\partial y_2}{\partial r} + (s - \varpi^2\varrho A)y_1\right)\cos\varpi t&\\
+ \left(EI\,\frac{\partial^4 y_2}{\partial r^4} + \varrho A v^2\,\frac{\partial^2 y_2}{\partial r^2} + 2v\varrho A\varpi\,\frac{\partial y_1}{\partial r} + (s - \varpi^2\varrho A)y_2\right)\sin\varpi t&\\
= F_0\delta(r)\cos\varpi t.&
\end{aligned}
$$

By the last equality we get

$$
\left\{
\begin{aligned}
&EI\,\frac{\partial^4 y_1}{\partial r^4} + \varrho A v^2\,\frac{\partial^2 y_1}{\partial r^2} - 2v\varrho A\varpi\,\frac{\partial y_2}{\partial r} + (s - \varpi^2\varrho A)y_1 = F_0\delta(r),\\
&EI\,\frac{\partial^4 y_2}{\partial r^4} + \varrho A v^2\,\frac{\partial^2 y_2}{\partial r^2} + 2v\varrho A\varpi\,\frac{\partial y_1}{\partial r} + (s - \varpi^2\varrho A)y_2 = 0.
\end{aligned}
\right. \tag{6}
$$

To solve (6) we use the Fourier transform to obtain

$$
\left\{
\begin{aligned}
&(EIu^4 - \varrho A v^2 u^2 + s - \varrho A\varpi^2)p_1 - 2v\varrho A\varpi i u p_2 = F_0,\\
&(EIu^4 - \varrho A v^2 u^2 + s - \varrho A\varpi^2)p_2 + 2v\varrho A\varpi i u p_1 = 0,
\end{aligned}
\right. \tag{7}
$$

where $p_1(u)$ and the $p_2(u)$ are the Fourier transforms of $y_1(r)$ and $y_2(r)$. From (7), we can easily determine $p_1(u)$ and $p_2(u)$. To calculate the inverse transforms of $p_1(u)$ and $p_2(u)$ we must decompose these functions into partial fractions. To do this, we calculate the complex roots of the equation $EIu^4 - \varrho Av^2u^2 + s - \varrho A\varpi^2 + 2v\varrho A\varpi u = 0$. Since the roots must form conjugate pairs, the four roots have the form $a \pm ib_1$ and $a \pm ib_2$, where the real and positive values a, b_1, b_2 satisfy (4). Calculating the inverse transforms of $p_1(u)$ and $p_2(u)$ we get (3).

2.2. *Mathematical model of infinite railway track oscillations with a finite number of sleepers under a fixed load.* The track is considered as a system consisting of a continuous elastic beam connected with discrete masses modelling the sleepers. We consider a finite railway track. We assume that the number of sleepers is also finite, and is equal to n. The vertical force is acting only at one fixed point $x = l$, and it is equal to $F_0e^{-\widetilde{\omega}t}\cos\varpi t$, where $\widetilde{\omega} > 0$. This means that the force has decreasing amplitude. By a suitable choice of $\widetilde{\omega}$, it should model the motion of the wheel which appears at one fixed point. It is important that we additionally assume that the railway track is oscillating together with the sleepers (see [4]).

For simplicity, we shall describe this model in the complex form. In this case the acting force is $F_0e^{\omega t}$, where $\omega = -\widetilde{\omega} + i\varpi$ and $\widetilde{\omega} > 0$. The equation of a continuous Bernoulli–Euler beam under a fixed load $F_0e^{\omega t}$, $\omega \in \mathbb{C}$, located at $x = l$, has the form

$$(8) \qquad IE\,\frac{\partial^4 z}{\partial x^4} + \varrho A\,\frac{\partial^2 z}{\partial t^2} + \sum_{j=1}^{n}\left(m_{pj}\,\frac{d^2 z_j}{dt^2} + k_{bj}\,\frac{dz_j}{dt} + s_{bj}z_j\right)\delta(x - l_j) = F_0e^{\omega t}\delta(x - l),$$

where

$z(x,t)$ — beam's deflection at the point x and time t,
$z_j(t)$ — position of the jth sleeper at vertical motion at the point l_j ($j = 1,\ldots,n$),
E — Young's modulus,
I — moment of inertia of the section of the beam, with respect to the horizontal axis,
ϱ — density of the beam,
A — area of the cross-section of the beam,
m_{pj} — mass of the jth sleeper,
s_{pj} — coefficient of elasticity of the jth spring joining the sleeper with the beam,
k_{pj} — drag coefficient of the jth spring joining the sleeper with the beam,
s_{bj} — coefficient of elasticity of the jth spring joining the sleeper with the ground,

k_{bj} — drag coefficient of the jth spring joining the sleeper with the ground.

The partial differential equation (8) is coupled with the system of ordinary differential equations

$$\begin{aligned}(9)\qquad m_{pj}\ddot{z}_j = &-k_{bj}\dot{z}_j - s_{bj}z_j - k_{pj}\dot{z}_j - s_{pj}z_j\\ &+\frac{\partial}{\partial t}z(l_j,t)k_{pj} + s_{pj}z(l_j,t)\quad \text{for } j=1,\dots,n.\end{aligned}$$

We are looking for a solution of system (8)–(9) satisfying the boundary conditions

$$(10)\qquad \lim_{|x|\to\infty} z^{(i)}(x,t)=0\qquad (i=0,\dots,4).$$

Before we formulate our theorem, we present the following lemma.

LEMMA 3. *The function $U(x)$ defined by*

$$(11)\qquad U(x)=\frac{1}{4\lambda^3 IE}\,(ie^{i\lambda|x|}-e^{-\lambda|x|}),\quad \textit{where}\quad \lambda^4=-\frac{\varrho A\omega^2}{IE},$$

is a solution of the equation

$$(12)\qquad EIU^{(IV)}(x)+\varrho A\omega^2 U(x)=\delta(x)$$

satisfying the conditions

$$(13)\qquad \lim_{|x|\to\infty} U^{(i)}(x)=0\qquad (i=0,\dots,4).$$

THEOREM 4. *The complex asymptotic solution of system* (8)–(10) *is given by*

$$(14)\qquad z(x,t)=e^{\omega t}\Big(F_0U(x-l)-\sum_{j=1}^{n}B_j\gamma_jU(x-l_j)\Big),$$

with

$$(15)\qquad z_j(t)=\beta_je^{\omega t}\qquad \textit{for } j=1,\dots,n,$$

and the constants γ_j, B_j, β_j are calculated from

$$(16)\qquad \gamma_j=\frac{(m_{pj}\omega^2+k_{bj}\omega+s_{bj})(k_{pj}\omega+s_{pj})}{m_{pj}\omega^2+k_{bj}\omega+s_{bj}+k_{pj}\omega+s_{pj}},$$

$$(17)\qquad B_i=F_0U(l_i-l)-\sum_{j=1}^{n}B_j\gamma_jU(l_i-l_j),$$

$$(18)\qquad \beta_j=\frac{B_j(k_{pj}\omega+s_{pj})}{m_{pj}\omega^2+k_{bj}\omega+s_{bj}+k_{pj}\omega+s_{pj}},$$

for $i,j=1,\dots,n$.

REMARK 5. Uniqueness of the asymptotic solution can be shown as in Section 2.1.

Remark 6. The real asymptotic solution $y(x,t)$ of this model will be given by $y(x,t) = \operatorname{Re} z(x,t)$.

Sketch of proof of Theorem 4. We assume that the complex asymptotic solution has the form

$$z(x,t) = B(x)e^{\omega t}, \tag{19}$$

$$z_j(t) = \beta_j e^{\omega t}. \tag{20}$$

Substituting (19)–(20) into (8) and (9) we obtain respectively

$$IEB^{(IV)}(x) + \varrho A\omega^2 B(x) = F_0\delta(x-l) - \sum_{j=1}^{n} \beta_j(\omega^2 m_{pj} + \omega k_{bj} + s_{bj})\delta(x-l_j) \tag{21}$$

and

$$\beta_j\omega^2 m_{pj} = -\beta_j\omega k_{bj} - \beta_j s_{bj} - \beta_j\omega k_{pj} - \beta_j s_{pj} + B_j\omega k_{pj} + B_j s_{pj}, \tag{22}$$

where $B_j = B(l_j)$. Calculating β_j from the above formula we get

$$\beta_j = \frac{B_j(k_{pj}\omega + s_{pj})}{m_{pj}\omega^2 + k_{bj}\omega + s_{bj} + k_{pj}\omega + s_{pj}} \quad \text{for } j = 1,\ldots,n. \tag{23}$$

From (21) and (23) we have

$$IEB^{(IV)}(x) + \varrho A\omega^2 B(x) = F_0\delta(x-l) - \sum^{n} j = 1\gamma_j B_j\delta(x-l_j), \tag{24}$$

with

$$\gamma_j = \frac{(m_{pj}\omega^2 + k_{bj}\omega + s_{bj})(k_{pj}\omega + s_{pj})}{(m_{pj}\omega^2 + k_{bj}\omega + s_{bj} + k_{pj}\omega + s_{pj})}, \tag{25}$$

where $B(x)$ satisfies

$$\lim_{|x|\to\infty} B^{(i)}(x) = 0 \quad (i = 0,\ldots,4). \tag{26}$$

Using Lemma 3 we obtain the solution of (24) satisfying (26) as follows:

$$B(x) = F_0 U(x-l) - \sum^{n} j = 1 B_j\gamma_j U(x-l_j). \tag{27}$$

Putting $x = l_k$ $(k = 1,\ldots,n)$ into (27) we can calculate B_k $(k = 1,\ldots,n)$ as a solution of a system of linear equations

$$\sum_{j=1}^{n}(\delta_{kj} + \gamma_j U(l_k - l_j))B_j = F_0 U(l_k - l) \quad \text{for } k = 1,\ldots,n. \tag{28}$$

This ends the proof.

3. Numerical experiment. We have performed computations using realistic data, delivered by the department of structural research of Polish National Railway Company.

type of rail	S42	S49	S60
application	small intensity of motion	average intensity of motion	large intensity of motion
ϱA [kg/dm^3]	7.85	7.85	7.85
E [N/m^2]	$2.1 \cdot 10^{11}$	$2.1 \cdot 10^{11}$	$2.1 \cdot 10^{11}$
I [cm^4]	1442	1815	3055
s, s_{pj} [N/m]	$3.3 \cdot 10^8$	$4.15 \cdot 10^8$	$7 \cdot 10^8$

In our numerical experiment we have made some simplifications. Each wagon consists of a wagon body and two lorry cars. We took into account only the motion of one of the lorry cars. Every lorry car has four wheels, but we consider only two of them, because we are interested in one side of the car. These two wheels are treated as one. So we used the following parameters in both models: $m = 1250$ kg (mass of one wheel), $F_0 = 12500$ N (gravity force acting on the track). Since our study is only a first step to construct a more general model, our first aim was to check the correctness of both models against reality. The second model seems to be more complicated to apply. Our second aim was to test whether numerical results obtained in both models are comparable for a special choice of $\widetilde{\omega}$, which substitutes the velocity v in the second model.

3.1. *Simulations for the first model.* In the first model we used additionally the parameters $\varpi = 50$ and $v = 10$ m/s. All simulations were made on the segment of length 35 m (as in the second model). We observed oscillations of the track at the fixed point $x_0 = 0$, and compared oscillations of different types. The oscillations are highest for S42 and lowest for S60, which is reasonable, because S42 is the lightest type of railway track and S60 is the heaviest one (Figs. 1–3).

3.2. *Simulations for the second model.* The remaining parameters used in these simulations are $r = 0.35$ m (distance between sleepers), $\omega = -1 + 50i$, $k_{pj} = 6.3 \cdot 10^4$ Ns/m, $k_{bj} = 8.2 \cdot 10^4$ Ns/m, $s_{bj} = 2.6 \cdot 10^8$ N/m (these coefficients are the same for every sleeper), $m_p = 75$ kg (mass of each sleeper), number of sleepers $n = 100$, place where a fixed force is acting (wheel) $l = 0$. The conclusions are the same as in model 1: S42 is oscillating stronger than S49 and S60 (Figs. 1–3).

3.3. *Conclusions and comments.* The numerical results (Figs. 1–3) demonstrate, in some sense, real behaviour of railway track. In both models the deflections are strongest for S42 and weakest for S60, which is reasonable. However, the deflections in both models are not too high. The reason is that we observe only one wheel. As mentioned earlier, the models considered are different. In the first one, we consider the moving wheel and we add a term with the elasticity coefficient to the equation. In the second model, we also study oscillations of particular sleepers but the motion of the wheel is modeled by a force acting at the fixed point. We also showed that for a suitable value of $\widetilde{\omega}$ the second, more complicated, model can give results similar to those given by the first one. This means that for numerical modelling of oscillation of track, we can also use the first, simpler model.

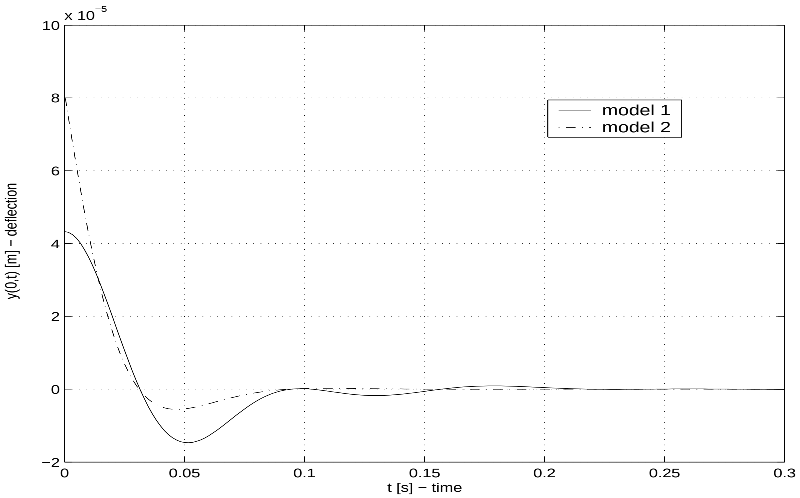

Fig. 1. Comparison of oscillations for both models for railway track S42

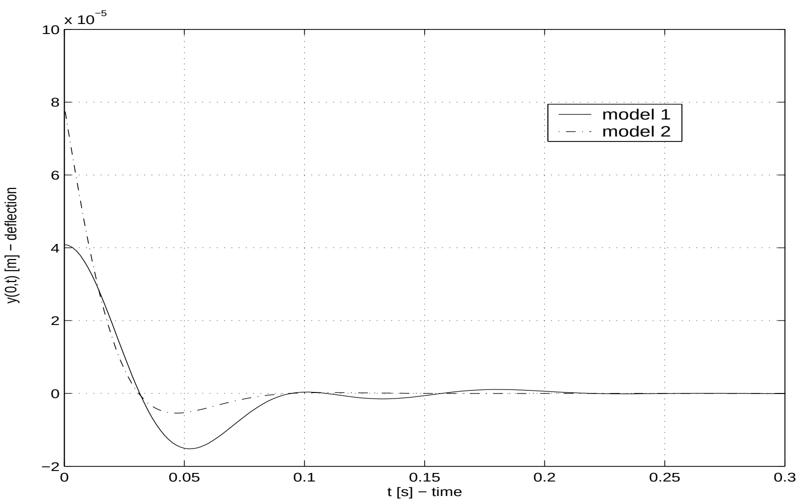

Fig. 2. Comparison of oscillations for both models for railway track S49

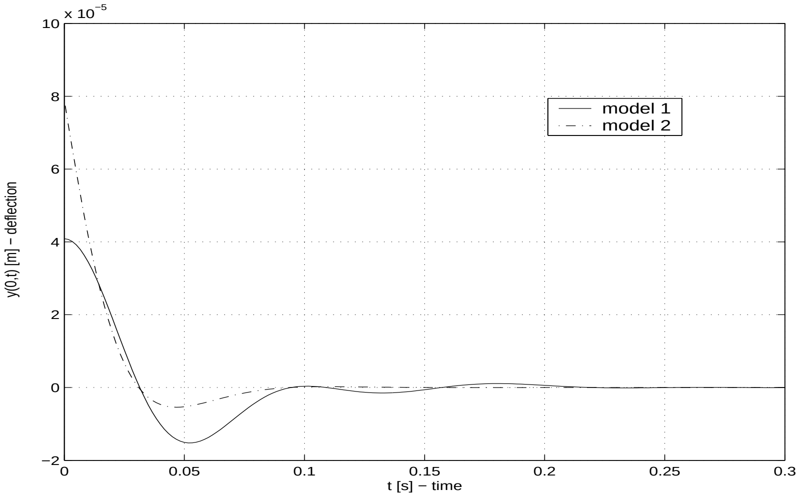

Fig. 3. Comparison of oscillations for both models for railway track S49

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