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## A NOTE ON ECONOMIC EQUILIBRIUM WITH NONSATIATED UTILITY FUNCTIONS

*Abstract.* The purpose of this paper is to prove the existence of a Walrasian equilibrium for the Arrow–Debreu and Arrow–Debreu–McKenzie models with positive price vector with nonsatiated utility functions of consumers by using variational inequalities. Moreover, the same technique is used to give an alternative proof of the existence of a Walrasian equilibrium for the Arrow–Debreu and Arrow–Debreu–McKenzie models with nonnegative, nonzero price vector with nonsatiated utility functions.

**1. Introduction.** K. J. Arrow and G. Debreu formulated the Walrasian general equilibrium model of economy in [4]. There are many mathematical methods of proving the existence of a Walrasian equilibrium, for example the fixed-point technique and topological methods (see [1], [2], [4], [6], [7], [13]). In this article the variational approach is used. Recently, this technique has been considered by many authors (see [3], [8], [9]–[12], [14], [15]).

Let us consider the Arrow–Debreu–McKenzie model with  $m$  consumers (indexed by  $j \in J := \{1, \dots, m\}$ ),  $s$  firms (indexed by  $i \in I := \{1, \dots, s\}$ ), and  $n$  goods (indexed by  $l \in L := \{1, \dots, n\}$ ). In such an economy, the society's initial endowments and technological possibilities (i.e., firms) are owned by consumers. The preference of consumer  $j$  is represented by a utility function, denoted by  $u_j$ . The initial endowment of consumer  $j$  is given by  $\omega_j \in \mathbb{R}_+^n$ . In addition, we suppose that consumer  $j$  owns a share  $\kappa_{ji}$  of firm  $i$ , where  $\sum_{j \in J} \kappa_{ji} = 1$ . Denote by  $Y_i \subset \mathbb{R}^n$  the production set associated with firm  $i$ . It will be assumed that  $Y_i$  is a closed, convex set containing 0 such that  $Y_i \cap \mathbb{R}_+^n = \{0\}$  and  $Y_i \cap (-Y_i) = \{0\}$ .

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Recall that an allocation  $(x_1^*, \dots, x_m^*, y_1^*, \dots, y_s^*)$ ,  $x_j^* \in \mathbb{R}_+^n$ ,  $j \in J$ ,  $y_i^* \in \mathbb{R}^n$ ,  $i \in I$ , and a price vector  $\pi \in \mathbb{R}_+^n \setminus \{0\}$  constitute a *competitive* (or *Walrasian*) *equilibrium* if the following conditions are satisfied ([4]):

- *Profit maximization*: For each firm  $i \in I$ ,  $y_i^*$  solves

$$\max_{y_i \in Y_i} \langle \pi, y_i \rangle.$$

- *Utility maximization*: For each consumer  $j \in J$ ,  $x_j^*$  solves

$$\max \left\{ u_j(x_j) : \langle \pi, x_j \rangle \leq \langle \pi, \omega_j \rangle + \sum_{i \in I} \kappa_{ij} \langle \pi, y_i^* \rangle, x_j \in \mathbb{R}_+^n \right\}.$$

- *Market balance*:

$$(1.1) \quad \sum_{j \in J} x_j^* - \sum_{j \in J} \omega_j - \sum_{i \in I} y_i^* \leq 0, \quad \left\langle \pi, \sum_{j \in J} x_j^* - \sum_{j \in J} \omega_j - \sum_{i \in I} y_i^* \right\rangle = 0.$$

The market balance condition states that the market clears for a commodity if its equilibrium price is positive. Otherwise, there may be an excess supply of the commodity at equilibrium and then its price is zero.

If  $Y_i = \{0\}$  for all  $i \in I$ , then the Arrow–Debreu–McKenzie model is the Arrow–Debreu model of pure exchange (cf. [1], [4], [13], [20]).

In our approach to the Arrow–Debreu–McKenzie model we introduce functions

$$V_j := -u_j, \quad \phi_j(\pi) := \langle \pi, \omega_j \rangle + \sum_{i \in I} \kappa_{ij} \sup_{y_i \in Y_i} \langle \pi, y_i \rangle, \quad j = 1, \dots, m,$$

$$\Phi(\pi) := \sum_{j \in J} \phi_j(\pi) = \left\langle \pi, \sum_{j \in J} \omega_j \right\rangle + \sum_{i \in I} \sup_{y_i \in Y_i} \langle \pi, y_i \rangle, \quad \pi \in \mathbb{R}_+^n.$$

For the Arrow–Debreu model we set  $V_j := -u_j$ ,  $\phi_j(\pi) := \langle \pi, \omega_j \rangle$ ,  $j = 1, \dots, m$ , and  $\Phi(\pi) := \sum_{j \in J} \phi_j(\pi) = \langle \pi, \sum_{j \in J} \omega_j \rangle$ .

In both models instead of (1.1) the following variational inequality, called the *balance condition*, will be considered:

$$\left\langle \tau - \pi, - \sum_{j \in J} x_j^* \right\rangle + \Phi(\tau) - \Phi(\pi) \geq 0, \quad \forall \tau \in \mathbb{R}_+^n.$$

Thus, we get the following *problem* (P): Find  $\pi \in \mathbb{R}_+^n$ ,  $\pi \neq 0$ , and  $x_j \in \mathbb{R}_+^n$ ,  $j = 1, \dots, m$ , such that

$$\begin{cases} V_j(x_j) = \min \{ V_j(x) : \langle \pi, x \rangle \leq \phi_j(\pi) \wedge x \in \mathbb{R}_+^n \}, \\ \left\langle - \sum_{j=1}^m x_j, \tau - \pi \right\rangle + \Phi(\tau) - \Phi(\pi) \geq 0, \quad \forall \tau \in \mathbb{R}_+^n. \end{cases}$$

The problem (P) was first considered under the assumption that  $\phi_j(\pi) \geq \gamma_j$  for some  $\gamma_j > 0$  (cf. [14]). Further, this problem was studied for a class of functions  $\phi_j$  which are nonnegative, continuous, positively homogeneous

of degree 1 and  $\Phi = \sum_{j=1}^m \phi_j$  (see [16]–[18]). The existence of a competitive equilibrium with positively price vector for consumers with strictly monotonic utility functions was proved in an alternative way in [19] by using results from [16], [17]. The strict monotonicity assumption is standard, guaranteeing positivity of the price vector. Some authors define a Walrasian equilibrium price as a positive vector: see for example [1], [13], [20].

In this paper we use the technique of [19] to prove analogous results for a market with consumers having utility functions fulfilling an assumption of nonsatiation. The notion of nonsatiation is weaker than strict monotonicity. Additional assumptions to get the existence of a positive or nonnegative equilibrium price vector with nonsatiated utility functions will be formulated. We compare assumptions which guarantee a nonnegative equilibrium price vector with the assumptions in [4].

**2. Statement of the problem and preliminaries.** Denote by  $\langle \cdot, \cdot \rangle$  the standard inner product in  $\mathbb{R}^n$ , and write

$$\mathbb{R}_+^n = \{x = [x_1, \dots, x_n] \in \mathbb{R}^n : x_i \geq 0, \forall i = 1, \dots, n\}.$$

Denote by  $\text{ind}_K$  the indicator function of a set  $K$ , i.e.

$$\text{ind}_K(y) = \begin{cases} 0 & \text{if } y \in K, \\ +\infty & \text{otherwise.} \end{cases}$$

Throughout the paper it will be assumed that the functions

$$V_j : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}, \quad j = 1, \dots, m,$$

are convex, proper and lower semicontinuous, and we let  $\bar{V}_j := V_j + \text{ind}_{\mathbb{R}_+^n}$ . Recall (see [5]) that a function  $V : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  is *proper* if its domain

$$\text{Dom } V := \{x \in \mathbb{R}^n : V(x) < +\infty\}$$

is nonempty. Assume that the functions

$$\phi_j : \mathbb{R}^n \rightarrow \mathbb{R}_+, \quad \phi_j \not\equiv 0, \quad j = 1, \dots, m,$$

are continuous and positively homogeneous of degree 1. Furthermore, suppose that

$$\Phi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}, \quad \Phi = \sum_{j=1}^m \phi_j,$$

is convex, proper, lower semicontinuous and positively homogeneous of degree 1.

We consider the following problem (P): Find  $\pi \in \mathbb{R}_+^n$ ,  $\pi \neq 0$ , and  $x_j \in \mathbb{R}_+^n$ ,  $j = 1, \dots, m$ , such that

$$\begin{cases} V_j(x_j) = \min\{V_j(x) : \langle \pi, x \rangle \leq \phi_j(\pi) \wedge x \in \mathbb{R}_+^n\}, \\ \left\langle -\sum_{j=1}^m x_j, \tau - \pi \right\rangle + \Phi(\tau) - \Phi(\pi) \geq 0, \quad \tau \in \mathbb{R}_+^n. \end{cases}$$

REMARK 2.1. The problem (P) has an equivalent form: Find  $(\pi, (x_j), (\alpha_j)) \in \mathbb{R}_+^n \times (\mathbb{R}_+^n)^m \times (\mathbb{R}_+)^m$  such that

$$\begin{cases} -\alpha_j \pi \in \partial \bar{V}_j(x_j), \\ \langle \pi, x_j \rangle - \phi_j(\pi) \in \partial \text{ind}_{\geq 0}(\alpha_j), \\ \Phi(\tau) - \Phi(\pi) \geq \left\langle \tau - \pi, \sum_{j=1}^m x_j \right\rangle \quad \forall \tau \in \mathbb{R}_+^n, \end{cases}$$

where  $(\alpha_j)$  are the Lagrange multipliers for the problem (P).

REMARK 2.2. Recall (see [5]) that if  $H$  is a Hilbert space and  $V : H \rightarrow \mathbb{R} \cup \{+\infty\}$  is a convex function, the *subdifferential*  $\partial V : H \rightarrow 2^H$  is defined by

$$\partial V(u) = \{w \in H : V(v) - V(u) \geq \langle w, v - u \rangle, \forall v \in H\}$$

whenever  $V(u) < +\infty$ , and  $\partial V(u) = \emptyset$  otherwise.

Let  $V : H \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper function. The *Fenchel conjugate*  $V^* : H \rightarrow \mathbb{R} \cup \{+\infty\}$  is defined by

$$\forall p \in H \quad V^*(p) = \sup_{v \in H} \{\langle p, v \rangle - V(v)\}.$$

REMARK 2.3. For a convex, lower semicontinuous function  $\Phi_+ = \Phi + \text{ind}_{\mathbb{R}_+^n}$ , positively homogeneous of degree 1, there exists a nonempty, convex, closed set  $W \subset \mathbb{R}^n$  such that  $\Phi_+(\tau) = \sup_{y \in W} \langle \tau, y \rangle$  for all  $\tau \in \mathbb{R}_+^n$  (see [5]).

**3. Existence of a competitive equilibrium.** The existence of an equilibrium with positive price vector for the Arrow–Debreu–McKenzie and Arrow–Debreu models was proved under one of the following assumptions on the function  $V_j = -u_j$ :

$(H_3^0)$   $V_j$  is *strictly decreasing* on  $\mathbb{R}_+^n$ , which means that

$$\forall x \in \mathbb{R}_+^n \quad \forall y \in \mathbb{R}_+^n \setminus \{0\} \quad V_j(x + y) < V_j(x),$$

or

$(H_3^1)$   $V_j$  is strictly decreasing on  $\text{Int } \mathbb{R}_+^n$  and

$$\forall x \in \text{Int } \mathbb{R}_+^n \quad \forall y \in \text{Fr } \mathbb{R}_+^n \quad V_j(x) < V_j(y).$$

Now, we consider a weaker assumption on the preferences:

$(H_4)$   $V_j$  is a *nonsatiated function* on  $\mathbb{R}_+^n$ , which means that

$$\forall x \in \mathbb{R}_+^n \quad \exists z \in \mathbb{R}_+^n \quad V_j(z) < V_j(x).$$

To prove the existence of a competitive equilibrium for the Arrow–Debreu–McKenzie and Arrow–Debreu models, similarly to [19], we use the Theorem of [16] for  $A_j = I$ , the identity matrix.

THEOREM 3.1 ([16, Theorem 3, p. 66]). *Suppose that for any  $j = 1, \dots, m$  the following conditions are satisfied:*

- (H<sub>1</sub>)  $0 \in \text{cl}(\text{Dom } \partial \bar{V}_j)$  and  $(\mathbb{R}_+^n \setminus \{0\}) \cap B(0, r_j) \subset \text{Int Dom } \bar{V}_j^*$  for some  $r_j > 0$ ;
- (H<sub>2</sub>)  $\{x \in \mathbb{R}_+^n : \{\langle x^*, x \rangle : x^* \in \partial \bar{V}_j(x)\} \cap \mathbb{R}_- \neq \emptyset\} \subset \bar{B}(0, M_j)$  for some  $M_j > 0$ ;
- (H<sub>5</sub>)  $\gamma_j := \min\{\phi_j(\tau) : \tau \in \mathbb{R}_+^n, |\tau| = 1\} > 0$ ;
- (H<sub>6</sub>)  $\sum_{j=1}^m x_j \notin \partial \Phi_+(0)$  for any  $x_j \in \partial \bar{V}_j^*(0)$ , where  $\Phi_+ = \Phi + \text{ind}_{\mathbb{R}_+^n}$ .

Then there exist a number  $s \geq 1$  and a system  $(\pi, (x_j), (\alpha_j)) \in \mathbb{R}_+^n \times (\mathbb{R}_+^n)^m \times (\mathbb{R}_+)^m$ ,  $\pi \neq 0$ , such that

$$\begin{cases} -\alpha_j \pi \in \partial \bar{V}_j(x_j), \\ \langle \pi, x_j \rangle - s \phi_j(\pi) \in \partial \text{ind}_{\geq 0}(\alpha_j), \\ \Phi(\tau) - \Phi(\pi) \geq \left\langle \tau - \pi, \sum_{j=1}^m x_j \right\rangle \quad \forall \tau \in \mathbb{R}_+^n. \end{cases}$$

Now we are in a position to prove a result which guarantees the existence of a competitive equilibrium with positive price vector for nonsatiated preferences of consumers.

**THEOREM 3.2.** *Suppose that for any  $j = 1, \dots, m$  the following conditions are satisfied:*

- (H<sub>1</sub><sup>1</sup>)  $0 \in \text{cl}(\text{Dom } \partial \bar{V}_j)$ ;
- (H<sub>4</sub>)  $V_j$  is a nonsatiated function on  $\mathbb{R}_+^n$ ;
- (H<sub>7</sub>)  $\phi_j(\tau) > 0$  for all  $\tau \in \text{Int } \mathbb{R}_+^n$ ;
- (H<sub>8</sub>) there exists  $\delta > 0$  such that  $\delta \|\tau\| \leq \Phi(\tau)$  for all  $\tau \in \mathbb{R}_+^n$ ;
- (H<sub>9</sub>) for all  $\tau \in \text{Fr } \mathbb{R}_+^n \setminus \{0\}$  and all  $z_j \in \mathbb{R}_+^n$  with  $\langle \tau, z_j \rangle = \phi_j(\tau)$ , there exists  $s_j \in \text{Fr } \mathbb{R}_+^n \setminus \{0\}$  such that  $\langle s_j, \tau \rangle = 0$ ,  $V_j(z_j + s_j) < V_j(z_j)$ .

Then there exists a solution of the following problem ( $\hat{P}$ ): Find  $\pi \in \text{Int } \mathbb{R}_+^n$  and  $x_j \in \mathbb{R}_+^n$ ,  $j = 1, \dots, m$ , such that

$$\begin{cases} V_j(x_j) = \min\{V_j(x) : \langle \pi, x \rangle \leq \phi_j(\pi), x \in \mathbb{R}_+^n\}, & j = 1, \dots, m, \\ \left\langle -\sum_{j=1}^m x_j, \tau - \pi \right\rangle + \Phi(\tau) - \Phi(\pi) \geq 0, & \forall \tau \in \mathbb{R}_+^n. \end{cases}$$

*Proof.* Let

$$(3.1) \quad \tilde{V}_j(\cdot) = \bar{V}_j(\cdot) + \text{ind}_{\bar{B}(0, K)}(\cdot), \quad K > \delta + m, \quad j = 1, \dots, m,$$

where  $\delta$  is the constant from (H<sub>8</sub>). We claim that the assumptions of Theorem 3.1 are satisfied for the system

$$(\tilde{V}_j(\cdot), \phi_j(\cdot) + \varepsilon \|\cdot\|, \Phi_+(\cdot) + \varepsilon m \|\cdot\|), \quad 0 < \varepsilon \leq 1,$$

Indeed, the function  $\Phi_+(\cdot) + \varepsilon m \|\cdot\|$  (where  $\Phi_+ = \Phi + \text{ind}_{\mathbb{R}_+^n}$ ) is convex, proper, l.s.c. and positively homogeneous of degree 1 for  $0 < \varepsilon \leq 1$ . Hence there exists

a convex, closed subset  $W^\varepsilon$  of  $\mathbb{R}^n$  such that  $(\Phi_+(\cdot) + \varepsilon m \|\cdot\|)^* = \text{ind}_{W^\varepsilon}$ . From  $(H_8)$  we see that  $W^\varepsilon \cap \mathbb{R}_+^n \subset \overline{B}(0, \delta + m)$ ,  $0 < \varepsilon \leq 1$ .

If  $\|x_j\| < K$  for some  $x_j \in \partial \widetilde{V}_j^*(0)$ , then there exists  $x_j^r \in [x_j, z_j] \cap \text{Dom } \overline{V}_j$  with  $\|x_j^r\| < K$  where  $z_j \in \text{Dom } \overline{V}_j$  such that  $\overline{V}_j(z_j) < \overline{V}_j(x_j)$  (the existence of  $z_j$  is a consequence of  $(H_4)$ ). From the convexity of  $\overline{V}_j$  we get  $\overline{V}_j(x_j^r) < \overline{V}_j(x_j)$ , which contradicts  $x_j \in \partial \widetilde{V}_j^*(0)$ . Hence, if  $x_j \in \partial \widetilde{V}_j^*(0)$ ,  $j = 1, \dots, m$ , then  $(H_4)$  implies that  $\|x_j\| = K$ ,  $j = 1, \dots, m$ . This fact and  $K > \delta + m$  imply that  $\sum_{j=1}^m x_j \notin \overline{B}(0, \delta + m) \cap \mathbb{R}_+^n$  and  $\sum_{j=1}^m x_j \notin \partial(\Phi_+ + \varepsilon m \|\cdot\|)(0)$  for  $x_j \in \partial \widetilde{V}_j^*(0)$ ,  $j = 1, \dots, m$ , and for all  $0 < \varepsilon \leq 1$ . Hence assumption  $(H_6)$  holds. It is easy to check assumptions  $(H_1)$ ,  $(H_2)$ ,  $(H_5)$ , because  $\text{Dom } \widetilde{V}_j^* = \mathbb{R}^n$  and  $\text{Dom } \partial \widetilde{V}_j \subset \overline{B}(0, K)$ .

Accordingly, from Theorem 3.1 it follows that for any  $0 < \varepsilon \leq 1$  there exist  $s^\varepsilon \geq 1$  and  $\pi^\varepsilon \in \mathbb{R}_+^n$ ,  $\pi^\varepsilon \neq 0$ ,  $x_j^\varepsilon \in \mathbb{R}_+^n$ ,  $\alpha_j^\varepsilon \in \mathbb{R}_+$ ,  $j = 1, \dots, m$ , such that

$$(3.2) \quad \begin{cases} -\alpha_j^\varepsilon \pi^\varepsilon \in \partial \widetilde{V}_j(x_j^\varepsilon), \\ \langle \pi^\varepsilon, x_j^\varepsilon \rangle - s^\varepsilon (\phi_j(\pi^\varepsilon) + \varepsilon \|\pi^\varepsilon\|) \in \partial \text{ind}_{\geq 0}(\alpha_j^\varepsilon), \\ \Phi(\tau) - \Phi(\pi^\varepsilon) + \varepsilon m (\|\tau\| - \|\pi^\varepsilon\|) \geq \left\langle \tau - \pi^\varepsilon, \sum_{j=1}^m x_j^\varepsilon \right\rangle, \quad \forall \tau \in \mathbb{R}_+^n. \end{cases}$$

It is obvious that  $\|x_j^\varepsilon\| \leq K$  for all  $0 < \varepsilon \leq 1$  and  $j \in \{1, \dots, m\}$ .

Notice that

$$\alpha_j^\varepsilon > 0, \quad \forall 0 < \varepsilon \leq 1, \forall j \in \{1, \dots, m\}.$$

Indeed, if  $\alpha_{j_0}^\varepsilon = 0$  for some  $0 < \varepsilon \leq 1$  and  $j_0 \in \{1, \dots, m\}$ , then from  $(3.2)_1$ ,  $(H_4)$  we obtain  $\|x_{j_0}^\varepsilon\| = K$ . From  $(3.2)_3$  and  $x_j^\varepsilon \in \mathbb{R}_+^n$ ,  $j = 1, \dots, m$ , we get  $\|\sum_{j=1}^m x_j^\varepsilon\| \leq \delta + m$ , which contradicts  $(3.1)$ .

Using the fact  $\alpha_j^\varepsilon > 0$ ,  $j = 1, \dots, m$ , we find that  $(3.2)_2$  is equivalent to

$$\langle \pi^\varepsilon, x_j^\varepsilon \rangle = s^\varepsilon (\phi_j(\pi^\varepsilon) + \varepsilon \|\pi^\varepsilon\|), \quad j = 1, \dots, m.$$

Summing up, from  $(3.2)_3$  we get

$$\begin{aligned} \Phi(\pi^\varepsilon) + \varepsilon m \|\pi^\varepsilon\| &= \left\langle \sum_{j=1}^m x_j^\varepsilon, \pi^\varepsilon \right\rangle = s^\varepsilon \sum_{j=1}^m (\phi_j(\pi^\varepsilon) + \varepsilon \|\pi^\varepsilon\|) \\ &= s^\varepsilon (\Phi(\pi^\varepsilon) + \varepsilon m \|\pi^\varepsilon\|). \end{aligned}$$

Hence  $s^\varepsilon = 1$  for all  $0 < \varepsilon \leq 1$ .

Let  $p^\varepsilon = \pi^\varepsilon / \|\pi^\varepsilon\|$ ,  $0 < \varepsilon \leq 1$ . There exists a sequence  $(\varepsilon_k)_{k \in \mathbb{N}}$  and  $p \in \mathbb{R}_+^n$ ,  $\|p\| = 1$ , such that  $\varepsilon_k \rightarrow 0$  and  $p^{\varepsilon_k} \rightarrow p$  as  $k \rightarrow \infty$ .

Moreover, we notice that

$$(3.3) \quad p \in \text{Int } \mathbb{R}_+^n.$$

Indeed, suppose that  $p \in \text{Fr } \mathbb{R}_+^n \setminus \{0\}$ . From  $(H_8)$ , we get  $\Phi(p) > 0$ , so there exists  $j_0 \in \{1, \dots, m\}$  such that  $\phi_{j_0}(p) > 0$ . We shall prove that  $(\alpha_{j_0}^{\varepsilon_k} \|\pi^{\varepsilon_k}\|)_{k \in \mathbb{N}}$  is bounded. Suppose to the contrary  $\alpha_{j_0}^{\varepsilon_k} \|\pi^{\varepsilon_k}\| \rightarrow \infty$  as  $k \rightarrow \infty$  (choosing a subsequence if necessary). Using the fact  $\partial \tilde{V}_{j_0}^*(0) \neq \emptyset$  we infer that there exists  $c_{j_0} \in \mathbb{R}$  such that

$$\tilde{V}_{j_0}(y) \geq -c_{j_0}, \quad \forall y \in \text{Dom } \tilde{V}_{j_0}.$$

From  $(3.2)_1$  and  $(3.2)_2$  we get

$$\phi_{j_0}(p^{\varepsilon_k}) \leq \frac{c_{j_0} + \tilde{V}_{j_0}(y)}{\alpha_{j_0}^{\varepsilon_k} \|\pi_k^\varepsilon\|} + \|y\|, \quad \forall y \in \text{Dom } \tilde{V}_{j_0}.$$

Letting  $k \rightarrow \infty$  we obtain

$$0 < \phi_{j_0}(p) \leq \|y\|, \quad \forall y \in \text{Dom } \tilde{V}_{j_0},$$

which contradicts the assumption  $0 \in \text{cl}(\text{Dom } \partial \tilde{V}_{j_0})$ .

Hence (passing to a subsequence) we can assume that there exist  $x_{j_0} \in \mathbb{R}_+^n$  and  $\tilde{\alpha}_{j_0} \in \mathbb{R}_+$  such that  $x_j^{\varepsilon_k} \rightarrow x_{j_0}$  and  $\alpha_{j_0}^{\varepsilon_k} \|\pi_k^\varepsilon\| \rightarrow \tilde{\alpha}_{j_0}$  as  $k \rightarrow \infty$ . From positive homogeneity of  $\phi_{j_0}$  of degree 1 we get an equivalent form of  $(3.2)_1$ ,  $(3.2)_2$ :

$$-\alpha_{j_0}^{\varepsilon_k} \|\pi^{\varepsilon_k}\| p^{\varepsilon_k} \in \partial \tilde{V}_{j_0}(x_{j_0}^{\varepsilon_k}), \quad \langle p^{\varepsilon_k}, x_{j_0}^{\varepsilon_k} \rangle - \phi_{j_0}(p^{\varepsilon_k}) - \varepsilon_k \in \partial \text{ind}_{\geq 0}(\alpha_{j_0}^{\varepsilon_k} \|\pi_k^\varepsilon\|).$$

Letting  $k \rightarrow \infty$  gives

$$-\tilde{\alpha}_{j_0} p \in \partial \tilde{V}_{j_0}(x_{j_0}), \quad \langle p, x_{j_0} \rangle - \phi_{j_0}(p) \in \partial \text{ind}_{\geq 0}(\tilde{\alpha}_{j_0}).$$

Hence

$$V_{j_0}(x_{j_0}) = \min\{V_{j_0}(x) : \langle p, x \rangle \leq \phi_{j_0}(p), x \in \mathbb{R}_+^n \cap \overline{B}(0, K)\}.$$

From  $(3.2)_3$  we get  $x_{j_0} \in \overline{B}(0, \delta + m)$ ,  $j = 1, \dots, m$ , hence  $\|x_{j_0}\| < K$ . It is easy to check that  $\tilde{\alpha}_{j_0} > 0$ . Hence  $\langle p, x_{j_0} \rangle = \phi_{j_0}(p)$ ,  $\partial \tilde{V}_{j_0}(x_{j_0}) = \partial \overline{V}_{j_0}(x_{j_0})$  and

$$(3.4) \quad V_{j_0}(x_{j_0}) = \min\{V_{j_0}(x) : \langle p, x \rangle \leq \phi_{j_0}(p), x \in \mathbb{R}_+^n\}.$$

From  $(H_9)$  we deduce that there exists  $s_{j_0} \in \text{Fr } \mathbb{R}_+^n \setminus \{0\}$  such that

$$\langle p, s_{j_0} \rangle = 0, \quad V_{j_0}(x_{j_0} + s_{j_0}) < V_{j_0}(x_{j_0}).$$

On the other hand,

$$x_{j_0} + s_{j_0} \in \{x \in \mathbb{R}_+^n : \langle p, x \rangle \leq \phi_{j_0}(p)\}.$$

We get a contradiction with (3.4), which means that  $p \in \text{Int } \mathbb{R}_+^n$ .

Since  $p \in \text{Int } \mathbb{R}_+^n$ , from assumption  $(H_7)$  we get  $\phi_j(p) > 0$ ,  $j = 1, \dots, m$ . Similarly to the above case, there exist  $x_j \in \mathbb{R}_+^n$  and  $\tilde{\alpha}_j \in \mathbb{R}_+$ ,  $j = 1, \dots, m$ , such that (passing to a subsequence)  $x_j^{\varepsilon_k} \rightarrow x_j$  and  $\alpha_j^{\varepsilon_k} \|\pi^{\varepsilon_k}\| \rightarrow \tilde{\alpha}_j$  as  $k \rightarrow \infty$  and

$$-\tilde{\alpha}_j p \in \partial \tilde{V}_j(x_j), \quad \langle p, x_j \rangle - \phi_j(p) \in \partial \text{ind}_{\geq 0}(\tilde{\alpha}_j), \quad j = 1, \dots, m,$$

and  $\|x_j\| < K$ . Hence  $\partial\tilde{V}_j(x_j) = \partial\bar{V}_j(x_j)$  and

$$(3.5) \quad V_j(x_j) = \min\{V_j(x) : \langle p, x \rangle \leq \phi_j(p), x \in \mathbb{R}_+^n\}, \quad j = 1, \dots, m.$$

From positive homogeneity of  $\Phi$  of degree 1, condition (3.2)<sub>3</sub> is equivalent to

$$(3.6) \quad \Phi(\tau) - \Phi(p^{\varepsilon_k}) + \varepsilon_k m (\|\tau\| - 1) \geq \left\langle \tau - p^{\varepsilon_k}, \sum_{j=1}^m x_j^{\varepsilon_k} \right\rangle, \quad \forall \tau \in \mathbb{R}_+^n,$$

(on substituting  $\tau/\|\pi_k^\varepsilon\|$  for  $\tau$ ). Letting  $k \rightarrow \infty$  we get

$$\Phi(\tau) - \Phi(p) \geq \left\langle \tau - p, \sum_{j=1}^m x_j \right\rangle, \quad \forall \tau \in \mathbb{R}_+^n.$$

Taking into account (3.3), (3.5), (3.6) we conclude that  $p \in \text{Int } \mathbb{R}_+^n$ ,  $x_j \in \mathbb{R}_+^n$ ,  $j = 1, \dots, m$ , is a solution of the problem (P) with positive price vector. ■

REMARK 3.3. The existence of a Walrasian equilibrium with positive price vector for strictly monotone utility functions  $((H_3^0), (H_3^1))$  was proved in [19] under assumptions  $(H_1)$ ,  $(H_7)$ ,  $(H_8)$ . For the Arrow–Debreu model,  $(H_7)$  and  $(H_8)$  mean the initial endowment  $\omega_j$  of each consumer is a nonnegative vector and the total endowment  $\sum_{j=1}^m \omega_j$  is a positive vector, so these conditions are not particularly restrictive. Notice that assumption  $(H_9)$  is not difficult to check.

The next result ensures the existence of a Walrasian equilibrium with nonnegative, nonzero price vector.

THEOREM 3.4. *Suppose that for any  $j = 1, \dots, m$  the following conditions are satisfied:*

- $(H_1^1)$   $0 \in \text{cl}(\text{Dom } \partial\bar{V}_j)$ ;
- $(H_4)$   $V_j$  is a nonsatiated function on  $\mathbb{R}_+^n$ ;
- $(H_5)$   $\gamma_j := \min\{\phi_j(\tau) : \tau \in \mathbb{R}_+^n, |\tau| = 1\} > 0$ .

Then there exists a solution of the problem (P).

*Proof.* The beginning of the proof is identical to the proof of Theorem 3.2. To show that  $\phi_j(p) > 0$ ,  $j = 1, \dots, m$ , we use assumption  $(H_5)$ . The rest of the proof is analogous to that of Theorem 3.2. ■

REMARK 3.5. To get the existence of a competitive equilibrium with nonsatiated utility functions with nonnegative, nonzero price vector we have made an additional assumption,  $(H_5)$ . For the Arrow–Debreu model assumption  $(H_5)$  means that all consumers provide to the market each good. It is worth mentioning that the existence of a competitive equilibrium in [4] in the first theorem was proved under the assumption

$$(3.7) \quad \forall j = 1, \dots, m \exists \hat{x}_i \in \mathbb{R}_+^n \quad \omega_i - \hat{x}_i \in \text{Int } \mathbb{R}_+^n,$$

which is equivalent to  $(H_5)$  in the Arrow–Debreu model and stronger than  $(H_5)$  in the Arrow–Debreu–McKenzie model. In the second theorem in [4] the assumption (3.7) is weakened, but the authors had to make other assumptions. Assumption  $(H_5)$  seems to be easier to check.

### References

- [1] C. D. Aliprantis, D. J. Brown and O. Burkinshaw, *Existence and Optimality of Competitive Equilibria*, Springer, Berlin, 1989.
- [2] C. D. Aliprantis, B. Cornet and R. Tourky, *Economic equilibrium: optimality and price decentralization*, *Positivity* 6 (2002), 205–241.
- [3] G. Anello, M. B. Donato and M. Milasi, *A quasi-variational approach to a competitive economic equilibrium problem without strong monotonicity assumption*, *J. Global Optim.* 48 (2010), 279–287.
- [4] K. J. Arrow and G. Debreu, *Existence of an equilibrium for a competitive economy*, *Econometrica* 22 (1954), 264–290.
- [5] J.-P. Aubin, *Optima and Equilibria*, Springer, Berlin, 1993.
- [6] G. Chichilnisky and G. Heal, *A unified treatment of finite and infinite economies: Limited arbitrage is necessary and sufficient for the existence of equilibrium and the core*, *Economic Theory* 12 (1998), 163–176.
- [7] S. Dafermos and A. Nagurney, *Sensitivity analysis for the general spatial economic equilibrium problem*, *Oper. Res.* 32 (1984), 1069–1086.
- [8] S. Dafermos and L. Zhao, *General economic equilibrium and variational inequalities*, *Oper. Res. Lett.* 10 (1991), 369–376.
- [9] M. B. Donato, M. Milasi and C. Vitanza, *Quasi-variational approach of a competitive economic equilibrium problem with utility function: existence of equilibrium*, *Math. Models Methods Appl. Sci.* 18 (2008), 351–367.
- [10] M. B. Donato, M. Milasi and C. Vitanza, *An existence result of a quasi-variational inequality associated to an equilibrium problem*, *J. Global Optim.* 40 (2008), 87–97.
- [11] A. Jofré, R. T. Rockafellar and R. J.-B. Wets, *A variational inequality scheme for determining an economic equilibrium of classical or extended type*, in: *Variational Analysis and Applications*, F. Giannessi and A. Maugeri (eds.), Springer, 2005, 553–577.
- [12] A. Jofré, R. T. Rockafellar and R. J.-B. Wets, *Variational inequalities and economic equilibrium*, *Math. Oper. Res.* 32 (2007), 32–50.
- [13] A. Mas-Colell, M. D. Whinston and J. R. Green, *Microeconomic Theory*, Oxford Univ. Press, 1995.
- [14] Z. Naniewicz, *On some optimization problem related to economic equilibrium*, *Control Cybernet.* 31 (2002), 141–165.
- [15] Z. Naniewicz, *Pseudomonotonicity and economic equilibrium problem in reflexive Banach space*, *Math. Oper. Res.* 32 (2007), 436–466.
- [16] Z. Naniewicz and M. Nockowska, *On variational approach to economic equilibrium-type problem*, *Ann. Math. Siles.* 20 (2006), 57–71.
- [17] Z. Naniewicz and M. Nockowska, *Systems of variational inequalities related to economic equilibrium*, *Control Cybernet.* 36 (2007), 889–909.
- [18] M. Nockowska, *Systems of variational inequalities in modeling economic equilibrium problems*, PhD dissertation, Cardinal Stefan Wyszyński Univ., 2006 (in Polish).

- [19] M. Nockowska-Rosiak, *Economic equilibrium through variational inequalities*, Appl. Math. 36 (2009), 43–58.
- [20] H. R. Varian, *Intermediate Microeconomics. A Modern Approach*, Norton, New York, London, 1999.

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