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Spaces of polynomial functions of bounded degrees on an embedded manifold and their duals

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Abstract. Let $\mathcal{O}(U)$ denote the algebra of holomorphic functions on an open subset $U \subset \mathbb{C}^n$ and $Z \subset \mathcal{O}(U)$ its finite-dimensional vector subspace. By the theory of least spaces of de Boor and Ron, there exists a projection $\boldsymbol{T_b}$ from the local ring $\mathcal{O}_{n,\boldsymbol{b}}$ onto the space $Z_{\boldsymbol{b}}$ of germs of elements of Z at $\boldsymbol{b}$. At a general point $\boldsymbol{b} \in U$ its kernel is an ideal and $\boldsymbol{T_b}$ induces the structure of an Artinian algebra on $Z_{\boldsymbol{b}}$. In particular, this holds at points where the kth jets of elements of Z form a vector bundle for each $k \in \mathbb{N}$. For an embedded manifold $X \subset \mathbb{C}^m$, we introduce a space of higher order tangents following Bos and Calvi. In the case of curve, using $\boldsymbol{T_b}$, we define the Taylor projector of order d at a general point $\boldsymbol{a} \in X$, generalising results of Bos and Calvi. It is a retraction of $\mathcal{O}_{X,\boldsymbol{a}}$ onto the set of polynomial functions on $X_{\boldsymbol{a}}$ of degree up to d. Using the ideal property stated above, we show that the transcendency index, defined by the author, of the embedding of a manifold $X \subset \mathbb{C}^m$ is not very high at a general point of X.

1. Introduction. The motivation of this paper is applications of least spaces of de Boor and Ron and a generalisation of the theory of Bos and Calvi of the Taylor projector on a plane algebraic curve. In particular, the main problem is to clarify the nature of "singularities of an affine embedding of a manifold" found by Bos and Calvi in the case of plane algebraic curves.

In the early nineties, there was substantial progress in the theory of multivariate polynomial interpolation with arbitrary interpolation nodes (see e.g. [GS], [Iz_3]). Outstanding methods are application of Gröbner bases (e.g. Marinari–Möller–Mora [MMM]), use of exponential polynomials for evaluation at interpolation points (Dyn–Ron [DR], de Boor–Ron [BR_1]) and the duality between the space of interpolating functions and its least space (de Boor–Ron [BR_1], [BR_2]). The first method is widely applied, treating the interpolation theory as algebraic geometry of 0-dimensional subschemes.

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The second is related to systems of PDEs with constant coefficients. The author feels that the third method is also applicable to general problems beyond interpolation. This is an attempt in such a direction.

The essence of the third method above is that, for a finite-dimensional vector space $Z_{\boldsymbol{b}} \subset \mathbb{C}\{\boldsymbol{t}\}$ of holomorphic function germs at $\boldsymbol{b}$, the initial forms of its elements with respect to the total degree generate a dual space with respect to a sesquilinear form, as follows.

Let U be an open subset of an affine space $\mathbb{C}^n$ and let $\mathcal{O}_n(U)$ denote the ring of holomorphic functions on U. Take $f(\boldsymbol{t}) \in \mathcal{O}_n(U)$ ($\boldsymbol{t} := (t_1, \dots, t_n)$) and $\boldsymbol{b} := (b_1, \dots, b_n) \in U$. The *least part* $f_{\boldsymbol{b}}{\downarrow}$ of f at $\boldsymbol{b}$ means the non-zero homogeneous part of lowest degree of the power series expansion of f with respect to the affine coordinates $\boldsymbol{t}' := \boldsymbol{t} - \boldsymbol{b}$ centred at $\boldsymbol{b}$. Since we shall consider these least parts as elements of a dual space of $\mathcal{O}_n(U)$, we replace the variable $\boldsymbol{t}'$ in the least parts by the corresponding Greek letter. In §2, this will be treated as a Schwartz distribution supported at $\boldsymbol{b}$. We have no need to attach the symbol $\boldsymbol{b}$ to $\boldsymbol{\tau}$. The polynomial $f_{\boldsymbol{b}}{\downarrow}$ may be considered as a homogeneous element of $\mathbb{C}[\boldsymbol{\tau}]$. For example

$$\big((t_1 - b_1)^p (t_2 - b_2)^q + (t_1 - b_1)^{p+1} (t_2 - b_2)^q\big)_{\boldsymbol{b}}{\downarrow} = \tau_1^p \tau_2^q \in \mathbb{C}[\boldsymbol{\tau}].$$

If Z is a vector subspace of $\mathcal{O}_{n,\boldsymbol{b}} := \mathbb{C}\{\boldsymbol{t} - \boldsymbol{b}\}$ or $\mathcal{O}(U)$, then $Z_{\boldsymbol{b}}{\downarrow} \subset \mathbb{C}[\boldsymbol{\tau}]$ denotes the linear span $\operatorname{Span}_{\mathbb{C}}(f_{\boldsymbol{b}}{\downarrow} : f \in Z)$ over $\mathbb{C}$ of the least parts of elements of Z. The mapping

$$\downarrow : Z \to Z_{\boldsymbol{b}}{\downarrow}, \qquad f \mapsto f_{\boldsymbol{b}}{\downarrow},$$

is called the *least operator*. Let

$$S_{n,\boldsymbol{b}} : \mathbb{C}[\boldsymbol{\tau}] \times \mathcal{O}_{n,\boldsymbol{b}} \to \mathbb{C}$$

denote the ordinary sesquilinear form (see §5). The restriction $S_Z : Z_{\boldsymbol{b}}{\downarrow} \times Z \to \mathbb{C}[\boldsymbol{t}]$ of $S_{n,\boldsymbol{b}}$ is found to be non-degenerate by de Boor and Ron [BR$_1$], [BR$_2$]. Let us define the projector

$$\mathcal{T}_{Z,\boldsymbol{b}} : \mathcal{O}_{n,\boldsymbol{b}} \to Z_{\boldsymbol{b}}$$

as the adjoint linear mapping of the inclusion $Z_{\boldsymbol{b}}{\downarrow} \to \mathcal{O}_{n,\boldsymbol{b}}{\downarrow} = \mathbb{C}[\boldsymbol{\tau}]$. This is a retraction, i.e. $\mathcal{T}_{Z,\boldsymbol{b}} \circ \kappa = \mathrm{id}_{Z_{\boldsymbol{b}}}$, where κ denotes the inclusion mapping. A vector subspace of $\mathbb{C}[\boldsymbol{\tau}]$ is defined to be *D-invariant* if it is closed with respect to the partial differentiations with respect to $\tau_1, \dots, \tau_n$. Let U_Z^{inv} denote the set of points where $Z_{\boldsymbol{b}}{\downarrow}$ ($\boldsymbol{b} \in U$) is D-invariant, and U_Z^{bdl} the set of points of U where, for each $k \in \mathbb{N}$, the k-jets of elements of Z form a vector bundle in a sufficiently small neighbourhood. The set U_Z^{bdl} is invariant under biholomorphic transformations of U. We prove the following.

THEOREM 4.5. *The set U_Z^{bdl} is non-empty and analytically open and $U_Z^{\mathrm{bdl}} \subset U_Z^{\mathrm{inv}}$, where a set is called* analytically open *if it is the complement of a closed analytic subset.*

This is the key theorem of this paper but it can be proved easily by describing $\mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow$ in the language of the "formal theory of differential equations". This method was suggested to the author by Tohru Morimoto.

THEOREM 8.3. *Let U be an open subset of $\mathbb{C}^n$ and Z a finite-dimensional vector subspace of $\mathcal{O}_n(U)$. Then U_Z^{inv} is invariant under biholomorphic transformations of U and the vector space $Z_{\boldsymbol{b}}$ has the structure of an Artinian algebra as a factor algebra of $\mathcal{O}_{n,\boldsymbol{b}}$ via the projector $T_{Z,\boldsymbol{b}} : \mathcal{O}_{n,\boldsymbol{b}} \to Z_{\boldsymbol{b}}$ at each $\boldsymbol{b} \in U_Z^{\mathrm{inv}}$. This structure is unique up to a canonical isomorphism as a contravariant tensor* (*see Remark* 9.7).

These results trace back to the very interesting theory of Bos and Calvi [BC$_1$]. Let X be a complex submanifold in an open subset $U \subset \mathbb{C}^m$ and $\mathcal{O}_{X,\boldsymbol{a}}$ the local algebra of germs of holomorphic functions on X at $\boldsymbol{a}$. The vector space of polynomials of degree at most d in $\boldsymbol{x}$ is denoted by $\mathbb{C}[\boldsymbol{x}]^d \subset \mathbb{C}[\boldsymbol{x}]$. We set

$$P^d(X_{\boldsymbol{a}}) := \mathbb{C}[\boldsymbol{x}]^d|_{X_{\boldsymbol{a}}} \subset \mathcal{O}_{X,\boldsymbol{a}},$$

the vector subspace of polynomial functions on X of degree at most d. Let

$$\boldsymbol{\Phi} := (\Phi_1, \ldots, \Phi_m) : \mathbb{C}^n_{\boldsymbol{b}} \to \mathbb{C}^m_{\boldsymbol{a}}$$

be a local parametrisation of X, which means that, if its range is restricted to the image $X_{\boldsymbol{a}}$, it is the germ of a biholomorphic mapping. In this paper, we indicate an analytic mapping germ by an upper case bold Greek letter, and the induced algebra homomorphism by the corresponding lower case letter, like

$$\varphi : \mathcal{O}_{m,\boldsymbol{a}} \to \mathcal{O}_{n,\boldsymbol{b}}, \qquad f \mapsto \varphi(f) := f \circ \boldsymbol{\Phi}.$$

Let

$$\mathbb{C}[\boldsymbol{\Phi}]^d := \varphi(\mathbb{C}[\boldsymbol{x}]^d) \subset \mathcal{O}_{n,\boldsymbol{b}}$$

denote the vector subspace of all pullbacks of elements of $\mathbb{C}[\boldsymbol{x}]^d$ by $\boldsymbol{\Phi}$, that is, the polynomials of degree $\leq d$ in the component functions $\Phi_1, \ldots, \Phi_m$. Following Bos and Calvi, we introduce a special set $D^{\varphi,d}_{\boldsymbol{a}} \subset \mathbb{C}[\boldsymbol{\xi}]$ of higher order tangents of X at $\boldsymbol{a}$ as the set of pushforwards of elements of $\mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow \subset \mathbb{C}[\boldsymbol{\tau}]$ by $\boldsymbol{\Phi}$, where $\mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow$ is considered as a set of higher order tangents of $\mathbb{C}^n$ at $\boldsymbol{b}$. It can be expressed as $D^{\varphi,d}_{\boldsymbol{a}} := {}^s\varphi(\mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow) \subset \mathbb{C}[\boldsymbol{\xi}]$, using the adjoint homomorphism ${}^s\varphi : \mathbb{C}[\boldsymbol{\tau}] \to \mathbb{C}[\boldsymbol{\xi}]$ of φ. It is a dual space of $P^d(X_{\boldsymbol{a}})$ with respect to the sesquilinear form induced by $S_{n,\boldsymbol{b}}$ and we call its elements *Bos–Calvi tangents*. We define the φ-Taylor projector

$$T^{\varphi,d}_{\boldsymbol{a}} : \mathcal{O}_{X,\boldsymbol{a}} \to P^d(X_{\boldsymbol{a}})$$

of degree d on $X \subset \mathbb{C}^m$ at $\boldsymbol{a}$ as the adjoint homomorphism of the inclusion $D^{\varphi,d}_{\boldsymbol{a}} \to \mathbb{C}[\tau]$. The mapping $T^{\varphi,d}_{\boldsymbol{a}}$ and the space $D^{\varphi,d}_{\boldsymbol{a}}$ are defined using a local parametrisation φ of X around $\boldsymbol{a}$. To grasp the complicated relations among these spaces and mappings at a glance, see Diagram 4 in §10.

Proposition 10.7. *Let X be a complex submanifold of an open subset of $\mathbb{C}^m$. For any point $\boldsymbol{a} \in X$, the following conditions are equivalent for each local parametrisation φ:*

(1) *The set $D_{\boldsymbol{a}}^{\varphi,d}$ of Bos–Calvi tangents is D-invariant at $\boldsymbol{a}$.*
(2) *The space of annihilators $(D_{\boldsymbol{a}}^{\varphi,d})^{\perp_X} = \operatorname{Ker} \mathcal{T}_{\boldsymbol{a}}^{\varphi,d}$ is an ideal of $\mathcal{O}_{X,\boldsymbol{a}}$.*

Remark 10.10. Let X be a complex submanifold of an open subset of $\mathbb{C}^m$. If $D_{\boldsymbol{a}}^{\psi,d}$ is D-invariant for some ψ, it is so also for all local parametrisations φ at $\boldsymbol{a}$, that is, $D_{\boldsymbol{a}}^{\varphi,d}$ is a covariant tensor.

Let us call a point $\boldsymbol{a} \in X$ *D-invariant of degree d* if it satisfies the condition (1) (or (2)) of Proposition 10.7 for any (or some) φ, and *D-invariant of degree* ∞ if it is D-invariant of degree d for all $d \in \mathbb{N}$. Theorem 4.5 implies that the set of D-invariant points of degree d contains a non-empty analytically open subset X. Thus the set of points which are not D-invariant of degree ∞ is contained in a countable union of thin analytic subsets of X, a set of first Baire's category with Lebesgue measure 0 in X.

In the case of a plane algebraic curve, Bos and Calvi [BC$_2$] prove that $\mathcal{T}_{\boldsymbol{a}}^{\varphi,d}$ is independent of the choice of φ if and only if the powers of monomials appearing in $\mathbb{C}[\boldsymbol{\Phi}]_{\boldsymbol{b}}^{d}\!\downarrow \subset \mathbb{C}[\tau]$ form a *gap-free* sequence and that this condition actually holds at all but a finite number of points on X, using the Wronskian. In this case, the gap-free property is equivalent to the D-invariance of $\mathbb{C}[\boldsymbol{\Phi}]_{\boldsymbol{b}}^{d}\!\downarrow$. Bos and Calvi [BC$_2$] give the name "Taylorian property" to the independence of $\mathcal{T}_{\boldsymbol{a}}^{\varphi,d}$ from the local parametrisation φ. In this paper, we generalise the theorem of Bos and Calvi to analytic curves of larger codimension as follows.

Theorem 11.1. *Let X be a 1-dimensional regular complex submanifold of an open subset of $\mathbb{C}^m$. Take a local parametrisation $\boldsymbol{\Phi} : \mathbb{C}_{\boldsymbol{b}}^n \to \mathbb{C}_{\boldsymbol{a}}^m$. Then, for any $d \in \mathbb{N}$, the following three properties of $\boldsymbol{a} \in X$ are equivalent:*

(1) *For all $k \in \mathbb{N}_0$, $\boldsymbol{a}$ is a bundle point of the k-jet spaces of $\mathbb{C}[\boldsymbol{\Phi}]_{\boldsymbol{b}}^{d}\!\downarrow$.*
(2) *The powers of monomials appearing in $\mathbb{C}[\boldsymbol{\Phi}]_{\boldsymbol{b}}^{d}\!\downarrow \subset \mathbb{C}[\tau]$ form a gap-free sequence.*
(3) *The point $\boldsymbol{a}$ is Taylorian of degree d.*

For a higher-dimensional submanifold, however, D-invariance does not mean the Taylorian property and the set $D_{\boldsymbol{a}}^{\varphi,d}$ of higher order tangents depends upon the local parametrisation $\boldsymbol{\Phi}$, as we will see in Example 9.6.

We can measure simplicity of embedding $X \subset \mathbb{C}^m$ by the set of Bos–Calvi tangents. Define

$$\theta_{\mathcal{O}_{n,\boldsymbol{b}},\boldsymbol{\Phi}}(d) := \max\{\deg p : p \in \mathbb{C}[\boldsymbol{\Phi}]_{\boldsymbol{b}}^{d}\!\downarrow \setminus \{0\}\}.$$

Let $\operatorname{ord}_{\boldsymbol{b}}(f)$ of $f \in \mathcal{O}_{n,\boldsymbol{b}}$ denote the vanishing order of f at $\boldsymbol{b}$. Since $\theta_{\mathcal{O}_{n,\boldsymbol{b}},\boldsymbol{\Phi}}(d)$ is equal to

$$\sup\{\operatorname{ord}_{\boldsymbol{b}} F(\Phi_1, \ldots, \Phi_m) : F \in \mathbb{C}[\boldsymbol{x}]^d,\ F(\Phi_1, \ldots, \Phi_m) \neq 0\},$$

its estimate as a function of d is called a *zero-estimate* of $\boldsymbol{\Phi}$. It is related to transcendence of the embedding $X \subset \mathbb{C}^m$ at $\boldsymbol{a}$. Zero-estimates are one of the most important methods in transcendental number theory. They have been found for exponential polynomials, for some number-theoretic functions or solutions of some good system of differential equations so far. Here, using D-invariance of degree ∞, we show an effective zero-estimate for a certain set of quite general holomorphic functions, but only at a general point. Let

$$\begin{aligned}\chi(\overline{X}_{\boldsymbol{a}}, d) &:= \dim_{\mathbb{C}} \mathbb{C}[\boldsymbol{\Phi}]^d - \dim_{\mathbb{C}} \mathbb{C}[\boldsymbol{\Phi}]^{d-1}\\ &= \dim_{\mathbb{C}} P^d(X_{\boldsymbol{a}}) - \dim_{\mathbb{C}} P^{d-1}(X_{\boldsymbol{a}})\end{aligned}$$

(with $\dim_{\mathbb{C}} P^{-1}(X_{\boldsymbol{a}}) = 0$) denote the Hilbert function of the smallest algebraic subset (the Zariski closure) $\overline{X}_{\boldsymbol{a}}$ of $\mathbb{C}^m$ that contains a representative of the germ $X_{\boldsymbol{a}}$. We have the following estimates.

THEOREM 12.5. *Let $\boldsymbol{\Phi} : \mathbb{C}^n_{\boldsymbol{b}} \to X_{\boldsymbol{a}} \subset \mathbb{C}^m_{\boldsymbol{a}}$ be an embedding of a complex manifold. Then*

$$\binom{n+d}{n} + \theta_{\mathcal{O}_{n,\boldsymbol{b}},\boldsymbol{\Phi}}(d) - d \le \dim_{\mathbb{C}} \mathbb{C}[\boldsymbol{\Phi}]^d = \sum_{i=0}^{d} \chi(\overline{X}_{\boldsymbol{a}}, i) \le \binom{m+d}{m}$$

for any D-invariant point $\boldsymbol{a}$ of degree d. Hence the transcendency index

$$\alpha(X_{\boldsymbol{a}}) := \limsup_{d\to\infty} \log_d \theta_{\mathcal{O}_{n,\boldsymbol{b}},\boldsymbol{\Phi}}(d),$$

defined in [Iz$_2$], *is majorised by* $\dim \overline{X}_{\boldsymbol{a}}$ ($\le m$) *at a D-invariant point of degree ∞.*

That is, the transcendency index of an embedding $X \subset \mathbb{C}^m$ of complex manifold is bounded above effectively except at points of a set expressed as a countable union of thin analytic subsets, even if X is quite general. This estimate has both merits and demerits in comparison with the zero-estimate obtained as a corollary of Gabrielov's multiplicity-estimate [Ga] for Noetherian functions on an integral manifold of a Noetherian vector field.

All results remain valid also in the real analytic category. In this paper we do not treat the multi-point interpolation problem as [BC$_1$]. The first half of this paper consists of detailed descriptions of basic facts which may be well-known to specialists of the respective fields. They are included because the author could not guess the fields of expertise of possible readers.

2. Least spaces. Here we recall the notion of least space of a vector space of holomorphic functions at a point. It is the graded space associated to the maximal-ideal-adic filtration. We use the term “least space” and the simple symbol $\downarrow$ of de Boor and Ron [BR$_1$], [BR$_2$] used in interpolation theory.

First we define the least operator and the least space in an intrinsic way. Let

$$\mathcal{O}_{n,\boldsymbol{b}} := \mathbb{C}\{\boldsymbol{t}-\boldsymbol{b}\} = \mathbb{C}\{t_1-b_1,\dots,t_n-b_n\}$$

denote the local algebra of convergent power series centred at $\boldsymbol{b} := (b_1,\dots,b_n) \in \mathbb{C}^n$ and

$$\mathfrak{m}_{n,\boldsymbol{b}} := (\boldsymbol{t}-\boldsymbol{b})\mathcal{O}_{n,\boldsymbol{b}} = (t_1-b_1,\dots,t_n-b_n)\mathcal{O}_{n,\boldsymbol{b}}$$

its maximal ideal. This algebra $\mathcal{O}_{n,\boldsymbol{b}}$ has a filtration

$$\mathcal{O}_{n,\boldsymbol{b}} = \mathfrak{m}^0_{n,\boldsymbol{b}} \supset \mathfrak{m}^1_{n,\boldsymbol{b}} \supset \cdots$$

satisfying the following conditions:

$$\bigcap_{i\in\mathbb{N}_0} \mathfrak{m}^i_{n,\boldsymbol{b}} = \{0\}, \qquad \dim_{\mathbb{C}} \frac{\mathfrak{m}^i_{n,\boldsymbol{b}}}{\mathfrak{m}^{i+1}_{n,\boldsymbol{b}}} = \frac{(n+i-1)!}{(n-1)!i!} < \infty.$$

Here the latter equality follows from the fact that the homogeneous polynomials of degree i form a representative system of residue classes of $\mathfrak{m}^i_{n,\boldsymbol{b}}/\mathfrak{m}^{i+1}_{n,\boldsymbol{b}}$. We define the *least space* of $\mathcal{O}_{n,\boldsymbol{b}}$ to be

$$\mathcal{O}_{n,\boldsymbol{b}}\!\downarrow := \bigoplus_{i\in\mathbb{N}_0} \frac{\mathfrak{m}^i_{n,\boldsymbol{b}}}{\mathfrak{m}^{i+1}_{n,\boldsymbol{b}}} \qquad (\mathbb{N}_0 := \{0,1,\dots\}).$$

An element contained in a single component $\mathfrak{m}^i_{n,\boldsymbol{b}}/\mathfrak{m}^{i+1}_{n,\boldsymbol{b}}$ is called *homogeneous.* Let us define the *order function*

$$\operatorname{ord}_{\boldsymbol{b}} : \mathcal{O}_{n,\boldsymbol{b}} \to \mathbb{N}_0, \qquad f \mapsto \operatorname{ord}_{\boldsymbol{b}} f,$$

by

$$\operatorname{ord}_{\boldsymbol{b}} f := \max\{i : f \in \mathfrak{m}^i_{n,\boldsymbol{b}}\} \qquad (\operatorname{ord}_{\boldsymbol{b}} 0 = +\infty).$$

If $\operatorname{ord}_{\boldsymbol{b}} f = i$, the *least part* $f_{\boldsymbol{b}}\!\downarrow$ of f is defined to be the residue class of f in $\mathfrak{m}^i_{n,\boldsymbol{b}}/\mathfrak{m}^{i+1}_{n,\boldsymbol{b}}$, i.e.

$$f_{\boldsymbol{b}}\!\downarrow := f \bmod \mathfrak{m}^{\alpha+1}_{n,\boldsymbol{b}} \qquad (\alpha := \operatorname{ord}_{\boldsymbol{b}} f,\ 0_{\boldsymbol{b}}\!\downarrow := 0).$$

The mapping

$$\downarrow : \mathcal{O}_{n,\boldsymbol{b}} \to \mathcal{O}_{n,\boldsymbol{b}}\!\downarrow, \qquad f \mapsto f_{\boldsymbol{b}}\!\downarrow,$$

is called the *least operator*. It is not a linear map. It is obvious that $\mathcal{O}_{n,\boldsymbol{b}}\!\downarrow = \operatorname{Span}_{\mathbb{C}}(f_{\boldsymbol{b}}\!\downarrow : f \in \mathcal{O}_{n,\boldsymbol{b}})$, the linear span of the least parts of all elements of $\mathcal{O}_{n,\boldsymbol{b}}$. These are the intrinsic definitions of the least part and the least space.

The original definition of the least part of f by de Boor and Ron is the non-zero homogeneous part $f^{\boldsymbol{t}}_{\boldsymbol{b}}\!\downarrow$ of f of smallest degree in the power series expansion of f with respect to some affine coordinate system $\boldsymbol{t}$. This homogeneous part $f^{\boldsymbol{t}}_{\boldsymbol{b}}\!\downarrow$ is a coordinate expression of $f_{\boldsymbol{b}}\!\downarrow$. Before §8, we fix an affine coordinate system and hence we omit the superscript $\boldsymbol{t}$ even in the coordinate expression $f^{\boldsymbol{t}}_{\boldsymbol{b}}\!\downarrow$. Let us adopt the multi-exponent notation:

$$\boldsymbol{\nu} := (\nu_1, \dots, \nu_n), \qquad (\boldsymbol{t} - \boldsymbol{b})^{\boldsymbol{\nu}} = (t_1 - b_1)^{\nu_1} \cdots (t_n - b_n)^{\nu_n}.$$

Since we consider elements of $\mathcal{O}_{n,\boldsymbol{b}}\!\downarrow$ as belonging to the dual space of $\mathcal{O}_{n,\boldsymbol{b}}$ in later sections, we write them as polynomials in Greek variables corresponding to the original as

$$\tau_i := (t_i - b_i)_{\boldsymbol{b}}\!\downarrow, \qquad \boldsymbol{\tau}^{\boldsymbol{\nu}} := (\boldsymbol{t} - \boldsymbol{b})^{\boldsymbol{\nu}}_{\boldsymbol{b}}\!\downarrow \in \mathfrak{m}^{|\boldsymbol{\nu}|}_{n,\boldsymbol{b}}/\mathfrak{m}^{|\boldsymbol{\nu}|+1}_{n,\boldsymbol{b}}.$$

Hence the least space $\mathcal{O}_{n,\boldsymbol{b}}\!\downarrow$ is identified with the polynomial algebra $\mathbb{C}[\boldsymbol{\tau}]$ with $\boldsymbol{\tau} := (\tau_1, \dots, \tau_n)$, $\tau_i := (t_i - b_i)_{\boldsymbol{b}}\!\downarrow$. The product in $\mathbb{C}[\boldsymbol{\tau}]$ is a natural operation as a consequence of the property

$$\mathfrak{m}^{i}_{n,\boldsymbol{b}}\mathfrak{m}^{j}_{n,\boldsymbol{b}} = \mathfrak{m}^{i+j}_{n,\boldsymbol{b}}.$$

Let $Z_{\boldsymbol{b}}$ be a finite-dimensional vector subspace of $\mathcal{O}_{n,\boldsymbol{b}}$. We put

$$Z_{\boldsymbol{b}}\!\downarrow := \mathrm{Span}_{\mathbb{C}}(f_{\boldsymbol{b}}\!\downarrow : f \in Z_{\boldsymbol{b}}) \subset \mathbb{C}[\boldsymbol{\tau}],$$

the linear span of $\{f_{\boldsymbol{b}}\!\downarrow : f \in Z_{\boldsymbol{b}}\}$, and call it the *least space* of $Z_{\boldsymbol{b}}$. We know the following (which will be strengthened in Theorem 5.5).

THEOREM 2.1 (de Boor–Ron [BR_2, Proposition 2.10]; cf. [Iz_3, Theorem 7.1]). *Let $Z_{\boldsymbol{b}}$ be a finite-dimensional vector subspace of $\mathcal{O}_{n,\boldsymbol{b}}$. Then*

$$\dim_{\mathbb{C}} Z_{\boldsymbol{b}}\!\downarrow = \dim_{\mathbb{C}} Z_{\boldsymbol{b}}.$$

3. Jet spaces and multivariate Wronskians. Let Z be a vector space of holomorphic functions on an open subset $U \subset \mathbb{C}^n$. The k-jets of elements of Z at a point of U form a vector space. If we gather such vector spaces only at good points of U, we get a holomorphic vector bundle, the k-jet bundle of Z. The theorem of Walker on Wronskians implies that the jet sections of elements of Z of order up to $\dim_{\mathbb{C}} Z - 1$ span those of any order on an analytically open subset. This order $\dim_{\mathbb{C}} Z - 1$ is minimal for a general Z.

Let $\mathcal{O}_n$ denote the sheaf of germs of holomorphic functions on $\mathbb{C}^n$. We call the sheaf of germs of holomorphic sections of a holomorphic vector bundle the *associated sheaf* or the *$\mathcal{O}_n$-module* associated to the bundle. It is indicated by the script style of the letter used for the bundle. Assigning to a holomorphic vector bundle on U the associated sheaf defines a bijective mapping of the set of isomorphism classes of holomorphic vector bundles of rank r over U onto the set of isomorphism classes of locally free $\mathcal{O}_U$-modules ($\mathcal{O}_U = \mathcal{O}_n|_U$) of rank r over U (see e.g. [PR, Proposition 3.3]). We use parenthesised $\boldsymbol{b}$ for values or sets of values at $\boldsymbol{b}$ (bundle fibre, e.g. $R_k(\boldsymbol{b})$), and subscript $\boldsymbol{b}$ for germs or sets of germs at $\boldsymbol{b} \in U$ (stalk, sheaf fibre, e.g. $\mathcal{L}^k_{\boldsymbol{b}}$)

or to indicate the centre of coordinates for which the least part is defined (as $Z_{\boldsymbol{b}}{\downarrow}$).

Let

$$\pi_k : J^k(\mathcal{O}_U) \to U, \qquad J^k(\mathcal{O}_U) \cong U \times \mathbb{C}^{N(n,k)},\ N(n,k) := \binom{n+k}{k},$$

denote the k-jet space of holomorphic functions on U, the holomorphic vector bundle of k-jets of holomorphic functions defined on open subsets of U. Its coordinates are denoted by

$$(\boldsymbol{t}, (u_{\boldsymbol{\nu}} : |\boldsymbol{\nu}| \leq k)) \qquad (\boldsymbol{t} := (t_1, \ldots, t_n) \in U,\, \boldsymbol{\nu} := (\nu_1, \ldots, \nu_n)).$$

Let $\mathcal{O}_n(V)$ denote the algebra of sections of $\mathcal{O}_n$ over $V \subset U$. The *k-jet extension* $j^k f$ of $f \in \mathcal{O}_n(V)$ is defined by

$$j^k f : V \to J^k(\mathcal{O}_U), \qquad \boldsymbol{t} \mapsto \big(\boldsymbol{t}, u_{\boldsymbol{\nu}}(j^k f) : |\boldsymbol{\nu}| \leq k\big),$$
$$u_{\boldsymbol{\nu}}(j^k f) := \frac{1}{\boldsymbol{\nu}!} \frac{\partial^{|\boldsymbol{\nu}|} f(\boldsymbol{t})}{\partial \boldsymbol{t}^{\boldsymbol{\nu}}}.$$

This is a section of the jet space $J^k(\mathcal{O}_U)$ over V. The coefficient $1/\boldsymbol{\nu}!$ of the $\boldsymbol{\nu}$th fibre coordinate is convenient in the calculation of prolongation below. The coordinates $u_{\boldsymbol{\nu}}$ ($|\boldsymbol{\nu}| \leq k$) are called the *fibre coordinates* corresponding to the normalised $\boldsymbol{\nu}$th derivative.

If Z is a finite-dimensional vector subspace of $\mathcal{O}_n(U)$, evaluation of the jet extension at $\boldsymbol{b} \in U$ defines the mapping

$$j^k|_Z(\boldsymbol{b}) : Z \to J^k(\mathcal{O}_U)(\boldsymbol{b}), \qquad f \mapsto j^k f(\boldsymbol{b}).$$

Let

$$(\boldsymbol{b};\, R_k(\boldsymbol{b})) := \{j^k f(\boldsymbol{b}) : f \in Z\}$$

denote its image. Then we have the natural commutative diagram of linear mappings of vector spaces:

$$\begin{tikzcd}
 & & Z \arrow[dl, "j^{k-1}|_Z(\boldsymbol{b})"'] \arrow[d, "j^k|_Z(\boldsymbol{b})"] \arrow[dr, "j^{k+1}|_Z(\boldsymbol{b})"] & & \\
\cdots & R_{k-1}(\boldsymbol{b}) \arrow[l] & R_k(\boldsymbol{b}) \arrow[l, "\Sigma^k(\boldsymbol{b})"'] & R_{k+1}(\boldsymbol{b}) \arrow[l, "\Sigma^{k+1}(\boldsymbol{b})"'] & \cdots \arrow[l]
\end{tikzcd}$$

Diagram 1. Jet spaces of $Z \subset \mathbb{C}[\boldsymbol{\tau}]$

Here, the horizontal mappings are projections defined by forgetting the coordinates corresponding to the highest order derivatives. The total image $\bigcup_{\boldsymbol{b} \in U} (\boldsymbol{b}, R_k(\boldsymbol{b}))$ is not an analytic subset of $J^k(\mathcal{O}_U)$ nor even a closed subset in general. The complement of a closed analytic subset in U is called *analytically open* in U. Set

$$r_k := \max\{\dim_{\mathbb{C}} R_k(\boldsymbol{t}) : \boldsymbol{t} \in U\}, \qquad U_Z^k := \{\boldsymbol{t} \in U : \dim_{\mathbb{C}} R_k(\boldsymbol{t}) = r_k\}.$$

Suppose that U is connected. Since the points of U_Z^k are characterised by the full rank condition of certain matrices with holomorphic elements, U_Z^k is a non-empty analytically open subset. Setting

$$R_k := \{(\boldsymbol{b}; R_k(\boldsymbol{b})) : \boldsymbol{b} \in U_Z^k\},$$

we have a holomorphic vector bundle

$$\pi_k|_{R_k} : R_k \to U_Z^k.$$

DEFINITION 3.1. We call a point of U_Z^k a *bundle point of the k-jet space* of Z and a point of $U_Z^{\mathrm{bdl}} := U_Z^0 \cap U_Z^1 \cap \cdots$ a *bundle point of all the jet spaces* of Z.

EXAMPLE 3.2. Define $Z := \mathrm{Span}(s^2, t^2, s^3)) \in \mathcal{O}_2(\mathbb{C}^2)$. Since the higher order derivatives of s^2, t^2, s^3 with respect to the multiple order

$$\boldsymbol{\nu} = (0,0),\ (1,0),\ (0,1),\ (2,0),\ (1,1),\ (0,2),\ (3,0),\ (2,1),\ (1,2),\ (0,3)$$

are listed as

$$\begin{pmatrix} s^2 & 2s & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ t^2 & 0 & 2t & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ s^3 & 3s^2 & 0 & 6s & 0 & 0 & 6 & 0 & 0 & 0 \end{pmatrix},$$

we have

$$\begin{gathered} r_0 = 1, \qquad r_1 = r_2 = \cdots = 3; \\ U_Z^0 = \mathbb{C}^2 \setminus \{(0,0)\} \supsetneq U_Z^1 = \mathbb{C}^2 \setminus \{st = 0\} \\ \subsetneq U_Z^2 = \mathbb{C}^2 \setminus \{s = 0\} \subsetneq U_Z^3 = U_Z^4 = \cdots = \mathbb{C}^2; \\ U_Z^{\mathrm{bdl}} = \mathbb{C}^2 \setminus \{st = 0\}. \end{gathered}$$

Hence there is no monotone inclusion relation among $U_Z^0, U_Z^1, U_Z^2, \ldots$ (see 3.6).

Now we recall a known fact on multivariate Wronskians. For multivariable polynomials, Siegel [Si] and Roth [Ro] found that their linear independence is decided by non-vanishing of a certain set of Wronskians, and applied this to the theory of rational approximation of algebraic numbers. Walker [Wa] has obtained the minimal set of Wronskians needed to decide linear independence. It allows us to write out a minimal finite system of PDEs explicitly whose solution space is a given finite-dimensional vector subspace $Z \subset \mathcal{O}_n(U)$. It also enables us to state the subsequent arguments efficiently.

Definition 3.3. Walker called $Y := \{\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_m\} \in (\mathbb{N}_0^n)^m$ *Young-like* if it satisfies the following condition:

$$(\boldsymbol{\nu} \in \mathbb{N}_0^n,\ \exists \boldsymbol{\nu}_i \in Y : \boldsymbol{\nu} \leq \boldsymbol{\nu}_i) \ \Rightarrow\ \boldsymbol{\nu} \in \boldsymbol{Y}.$$

Here $\leq$ denotes the product order of the usual order of $\mathbb{N}_0$ defined as

$$\boldsymbol{\nu} := (\nu_1, \dots, \nu_n) \leq \boldsymbol{\mu} := (\mu_1, \dots, \mu_n) \ \Leftrightarrow\ \nu_1 \leq \mu_1, \dots, \nu_1 \leq \mu_n.$$

Young-likeness of Y is equivalent to D-invariance of $\mathrm{Span}_{\mathbb{C}}(\boldsymbol{\tau}^{\boldsymbol{\nu}_1}, \dots, \boldsymbol{\tau}^{\boldsymbol{\nu}_m})$ in the sense defined in §4. Set

$$\mathcal{Y}_m := \{Y \in (\mathbb{N}_0^n)^m : Y \text{ is Young-like}\}.$$

Young-like sets are also called *order-closed* [BR$_2$], *monotone* or *lower sets.*

Theorem 3.4 (Siegel [Si]; Roth [Ro]; Walker [Wa, Theorem 3.1, Theorem 3.4, Remark in §3]).

(1) *Let $f_1, \dots, f_m$ be meromorphic functions on a connected open subset $U \subset \mathbb{C}^n$. Then they are linearly independent if and only if there exists at least one $\{\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_m\} \in \mathcal{Y}_m$ such that*

$$W(f_1, \dots, f_m; \boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_m) := \begin{vmatrix} f_1^{(\boldsymbol{\nu}_1)} & f_1^{(\boldsymbol{\nu}_2)} & \cdots & f_1^{(\boldsymbol{\nu}_m)} \\ f_2^{(\boldsymbol{\nu}_1)} & f_2^{(\boldsymbol{\nu}_2)} & \cdots & f_2^{(\boldsymbol{\nu}_m)} \\ \vdots & \vdots & \vdots & \vdots \\ f_m^{(\boldsymbol{\nu}_1)} & f_m^{(\boldsymbol{\nu}_2)} & \cdots & f_m^{(\boldsymbol{\nu}_m)} \end{vmatrix}$$

does not vanish identically.

(2) *The set $\mathcal{Y}_m$ is the least set with the above property in the following sense. If $\mathcal{Y}' \subset (\mathbb{N}_0^n)^m$ and $\mathcal{Y}' \subsetneq \mathcal{Y}_m$, then there exist linearly independent monomials $f_1, \dots, f_m$ such that*

$$W(f_1, \dots, f_m;\ \boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_m) = 0$$

for all $(\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_m) \in \mathcal{Y}'$.

We use the following immediate consequence of this theorem.

Corollary 3.5. *Let $\{f_1, \dots, f_m\}$ be a basis of a vector subspace $Z \subset \mathcal{O}_n(U)$. Then Z is the space of holomorphic solutions of the system of PDEs*

$$W(f_1, \dots, f_m, y;\ \boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_{m+1}) = 0 \qquad (\{\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_{m+1}\} \in \mathcal{Y}_{m+1}),$$

where $y = y(\boldsymbol{t}) \in \mathcal{O}_n(U)$ denotes the unknown function.

Lemma 3.6. *Let U be a connected open subset of $\mathbb{C}^n$ and let $\{f_1, \dots, f_m\}$ $(m \geq 1)$ be a basis of a vector subspace $Z \subset \mathcal{O}_n(U)$.*

(1) *If $Y = \{\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_m\} \in \mathcal{Y}_m$ and $W(f_1, \dots, f_m; Y)(\boldsymbol{b}) \neq 0$, then $\boldsymbol{b} \in U_Z^{m-1}$ and the vectors $(f_1^{(\boldsymbol{\nu}_i)}(\boldsymbol{b}), \dots, f_m^{(\boldsymbol{\nu}_i)}(\boldsymbol{b}))$ $(i = 1, \dots, m)$*

span all $(f_1^{(\boldsymbol{\nu})}(\boldsymbol{b}), \ldots, f_m^{(\boldsymbol{\nu})}(\boldsymbol{b}))$ $(\boldsymbol{\nu} \in \mathbb{N}_0^m)$. *Hence the fibre coordinates* $u_{\boldsymbol{\nu}_1}, \ldots, u_{\boldsymbol{\nu}_m}$ *of* $J^{m-1}(\mathcal{O}_U)$ *form a fibre coordinate system of* R_{m-1} *over a neighbourhood of* $\boldsymbol{b}$.

(2) *We have*

$$1 = r_0 \le r_1 \le \cdots \le r_{m-1} = r_m = \cdots = m,$$
$$U_Z^{m-1} \subset U_Z^m \subset \cdots, \qquad U_Z^{\mathrm{bdl}} = U_Z^1 \cap \cdots \cap U_Z^{m-1}.$$

(3) *The vector bundles* R_k $(k \ge m-1)$ *are all isomorphic when restricted to* U_Z^{m-1}.

(4) *The sets* U_Z^k *and* U_Z^{bdl} *are non-empty analytically open subsets of* U *and independent of the choice of local coordinates of* U.

Proof. (1) If $\boldsymbol{\nu}_i \in Y \in \mathcal{Y}_m$, there is a chain Z connecting it to $(0, \ldots, 0)$. It is not longer than $\#Y - 1 = m - 1$ $(1 \le i \le m)$. Then the coordinates $u_{\boldsymbol{\nu}_1}, \ldots, u_{\boldsymbol{\nu}_m}$ form a subset of the fibre coordinate system of $J^{m-1}(\mathcal{O}_U)(\boldsymbol{b})$. The assumption $W(f_1, \ldots, f_m; Y)(\boldsymbol{b}) \ne 0$ implies that $R_{m-1}(\boldsymbol{b})$ is an m-dimensional subspace of $J^{m-1}(\mathcal{O}_U)(\boldsymbol{b})$. Since $R_k(\boldsymbol{b})$ is spanned by m vectors, it is maximal and we see that $\boldsymbol{b} \in U_Z^{m-1}$. The rest is now obvious.

(2) The fact that the r_i are non-decreasing is obvious. By Theorem 3.4, there exist $\boldsymbol{b}$ and $Y := \{\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_m\} \in \mathcal{Y}_m$ with $W(f_1, \ldots, f_m; Y)(\boldsymbol{b}) \ne 0$. Then $r_{m-1} = m$ by (1). These prove the first expression. Since $r_{m-1} = r_m = \cdots = m$ and the R_k are increasing, we have $U_Z^{m-1} \subset U_Z^m \subset \cdots$. Therefore $U_Z^{\mathrm{bdl}} = U_Z^1 \cap U_Z^2 \cap \cdots = U_Z^1 \cap \cdots \cap U_Z^{m-1}$.

(3) Property (1) implies that u_ν $(|\boldsymbol{\nu}| \le m-1)$ is a linear combination of $u_{\boldsymbol{\nu}_1}, \ldots, u_{\boldsymbol{\nu}_m}$ in R_{m-1} with coefficients in $\mathcal{O}_n$. Thus R_k $(k \ge m-1)$ are not proper extensions of R_{m-1}.

(4) We have already stated that U_Z^k are open in their definition above. Independence from coordinate changes is a consequence of the fact that the jet spaces are contravariant geometric objects (see Remark 9.7), which will be detailed in §8. ∎

4. Generic D-invariance of least spaces. Let Z be a vector space of holomorphic functions on an open subset $U \subset \mathbb{C}^n$. The vector space of germs of elements of Z at $\boldsymbol{b} \in U$ is denoted by $Z_{\boldsymbol{b}}$. The least space $Z_{\boldsymbol{b}}\!\downarrow$ of $Z_{\boldsymbol{b}}$ is identified as a vector subspace of $\mathbb{C}[\boldsymbol{\tau}]$ (as stated in the introduction). Our main purpose here is to prove that $Z_{\boldsymbol{b}}\!\downarrow$ is closed under partial differentiation by τ_i $(1 \le i \le n)$ at a general point of U. The point of the proof is the prolongation of PDEs annihilating the jets of elements of Z, which was suggested to the author by Tohru Morimoto.

Definition 4.1. A vector subspace $Q \subset \mathbb{C}[\boldsymbol{\tau}]$ $(\boldsymbol{\tau} := (\tau_1, \ldots, \tau_n))$ is *D-invariant* if it is closed with respect to the partial differentiations with respect to $\tau_1, \ldots, \tau_n$. Let Z be a vector space of holomorphic functions on an open subset $U \subset \mathbb{C}^n$. Define

$$U_Z^{\mathrm{inv}} := \{\boldsymbol{b} : Z_{\boldsymbol{b}}{\downarrow} \text{ is } D\text{-invariant}\}.$$

PROPOSITION 4.2. *If $0 < \dim_{\mathbb{C}} Z_{\boldsymbol{b}} < \infty$ and $\boldsymbol{b} \in U_Z^{\mathrm{inv}}$, then there exists $f \in Z_{\boldsymbol{b}}$ such that $f(\boldsymbol{b}) \neq 0$.*

Proof. If $f(\boldsymbol{b}) = 0$ for all $f \in Z_{\boldsymbol{b}}$, we have $1 \notin Z_{\boldsymbol{b}}{\downarrow}$. This contradicts the assumption of D-invariance of $Z_{\boldsymbol{b}}{\downarrow}$. ∎

Therefore, at a point $\boldsymbol{b}$ of the simultaneous vanishing locus of the elements of Z, the least space $Z_{\boldsymbol{b}}{\downarrow}$ is not D-invariant. The converse does not hold, as we will see in Example 7.6.

PROPOSITION 4.3. *If $Q \subset \mathbb{C}[\boldsymbol{\tau}]$ is D-invariant, it is translation invariant, i.e. $p(\boldsymbol{\tau}) \in Q$ implies $p(\boldsymbol{\tau} + \boldsymbol{b}) \in Q$ for any constant vector $\boldsymbol{b}$.*

Proof. This is obvious from the ordinary Taylor formula:

$$p(\boldsymbol{\tau} + \boldsymbol{b}) = \sum_{|\boldsymbol{\nu}| \leq d} \frac{1}{\boldsymbol{\nu}!} \frac{\partial^{|\boldsymbol{\nu}|} p(\boldsymbol{\tau})}{\partial \boldsymbol{\tau}^{\boldsymbol{\nu}}} \boldsymbol{b}^{\boldsymbol{\nu}} \qquad (d := \deg p). \ ∎$$

Let $\mathcal{L}^k$ denote the sheaf of germs of holomorphic functions on the manifold $J^k(\mathcal{O}_{U_Z^{\mathrm{bdl}}})$ vanishing on the submanifold R_k which are linear in the fibre coordinates $u_{\boldsymbol{\nu}}$. This is an $\mathcal{O}_{U_Z^{\mathrm{bdl}}}$-module on U_Z^{bdl}. The local sections of $\mathcal{L}^k$ are functions

$$A(\boldsymbol{t}, \boldsymbol{u}) := \sum_{|\boldsymbol{\nu}| \leq k} \alpha_{\boldsymbol{\nu}}(\boldsymbol{t}) \cdot u_{\boldsymbol{\nu}}$$

which are homogeneous linear in fibre coordinates $u_{\boldsymbol{\nu}}$ of $J^k(\mathcal{O}_{U_Z^{\mathrm{bdl}}})$ with coefficients $\alpha_{\boldsymbol{\nu}}(\boldsymbol{t}) \in \mathcal{O}_n(V)$ $(V \subset U_Z^{\mathrm{bdl}})$. If the $u_{\boldsymbol{\nu}}$ are replaced by the corresponding differential operators $(1/\boldsymbol{\nu}!) \cdot \partial^{|\boldsymbol{\nu}|}/\partial \boldsymbol{t}^{\boldsymbol{\nu}}$, then elements of $\mathcal{L}^k(V)$ become linear partial differential operators which annihilate the functions of Z on V.

DEFINITION 4.4. Take a local section

$$A(\boldsymbol{t}, \boldsymbol{u}) := \sum_{|\boldsymbol{\nu}| \leq k-1} \alpha_{\boldsymbol{\nu}}(\boldsymbol{t}) \cdot u_{\boldsymbol{\nu}} \in \mathcal{L}^{k-1}(V) \qquad (\alpha_{\boldsymbol{\nu}}(\boldsymbol{t}) \in \mathcal{O}_n(V))$$

over $V \subset U$. Differentiating the relation

$$\sum_{|\boldsymbol{\nu}| \leq k-1} \frac{1}{\boldsymbol{\nu}!} \alpha_{\boldsymbol{\nu}}(\boldsymbol{t}) \cdot \frac{\partial^{|\boldsymbol{\nu}|} f}{\partial \boldsymbol{t}^{\nu}} = 0 \qquad (f \in Z)$$

with respect to t_i, we have

$$\sum_{|\boldsymbol{\nu}| \leq k-1} \frac{1}{\boldsymbol{\nu}!} \left(\alpha_{\boldsymbol{\nu}}^{(\boldsymbol{e}_i)}(\boldsymbol{t}) \cdot \frac{\partial^{|\boldsymbol{\nu}|}}{\partial \boldsymbol{t}^{\boldsymbol{\nu}}} + \alpha_{\boldsymbol{\nu}}(\boldsymbol{t}) \cdot \frac{\partial^{|\boldsymbol{\nu} + \boldsymbol{e}_i|}}{\partial \boldsymbol{t}^{\boldsymbol{\nu} + \boldsymbol{e}_i}} \right) f = 0 \qquad (f \in Z),$$

where $\boldsymbol{e}_i := (\delta_{1i}, \dots, \delta_{ni})$ denotes the ith unit vector. Hence $\alpha_{\boldsymbol{\nu}}^{(\boldsymbol{e}_i)}$ expresses the derivative of $\alpha_{\boldsymbol{\nu}}$ with respect to t_i. This equation implies that

$$\sum_{|\boldsymbol{\nu}|\le k-1} \left(\alpha_{\boldsymbol{\nu}}^{(\boldsymbol{e}_i)}(\boldsymbol{t}) \cdot u_{\boldsymbol{\nu}} + (\nu_i+1)\alpha_{\boldsymbol{\nu}}(\boldsymbol{t}) \cdot u_{\boldsymbol{\nu}+\boldsymbol{e}_i}\right) \in \mathcal{L}^k(V) \quad (i = 1, \dots, n).$$

These are called the *first prolongations* of $A(\boldsymbol{t}, \boldsymbol{u})$.

THEOREM 4.5. *Let $\mathcal{O}_n(U)$ be the algebra of holomorphic functions on a connected open subset $U \subset \mathbb{C}^n$, Z its finite-dimensional vector subspace and $U_Z^{\mathrm{bdl}} \subset U$ the set of bundle points of all the jet spaces of Z (Definition 3.1). Then U_Z^{bdl} is a non-empty analytically open subset and $U_Z^{\mathrm{bdl}} \subset U_Z^{\mathrm{inv}}$.*

Proof. We have seen that U_Z^{bdl} is a non-empty analytically open set in Lemma 3.6. The canonical projections

$$\Pi^k : J_k(\mathcal{O}_U) \to J_{k-1}(\mathcal{O}_U), \qquad \Sigma^k : R_k \to R_{k-1}$$

are constant rank homomorphisms over U_Z^{bdl} and they induce the inclusion $i^k : \mathrm{Ker}\,\Sigma^k \to \mathrm{Ker}\,\Pi^k$ of locally free analytic sheaves by the following diagram:

$$\begin{tikzcd}
\mathcal{R}_{k-1} \arrow[d] & \mathcal{R}_k \arrow[l, "\Sigma^k"'] \arrow[d] & \mathrm{Ker}\,\Sigma^k \arrow[l] \arrow[d, dotted, "i^k"] & 0 \arrow[l] \\
\mathcal{J}_{k-1}(\mathcal{O}_U) & \mathcal{J}_k(\mathcal{O}_U) \arrow[l, "\Pi^k"'] & \mathrm{Ker}\,\Pi^k \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

A local section of $\mathrm{Ker}\,\Pi^k$ is expressed as

$$f(\boldsymbol{t}, \boldsymbol{\tau}) = \sum_{|\boldsymbol{\nu}|=k} \beta_{\boldsymbol{\nu}}(\boldsymbol{t})\boldsymbol{\tau}^{\boldsymbol{\nu}}, \qquad \beta_{\boldsymbol{\nu}}(\boldsymbol{t}) := \frac{1}{\boldsymbol{\nu}!} \frac{\partial^{|\boldsymbol{\nu}|} f}{\partial \boldsymbol{\tau}^{\boldsymbol{\nu}}}(\boldsymbol{t}, \boldsymbol{0}),$$

where the monomials $\boldsymbol{\tau}^{\boldsymbol{\nu}}$ stand for the base of the fibre corresponding to the coordinate $u_{\boldsymbol{\nu}}$. (We may write $\boldsymbol{\tau}^{\boldsymbol{\nu}}$ as $(d\boldsymbol{t})^{\odot\boldsymbol{\nu}} = (dt_1)^{\odot\nu_1} \odot \cdots \odot (dt_n)^{\odot\nu_n}$, using the symmetric tensor product $\odot$, see Theorem 8.3(1).)

The least space $Z_{\boldsymbol{b}}\!\downarrow$ is obtained by evaluating $\mathrm{Ker}\,\Sigma^k$ at $\boldsymbol{b}$. Hence D-invariance of $Z_{\boldsymbol{b}}\!\downarrow$ at degree k reduces to the implication

$$f \in \mathrm{Ker}\,\Sigma^k \;\Rightarrow\; \frac{\partial f}{\partial \tau_i} \in \mathrm{Ker}\,\Sigma^{k-1}.$$

Take $f(\boldsymbol{t}, \boldsymbol{\tau}) \in \mathrm{Ker}\,\Sigma^k$ and any defining equation

$$A(\boldsymbol{t}, \boldsymbol{u}) := \sum_{|\boldsymbol{\nu}|\le k-1} \alpha_{\boldsymbol{\nu}}(\boldsymbol{t}) \cdot u_{\boldsymbol{\nu}} \in \mathcal{L}^{k-1}(V)$$

of $\mathcal{R}_{k-1}$ over a neighbourhood V of $\boldsymbol{b}$. We have

$$
\begin{aligned}
A\left(j^k\left(\frac{\partial f}{\partial \tau_i}\right)\right) &= \sum_{|\boldsymbol{\nu}|\le k-1} \alpha_{\boldsymbol{\nu}}(\boldsymbol{t})\cdot u_{\boldsymbol{\nu}}\left(j^k\left(\frac{\partial f}{\partial \tau_i}\right)\right)\\
&= \sum_{|\boldsymbol{\nu}|\le k-1} (\nu_i+1)\alpha_{\boldsymbol{\nu}}(\boldsymbol{t})\cdot u_{\boldsymbol{\nu}+\boldsymbol{e}_i}(j^k f)\\
&= \sum_{|\boldsymbol{\nu}|\le k-1} \bigl(\alpha_{\boldsymbol{\nu}}^{(\boldsymbol{e}_i)}(\boldsymbol{t})\cdot u_{\boldsymbol{\nu}} + (\nu_i+1)\alpha_{\boldsymbol{\nu}}(\boldsymbol{t})\cdot u_{\boldsymbol{\nu}+\boldsymbol{e}_i}\bigr)(j^k f) = 0.
\end{aligned}
$$

Here, since $|\boldsymbol{\nu}| \le k-1$ implies $u_{\boldsymbol{\nu}}(j^k f) = 0$, the third equality follows. The last equality follows from the first prolongations stated above. This proves that $\partial f/\partial\tau_i \in \mathcal{R}_{k-1,\boldsymbol{b}}$. The inclusion $\partial f/\partial\tau_i \in \operatorname{Ker} \Sigma_{\boldsymbol{b}}^{k-1}$ follows from $f(\boldsymbol{t},\boldsymbol{\tau}) \in \operatorname{Ker} \Sigma^k$ (homogeneity of f). Evaluating at $\boldsymbol{b} \in U_Z^k \cap U_Z^{k-1}$, we have shown that $Z_{\boldsymbol{b}}{\downarrow}$ is D-invariant at degree k. Altogether, $Z_{\boldsymbol{b}}{\downarrow}$ is D-invariant on the open subset

$$U_Z^{\mathrm{bdl}} := U_Z^0 \cap U_Z^1 \cap \cdots = U_Z^0 \cap \cdots \cap U_Z^{m-1}$$

$(m := \dim_{\mathbb{C}} Z)$ described in Proposition 3.6. ∎

By Corollary 3.5, there exists $k \le m-1$ such that a system of linear PDEs of order $k+1$ is sufficient to select the sections of Z, namely Z is defined by $\mathcal{L}^k$. We may call the sheaf $\mathcal{L}^k$ for such k the *defining system of PDEs* for Z. In particular, $\mathcal{L}^{m-1}$ is a defining system of Z.

5. Sesquilinear forms and weak topologies. Here we recall the sesquilinear form on the product $\mathbb{C}[\boldsymbol{\tau}]\times\mathbb{C}\{\boldsymbol{t}\}$ of the space of polynomials and the convergent power series algebra. The restriction of this sesquilinear form to the product of a finite-dimensional subspace $Z \subset \mathbb{C}\{\boldsymbol{t}\}$ and its least space $Z_{\boldsymbol{b}}{\downarrow} \subset \mathbb{C}[\boldsymbol{\tau}]$ was proved by de Boor–Ron [BR$_1$], [BR$_2$] to be non-degenerate. We choose the notation and prove their properties via the statements on bilinear forms in Bourbaki [Bo$_2$].

Let us define a complex bilinear form

$$B_{n,\boldsymbol{b}} : \mathbb{C}[\boldsymbol{\tau}] \times \mathcal{O}_{n,\boldsymbol{b}} \to \mathbb{C}, \qquad (p,f) \mapsto B_{n,\boldsymbol{b}}\langle p \parallel f\rangle,$$

by

$$B_{n,\boldsymbol{b}}\Bigl\langle \sum_{\text{finite}} a_{\boldsymbol{\nu}}\boldsymbol{\tau}^{\boldsymbol{\nu}} \Bigm\| \sum b_{\boldsymbol{\mu}}(\boldsymbol{t}-\boldsymbol{b})^{\boldsymbol{\mu}}\Bigr\rangle := \sum_{\text{finite}} \boldsymbol{\nu}!\, a_{\boldsymbol{\nu}} b_{\boldsymbol{\nu}},$$

where $\boldsymbol{\nu}! = \nu_1!\cdots\nu_m!$. In particular,

$$
\begin{aligned}
B_{n,\boldsymbol{b}}\langle \boldsymbol{\tau}^{\boldsymbol{\nu}} \parallel (\boldsymbol{t}-\boldsymbol{b})^{\boldsymbol{\mu}}\rangle &= \frac{\partial^{|\boldsymbol{\nu}|}(\boldsymbol{t}-\boldsymbol{b})^{\boldsymbol{\mu}}}{\partial \boldsymbol{t}^{\boldsymbol{\nu}}}(\boldsymbol{b})\\
&= \frac{\partial^{\nu_1+\cdots+\nu_n}(\boldsymbol{t}-\boldsymbol{b})^{\boldsymbol{\mu}}}{\partial t_1^{\nu_1}\ldots\partial t_n^{\nu_n}}(\boldsymbol{b}) = \begin{cases} \boldsymbol{\nu}! & (\boldsymbol{\nu}=\boldsymbol{\mu}),\\ 0 & (\boldsymbol{\nu}\ne\boldsymbol{\mu}).\end{cases}
\end{aligned}
$$

Thus the monomial $\boldsymbol{\tau}^{\boldsymbol{\nu}}$ can be identified with the signed higher order derivative $(-1)^{|\boldsymbol{\nu}|}\delta_{\boldsymbol{b}}^{(\boldsymbol{\nu})}$ of the Dirac delta function supported at $\boldsymbol{b} \in \mathbb{C}^m$. The sign $(-1)^{|\boldsymbol{\nu}|}$ here originates from partial integration in Schwartz's distribution theory [Sc, (II, 1;7)].

Now let u denote the complex conjugations:

$$u : \mathbb{C}[\boldsymbol{\tau}] \to \mathbb{C}[\boldsymbol{\tau}], \qquad \sum_{\text{finite}} a_{\boldsymbol{\nu}} \boldsymbol{\tau}^{\boldsymbol{\nu}} \mapsto \sum_{\text{finite}} \bar{a}_{\boldsymbol{\nu}} \boldsymbol{\tau}^{\boldsymbol{\nu}},$$

$$u : \mathcal{O}_{n,\boldsymbol{b}} \to \mathcal{O}_{n,\boldsymbol{b}}, \qquad \sum b_{\boldsymbol{\mu}} (\boldsymbol{t}-\boldsymbol{b})^{\boldsymbol{\mu}} \mapsto \sum \bar{b}_{\boldsymbol{\mu}} (\boldsymbol{t}-\boldsymbol{b})^{\boldsymbol{\mu}}.$$

The sesquilinear form

$$S_{n,\boldsymbol{b}} : \mathbb{C}[\boldsymbol{\tau}] \times \mathcal{O}_{n,\boldsymbol{b}} \to \mathbb{C}, \qquad (p, f) \mapsto S_{n,\boldsymbol{b}}\langle p \mid f\rangle,$$

is defined by

$$S_{n,\boldsymbol{b}}\langle p \mid f\rangle := B_{n,\boldsymbol{b}}\langle p \parallel u(f)\rangle.$$

This can also be written as

$$S_{n,\boldsymbol{b}} := B_{n,\boldsymbol{b}} \circ (\mathrm{id}_{\mathbb{C}[\boldsymbol{\tau}]}, u).$$

Explicitly, we have

$$S_{n,\boldsymbol{b}}\Big\langle \sum_{\text{finite}} a_{\boldsymbol{\nu}} \boldsymbol{\tau}^{\boldsymbol{\nu}} \,\Big|\, \sum b_{\boldsymbol{\mu}} (\boldsymbol{t}-\boldsymbol{b})^{\boldsymbol{\mu}} \Big\rangle = \sum \boldsymbol{\nu}! a_{\boldsymbol{\nu}} \bar{b}_{\boldsymbol{\nu}}.$$

The *weak topology* of $\mathbb{C}[\boldsymbol{\tau}]$ with respect to $B_{n,\boldsymbol{b}}$ is the coarsest topology such that the linear functionals

$$b_{\|f} : \mathbb{C}[\boldsymbol{\tau}] \to \mathbb{C}, \qquad p \mapsto B_{n,\boldsymbol{b}}\langle p \parallel f\rangle,$$

are continuous for all $f \in \mathcal{O}_{n,\boldsymbol{b}}$. Similarly the *weak topology* of $\mathcal{O}_{n,\boldsymbol{b}}$ with respect to $B_{n,\boldsymbol{b}}$ is the coarsest topology such that the linear functionals

$$b_{p\|} : \mathcal{O}_{n,\boldsymbol{b}} \to \mathbb{C}, \qquad f \mapsto B_{n,\boldsymbol{b}}\langle p \parallel f\rangle,$$

are continuous for all $p \in \mathbb{C}[\boldsymbol{\tau}]$. With these topologies, $\mathbb{C}[\boldsymbol{\tau}]$ and $\mathcal{O}_{n,\boldsymbol{b}}$ become topological vector spaces.

REMARK 5.1. We have chosen the sesquilinear form expressed by the diagonal matrix with diagonal elements $\boldsymbol{\nu}!$ but note that this does not have intrinsic legitimacy. This form may be transformed to a different positive definite Hermitian matrix in other affine coordinates but the weak topologies remain unchanged.

If L is a vector subspace of $\mathbb{C}[\boldsymbol{\tau}]$, its annihilator space (orthogonal space) with respect to $B_{n,\boldsymbol{b}}$ is denoted by $L^\top$:

$$L^\top := \{f : B_{n,\boldsymbol{b}}\langle p \parallel f\rangle = 0 \text{ for all } p \in L\}.$$

Similarly, if K is a vector subspace of $\mathcal{O}_{n,\boldsymbol{b}}$, its annihilator space with respect to $B_{n,\boldsymbol{b}}$ is denoted by $K^{\top}$:

$$K^{\top} := \{p : B_{n,\boldsymbol{b}}\langle p \parallel f\rangle = 0 \text{ for all } f \in K\}.$$

These are vector subspaces. We adopt the abbreviation $L^{\top\top} = (L^{\top})^{\top}$ and $K^{\top\top} = (K^{\top})^{\top}$, etc. Recall that $L^{\top\top}$ and $K^{\top\top}$ are the weak closures of L and K respectively [Bo$_2$, 4, §1, n°3, 4].

The *weak topologies* of $\mathbb{C}[\boldsymbol{\tau}]$ and $\mathcal{O}_{n,\boldsymbol{b}}$ with respect to $S_{n,\boldsymbol{b}}$ are defined in a similar way to those with respect to $B_{n,\boldsymbol{b}}$ and they yield topological vector spaces. The weak topology of $\mathcal{O}_{n,\boldsymbol{b}}$ is nothing but that of coefficient-wise convergence. If $L \subset \mathbb{C}[\boldsymbol{\tau}]$ and $K \subset \mathcal{O}_{n,\boldsymbol{b}}$ are vector subspaces, their annihilator spaces with respect to $S_{n,\boldsymbol{b}}$ are denoted by $L^{\perp}$, $K^{\perp}$:

$$L^{\perp} := \{f : S_{n,\boldsymbol{b}}\langle p \mid f\rangle = 0 \text{ for all } p \in L\},$$
$$K^{\perp} := \{p : S_{n,\boldsymbol{b}}\langle p \mid f\rangle = 0 \text{ for all } f \in K\}.$$

These are also vector subspaces. We write $L^{\perp\perp} = (L^{\perp})^{\perp}$ and $K^{\perp\perp} = (K^{\perp})^{\perp}$, etc. It is obvious that

$$S_{n,\boldsymbol{b}}\langle p \mid f\rangle = B_{n,\boldsymbol{b}}\langle p \parallel u(f)\rangle = \overline{B_{n,\boldsymbol{b}}\langle u(p) \parallel f\rangle},$$
$$B_{n,\boldsymbol{b}}\langle p \parallel f\rangle = S_{n,\boldsymbol{b}}\langle p \mid u(f)\rangle = \overline{S_{n,\boldsymbol{b}}\langle u(p) \mid f\rangle}.$$

Defining $s_{|f}$, $s_{p|}$ in a similar way to $b_{||f}$, $b_{p||}$, we have

$$s_{|u(f)} = b_{\|f}, \qquad s_{|f} = b_{\|u(f)}, \qquad s_{p|} = b_{p\|} \circ u, \qquad b_{p\|} = s_{p|} \circ u.$$

Since u is an involution (i.e. $u \circ u$ is the identity) and commutes with the operations $^{\perp}$ and $^{\top}$, we have the equalities

$$\{s_{|f} : f \in \mathcal{O}_{n,\boldsymbol{b}}\} = \{b_{|f} : f \in \mathcal{O}_{n,\boldsymbol{b}}\}, \qquad \{s_{p\|} : p \in \mathbb{C}[\boldsymbol{\tau}]\} = \{b_{p\|} : p \in \mathbb{C}[\boldsymbol{\tau}]\}.$$

Hence we obtain the following.

Proposition 5.2. *The weak topologies on $\mathbb{C}[\boldsymbol{\tau}]$ (resp. $\mathcal{O}_{n,\boldsymbol{b}}$) with respect to $B_{n,\boldsymbol{b}}$ and $S_{n,\boldsymbol{b}}$ coincide. If L (resp. K) is a vector subspace of $\mathbb{C}[\boldsymbol{\tau}]$ (resp. $\mathcal{O}_{n,\boldsymbol{b}}$), then*

$$L^{\perp} = u(L^{\top}) = (u(L))^{\top}, \qquad K^{\top} = u(K^{\perp}) = (u(K))^{\perp},$$
$$K^{\perp} = u(K^{\top}) = (u(K))^{\top}, \qquad L^{\top} = u(L^{\perp}) = (u(L))^{\perp},$$

and hence

$$L^{\perp\perp} = L^{\top\top}, \qquad K^{\top\top} = K^{\perp\perp}.$$

By the first assertion, we have no need to refer to forms $B_{n,\boldsymbol{b}}$ and $S_{n,\boldsymbol{b}}$ for weak topologies. Now we can deduce a few properties of the space of annihilators with respect to the sesquilinear forms from those with respect to bilinear forms [Bo$_2$, 4, §1, n°4].

Corollary 5.3. *Let L (resp. K) be a vector subspace of $\mathbb{C}[\boldsymbol{\tau}]$ (resp. $\mathcal{O}_{n,\boldsymbol{b}}$). Then:*

(1) *$K^{\perp}$ and $L^{\perp}$ are all weakly closed.*
(2) *$L^{\perp\perp}$ is the weak closure of L in $\mathbb{C}[\boldsymbol{\tau}]$ and $K^{\perp\perp}$ is the weak closure of K in $\mathcal{O}_{n,\boldsymbol{b}}$.*
(3) *$L^{\perp\perp\perp} = L^{\perp}$ and $K^{\perp\perp\perp} = K^{\perp}$.*

We can easily prove the following.

Proposition 5.4. *The form $S_{n,\boldsymbol{b}} : \mathbb{C}[\boldsymbol{\tau}] \times \mathcal{O}_{n,\boldsymbol{b}} \to \mathbb{C}$ is a non-degenerate sesquilinear form, i.e. $\mathbb{C}[\boldsymbol{\tau}]^{\perp} = \{0\}$ and $\mathcal{O}_{n,\boldsymbol{b}}^{\perp} = \{0\}$.*

Following the notation of §2, write

$$\boldsymbol{\tau}^{\boldsymbol{\nu}} := (\boldsymbol{t} - \boldsymbol{b})_{\boldsymbol{b}}^{\boldsymbol{\nu}}{\downarrow} \in \mathfrak{m}_{n,\boldsymbol{b}}^{|\boldsymbol{\nu}|}/\mathfrak{m}_{n,\boldsymbol{b}}^{|\boldsymbol{\nu}|+1} \subset \mathbb{C}[\boldsymbol{\tau}] = \mathcal{O}_{n,\boldsymbol{b}}{\downarrow}$$

for elements of the least spaces. The following is a most natural and efficient construction of a dual space of a finite-dimensional subspace of $\mathcal{O}_{n,\boldsymbol{b}}$.

Theorem 5.5 (de Boor–Ron [BR$_2$, Theorem 5.8]). *Let Z be a vector subspace of $\mathcal{O}_{n,\boldsymbol{b}}$. Then the sesquilinear form*

$$S_Z : Z_{\boldsymbol{b}}{\downarrow} \times Z \to \mathbb{C}$$

obtained as the restriction of $S_{n,\boldsymbol{b}} : \mathbb{C}[\boldsymbol{\tau}] \times \mathcal{O}_{n,\boldsymbol{b}} \to \mathbb{C}$ is non-degenerate, i.e. $\mathbb{C}[\boldsymbol{\tau}]^{\perp_Z} = \{0\}$ and $\mathcal{O}_{n,\boldsymbol{b}}^{\perp_Z} = \{0\}$, where $\perp_Z$ denotes the space of annihilators with respect to S_Z.

Proof. Suppose that $f \neq 0$ belongs to Z and annihilates $Z_{\boldsymbol{b}}{\downarrow}$. We can write f as

$$f = \sum_{|\boldsymbol{\nu}| \geq d} a_{\boldsymbol{\nu}} \boldsymbol{\tau}^{\boldsymbol{\nu}}, \qquad d := \operatorname{ord}_{\boldsymbol{b}} f < \infty.$$

Then we have

$$0 = S_Z(f_{\boldsymbol{b}}{\downarrow}, f) = \sum_{|\boldsymbol{\nu}| = d} \boldsymbol{\nu}! |a_{\boldsymbol{\nu}}|^2 \neq 0,$$

a contradiction. This proves that $(Z_{\boldsymbol{b}}{\downarrow})^{\perp_Z} = \{0\}$.

Suppose that $p \neq 0$ belongs to $Z_{\boldsymbol{b}}{\downarrow}$ and annihilates Z. Let $p_{\boldsymbol{b}}{\uparrow} \neq 0$ denote the highest degree homogeneous part of p at $\boldsymbol{b}$. Since $Z_{\boldsymbol{b}}{\downarrow}$ is generated by homogeneous elements, there exists $g \in Z$ such that $p_{\boldsymbol{b}}{\uparrow} = g_{\boldsymbol{b}}{\downarrow}$. Then

$$0 < S_n \langle p_{\boldsymbol{b}}{\uparrow} \mid p_{\boldsymbol{b}}{\uparrow} \rangle = S_n \langle p_{\boldsymbol{b}}{\uparrow} \mid g_{\boldsymbol{b}}{\downarrow} \rangle = S_Z \langle p \mid g \rangle = 0,$$

a contradiction. This proves that $Z^{\perp} = \{0\}$. ∎

Let us recall an elementary fact: existence of the *adjoint mapping.*

Lemma 5.6. *Let $S : Q \times Z \to \mathbb{C}$ and $S' : Q' \times Z' \to \mathbb{C}$ be non-degenerate sesquilinear forms defined on products of topological vector spaces equipped* *with the weak topologies with respect to S and S' respectively, and let $\kappa : Z \to Z'$ be a linear mapping. Then the following are equivalent:*

(1) *κ is weakly continuous.*
(2) *There exists a linear mapping ${}^s\kappa : Q' \to Q$ (which we call the adjoint of κ) such that*

$$\forall p \in Q, \forall f \in Z' : \quad S\langle \kappa(p) \mid f\rangle = S'\langle p \mid {}^s\kappa(f)\rangle.$$

The linear mapping ${}^s\kappa$ which satisfies (2) is unique for κ. Each of these conditions implies that ${}^s\kappa$ is weakly continuous and ${}^{ss}\kappa := {}^s({}^s\kappa) = \kappa$.

Proof. Let u denote the complex conjugation $\sum a_{\boldsymbol{\mu}}(\boldsymbol{t}-\boldsymbol{b})^{\boldsymbol{\mu}} \mapsto \sum \bar{a}_{\boldsymbol{\mu}}(\boldsymbol{t}-\boldsymbol{b})^{\boldsymbol{\mu}}$ and define the bilinear forms $B : Q \times Z \to \mathbb{C}$ and $B' : Q' \times Z' \to \mathbb{C}$ by

$$B\langle p \parallel f\rangle := S\langle p \mid u(f)\rangle, \qquad B'\langle p \parallel f\rangle := S'\langle p \mid u(f)\rangle$$

(cf. the third paragraph of §5). If κ is weakly continuous, it has the weakly continuous transpose mapping, i.e. there exists a linear mapping ${}^t\kappa : Q' \to Q$ such that

$$\forall p \in Q, \forall f \in Z' : \quad B\langle \kappa(p) \parallel f\rangle = B'\langle p \parallel {}^t\kappa(f)\rangle,$$

by [Bo$_2$, 4, §4, n°1]. We have only to set ${}^s\kappa := u \circ {}^t\kappa \circ u$. The rest follows from the corresponding properties of bilinear forms. ∎

Lemma 5.7. *Multiplication by $p(\boldsymbol{\tau})$ in $\mathbb{C}[\boldsymbol{\tau}]$ is the adjoint of the differential operator $u(p)(\boldsymbol{\partial_t})$ ($\boldsymbol{\partial_t} := (\partial/\partial t_1, \ldots, \partial/\partial t_n)$) on $\mathcal{O}_{n,\boldsymbol{b}}$ with respect to $S_{n,\boldsymbol{b}}$. In particular, multiplication by τ_i is the adjoint of differentiation in t_i. Similarly, multiplication by $f(\boldsymbol{t})$ in $\mathcal{O}_{n,\boldsymbol{b}}$ is the adjoint of the infinite differential operator $u(f)(\boldsymbol{\partial_\tau})$ ($\boldsymbol{\partial_\tau} := (\partial/\partial \tau_1, \ldots, \partial/\partial \tau_n)$) on $\mathbb{C}[\boldsymbol{\tau}]$ with respect to $S_{n,\boldsymbol{b}}$. (Such an operator is valid for polynomials.) In particular, multiplication by t_i is the adjoint of differentiation in τ_i.*

Proof. First note that these operations are weakly continuous and hence they have weakly continuous adjoinds by Lemma 5.6. The second assertion is obvious from the direct calculations:

$$\begin{aligned} S_{n,\boldsymbol{b}}\Big\langle \sum_{\boldsymbol{\lambda}} c_{\boldsymbol{\lambda}} \boldsymbol{\partial}_{\boldsymbol{\tau}}^{\boldsymbol{\lambda}} \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}} \boldsymbol{\tau}^{\boldsymbol{\nu}} \,\Big|\, \sum_{\boldsymbol{\mu}} b_{\boldsymbol{\mu}} \boldsymbol{t}^{\boldsymbol{\mu}} \Big\rangle &= \sum_{\boldsymbol{\nu},\boldsymbol{\lambda}} (\boldsymbol{\lambda}+\boldsymbol{\nu})! c_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}+\boldsymbol{\nu}} \bar{b}_{\boldsymbol{\nu}} \\ = \sum_{\boldsymbol{\nu}} \sum_{\boldsymbol{\mu} \ge \boldsymbol{\nu}} \boldsymbol{\mu}! a_{\boldsymbol{\mu}} \bar{b}_{\boldsymbol{\nu}} c_{\boldsymbol{\mu}-\boldsymbol{\nu}} &= S_{n,\boldsymbol{b}}\Big\langle \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\nu}} \boldsymbol{\tau}^{\boldsymbol{\nu}} \,\Big|\, \sum_{\boldsymbol{\lambda}} \bar{c}_{\boldsymbol{\lambda}} \boldsymbol{t}^{\boldsymbol{\lambda}} \sum_{\boldsymbol{\mu}} b_{\boldsymbol{\mu}} \boldsymbol{t}^{\boldsymbol{\mu}} \Big\rangle, \end{aligned}$$

where $\ge$ denotes the product order of the order of the set of integers. The proof of the first inequality is quite similar. ∎

6. Weak topologies of analytic algebras. Here we recall some topological properties of analytic algebras over $\mathbb{C}$ and their homomorphisms. The main reference is Grauert–Remmert [GR].

An algebra isomorphic to a factor algebra of a convergent power series algebra by a proper ideal is called a (*local*) *analytic algebra*. Take an analytic algebra $A := \mathcal{O}_{n,\boldsymbol{b}}/I$. This is a local $\mathbb{C}$-algebra in the sense it has a unique maximal ideal $\mathfrak{m}_A$, which consists of the residue classes of elements of $\mathfrak{m}_{n,\boldsymbol{b}}$, and A is a vector space over the subalgebra $\mathbb{C} \subset A$ such that the canonical homomorphism

$$\mathbb{C} \to A/\mathfrak{m}_A, \qquad \lambda \mapsto \lambda \cdot 1 \bmod \mathfrak{m}_A,$$

of fields is an isomorphism. A homomorphism of algebras is always assumed to be unitary: $1 \mapsto 1$. Then any homomorphism $\varphi : B \to A$ of local $\mathbb{C}$-algebras is local: $\varphi(\mathfrak{m}_B) \subset \mathfrak{m}_A$.

Following Grauert–Remmert [GR], let us see that the weak topology on A is independent of the expression $A = \mathcal{O}_{n,\boldsymbol{b}}/I$. Let $\pi_i : A \to A/\mathfrak{m}_A^i$ ($i \in \mathbb{N}$) be the factor epimorphism. The analytic algebra $A/\mathfrak{m}_A^i$ is a finite-dimensional $\mathbb{C}$-vector space and hence it has a unique structure of a topological vector space by [Bo$_2$, 1, §2, n°3, Th. 2]. We give A the coarsest topology such that all the π_i ($i \in \mathbb{N}$) are continuous and call it the *projective topology*. Of course this is independent of the expression $\mathcal{O}_{n,\boldsymbol{b}}/I$. Note that the projective topology is called "schwache Topologie" in [GR], which will be proved to be the same as our weak topology in the following lemma.

LEMMA 6.1. *Let $I \subset \mathcal{O}_{n,\boldsymbol{b}}$ be an ideal and $A := \mathcal{O}_{n,\boldsymbol{b}}/I$ the analytic algebra. Then:*

(1) *On a regular analytic algebra $\mathcal{O}_{n,\boldsymbol{b}}$, the projective topology and the weak topology coincide.*
(2) *The ideal I is closed with respect to both the projective topology and the weak topology with respect to $S_{n,\boldsymbol{b}} : \mathbb{C}[\boldsymbol{\tau}] \times \mathcal{O}_{n,\boldsymbol{b}} \to \mathbb{C}$.*
(3) *The sesquilinear form $S_{n,\boldsymbol{b}}$ induces a non-degenerate form*

$$S_A : I^{\perp_n} \times A \to \mathbb{C}.$$

(4) *The weak topology of $I^{\perp_n}$ (resp. of A) with respect to S_A coincides with the relative topology from $\mathcal{O}_{n,\boldsymbol{b}}$ (resp. the factor topology of $\mathcal{O}_{n,\boldsymbol{b}}$).*
(5) *The projective topology of $A := \mathcal{O}_{n,\boldsymbol{b}}/I$ coincides with the factor topology of $\mathcal{O}_{n,\boldsymbol{b}}$ with the projective topology.*
(6) *The weak topology and the projective topology on A coincide and the factor epimorphism $\mathcal{O}_{n,\boldsymbol{b}} \to A$ is always a weakly continuous and open mapping.*

Proof. The first assertion is easy to see. Closeness is proved in [GR, II, §1, Satz 2], and hence $I = I^{\perp_n \perp_n}$. Then $S_{n,\boldsymbol{b}}$ induces a non-degenerate sesquilinear form S_A on $I^{\perp_n} \times A \cong I^{\perp_n} \times (\mathcal{O}_{n,\boldsymbol{b}}/I^{\perp_n \perp_n})$ by [Bo$_2$, 4, §1, n°5, Prop. 5]. This pairing defines a weak topology on A. It coincides with the factor topology of $\mathcal{O}_{n,\boldsymbol{b}}$ by [Bo$_2$, 4, §1, n°5, Prop. 7]. The weak topology on $I^{\perp_n}$ coincides with the relative one by [Bo$_2$, 4, §1, n°5, Prop. 6]. The projective topology on A coincides with the factor topology of $\mathcal{O}_{n,\boldsymbol{b}}$ by [GR, II, §1, Satz 3]. Assertions (1), (4) and (5) imply that the two topologies on A coincide. Continuity and openness are obvious properties of a factor morphism of topological groups. ■

Corollary 6.2. *Let $\varphi : B \to A$ be a homomorphism of analytic algebras. Then φ is weakly continuous.*

Proof. Suppose that $A = \mathcal{O}_{m,\boldsymbol{b}}/I$ and $B = \mathcal{O}_{n,\boldsymbol{b}}/J$, and let $\pi_A : \mathcal{O}_{m,\boldsymbol{a}} \to A$ and $\pi_B : \mathcal{O}_{n,\boldsymbol{b}} \to B$ denote the factor epimorphisms. Then φ lifts to $\tilde{\varphi} : \mathcal{O}_{n,\boldsymbol{b}} \to \mathcal{O}_{m,\boldsymbol{a}}$ (see e.g. [GR, II, §0, Satz 3]). Since $\tilde{\varphi}$ is obtained by substituting for y_i elements of the maximal ideal of $\mathcal{O}_{n,\boldsymbol{b}}$, it is easy to see that $\tilde{\varphi}$ is weakly continuous. Then the inverse image $(\pi_A \circ \tilde{\varphi})^{-1}(U)$ of an open subset $U \subset A$ is open by Lemma 6.1. This implies that $\varphi^{-1}(U)$ is open again by Lemma 6.1 and so φ is weakly continuous. ■

7. Projector to a vector subspace. Bos and Calvi used the least space of de Boor and Ron to define the Taylor projector. Their construction can be abstracted as follows. Let $Z_{\boldsymbol{b}}$ be a finite-dimensional vector subspace of the local analytic algebra $\mathcal{O}_{n,\boldsymbol{b}}$. Then there exists a retraction (projector) $\mathcal{T}_{Z,\boldsymbol{b}} : \mathcal{O}_{n,\boldsymbol{b}} \to Z_{\boldsymbol{b}}$ of vector spaces. By the results of former sections, at a general point $\boldsymbol{b}$, the kernel of $\mathcal{T}_{Z,\boldsymbol{b}}$ is an ideal and the space $Z_{\boldsymbol{b}}$ has the structure of an Artinian algebra as a quotient of $\mathcal{O}_{n,\boldsymbol{b}}$.

Let $\varphi : \mathcal{O}_{m,\boldsymbol{a}} \to \mathcal{O}_{n,\boldsymbol{b}}$ be a homomorphism of analytic algebras. Its adjoint

$$^{s}\varphi : \mathbb{C}[\boldsymbol{\tau}] = \mathcal{O}_{n,\boldsymbol{b}}{\downarrow} \to \mathbb{C}[\boldsymbol{\xi}] = \mathcal{O}_{m,\boldsymbol{a}}{\downarrow}$$

exists by Lemma 5.6. This is nothing but the push-forward of derivatives of Dirac delta (a distribution with one-point support $\{\boldsymbol{a}\}$, see Schwartz [Sc, (III, 10;1), (IX, 4;1)]). Its concrete forms are given by multivariate versions of Faà di Bruno's formula (the formula for the higher order derivatives of composite multivariate functions, see e.g. [LP], [Ma]). Note that the image of a homogeneous element by $^{s}\varphi$ is not always homogeneous. It is a troublesome task to calculate this formula by hand.

Definition 7.1. For a finite-dimensional vector subspace $Z_{\boldsymbol{b}} \subset \mathcal{O}_{n,\boldsymbol{b}}$, we have defined a non-degenerate sesquilinear form

$$S_{Z,\boldsymbol{b}} : Z_{\boldsymbol{b}}{\downarrow} \times Z_{\boldsymbol{b}} \to \mathbb{C}$$

induced from $S_{n,\boldsymbol{b}}$ using some fixed coordinates $\boldsymbol{t} = (t_1, \ldots, t_n)$ (Theorem 5.5). The topology of $Z_{\boldsymbol{b}}$ (resp. $Z_{\boldsymbol{b}}{\downarrow}$) as a topological vector space is unique by [Bo$_2$, 1, §2, n°3, Th. 2]. Let $\iota : Z_{\boldsymbol{b}}{\downarrow} \to \mathcal{O}_{n,\boldsymbol{b}}{\downarrow}$ and $\kappa : Z_{\boldsymbol{b}} \to \mathcal{O}_{n,\boldsymbol{b}}$ denote the inclusion mappings. Since all linear mappings defined on a finite-dimensional space are continuous by [Bo$_2$, 1, §2, n°3, Cor. 2], the mappings ι and κ are continuous. Then we have the weakly continuous adjoint linear mappings

$\mathcal{T}_{Z,\boldsymbol{b}} := {}^s\iota : \mathcal{O}_{n,\boldsymbol{b}} \to Z_{\boldsymbol{b}}$ of ι and ${}^s\kappa : \mathcal{O}_{n,\boldsymbol{b}} \to Z_{\boldsymbol{b}}\!\downarrow$ of κ by Lemma 5.6. Thus we have the following diagram, where the bold vertical lines indicate sesquilinear pairings:

$$\begin{tikzcd} Z_{\boldsymbol{b}} \arrow[r, hook, shift left, "\kappa"] \arrow[d, dash, very thick, "S_{Z,\boldsymbol{b}}"'] & \mathcal{O}_{n,\boldsymbol{b}} = \mathbb{C}\{\boldsymbol{t}-\boldsymbol{b}\} \arrow[l, dashed, shift left, "\mathcal{T}_{Z,\boldsymbol{b}}"] \arrow[d, dash, very thick, "S_{n,\boldsymbol{b}}"] \\ Z_{\boldsymbol{b}}\!\downarrow \arrow[r, hook, shift left, "\iota"] & \mathcal{O}_{n,\boldsymbol{b}}\!\downarrow = \mathbb{C}[\boldsymbol{\tau}] \arrow[l, dashed, shift left, "{}^s\kappa"] \end{tikzcd}$$

Diagram 2. Projector

PROPOSITION 7.2. *Let $Z_{\boldsymbol{b}} \subset \mathcal{O}_{n,\boldsymbol{b}}$ be a finite-dimensional vector subspace. Then:*

(1) *The mappings $\mathcal{T}_{Z,\boldsymbol{b}}$ and ${}^s\kappa$ are weakly continuous.*

(2) *We have the equalities*

$$\operatorname{Ker} \mathcal{T}_{Z,\boldsymbol{b}} = (Z_{\boldsymbol{b}}\!\downarrow)^{\perp_n}, \qquad (\operatorname{Ker} \mathcal{T}_{Z,\boldsymbol{b}})^{\perp_n} = Z_{\boldsymbol{b}}\!\downarrow,$$
$$\operatorname{Ker} {}^s\kappa = (Z_{\boldsymbol{b}})^{\perp_n}, \qquad (\operatorname{Ker} {}^s\kappa)^{\perp_n} = Z_{\boldsymbol{b}},$$

where $\perp_n$ denotes the subspace of annihilators with respect to $S_{n,\boldsymbol{b}}$.

(3) *The mappings $\mathcal{T}_{\boldsymbol{a},d}$ and ${}^s\kappa$ are retractions of vector spaces, i.e. $\mathcal{T}_{Z,\boldsymbol{b}} \circ \kappa$ and ${}^s\kappa \circ \iota$ are the identities.*

Proof. Property (1) is already stated above. The first equality of (2) follows from the fact that the sesquilinear form $S_{n,\boldsymbol{b}}$ is non-degenerate. Since $Z_{\boldsymbol{b}}\!\downarrow$ is finite-dimensional, it is weakly closed and $Z_{\boldsymbol{b}}\!\downarrow = (Z_{\boldsymbol{b}}\!\downarrow)^{\perp_n\perp_n}$. Then the first equality implies the second. The remaining assertions of (2) are proved quite similarly. If $f \in Z_{\boldsymbol{b}}$ and $p \in Z_{\boldsymbol{b}}\!\downarrow$, we have

$$S_{Z,\boldsymbol{b}}\langle p \mid f\rangle = S_{n,\boldsymbol{b}}\langle \iota(p) \mid \kappa(f)\rangle = S_{Z,\boldsymbol{b}}\langle p \mid \mathcal{T}_{Z,\boldsymbol{b}} \circ \kappa(f)\rangle.$$

Since $S_{Z,\boldsymbol{b}}$ is non-degenerate, this implies that $\mathcal{T}_{Z,\boldsymbol{b}} \circ \kappa$ is the identity. Similarly ${}^s\kappa \circ \iota$ is also the identity. ∎

REMARK 7.3. A retraction of a space F of smooth functions is closely related to interpolation. Let Z be a finite-dimensional vector subspace of F. An interpolation problem is the task of finding a function $f \in Z$ which, together with its higher order derivatives, takes the prescribed values at a finite number of points $\boldsymbol{b}_i$. These quantities, the values of the function itself or its higher order derivatives at $\boldsymbol{b}_i$, can be expressed by Schwartz distributions p_i $(1 \le i \le q)$ supported at $\boldsymbol{b}_i$ as we have seen in §5, and can be expressed as elements of $\mathbb{C}[\boldsymbol{\tau}]$. For simplicity, consider the case when the support consists of a single point $\boldsymbol{b}$. If we apply the distributions p_i to $f \in \mathcal{O}_{n,\boldsymbol{b}}$, we get "interpolation data" $S_{n,\boldsymbol{b}}\langle p_i \mid f\rangle \in \mathbb{C}$ $(1 \le i \le q)$. Take a finite-dimensional space $Z_{\boldsymbol{b}} \subset \mathcal{O}_{n,\boldsymbol{b}}$. Since

$$S_{n,\boldsymbol{b}}\langle p_i \mid f\rangle = S_{Z,\boldsymbol{b}}\langle p_i \mid \mathcal{T}_{Z,\boldsymbol{b}}(f)\rangle \quad (f \in \mathcal{O}_{n,\boldsymbol{b}}),$$

f and $T_{Z,\boldsymbol{b}}(f)$ have the same data for the interpolation quantities $p_i \in Z_{\boldsymbol{b}}\downarrow$. Thus the data of f is interpolated by an element $T_{Z,\boldsymbol{b}}(f) \in Z_{\boldsymbol{b}}$.

LEMMA 7.4. *Let $Z_{\boldsymbol{b}} \subset \mathcal{O}_{n,\boldsymbol{b}}$ be a finite-dimensional vector subspace. Then the following conditions are equivalent:*

(1) *The least space $Z_{\boldsymbol{b}}\downarrow$ is D-invariant.*
(2) $\operatorname{Ker} T_{Z,\boldsymbol{b}} = (Z_{\boldsymbol{b}}\downarrow)^{\perp_n}$ *is an ideal.*

Condition (2) is the property of an ideal projector of Birkhoff [Bi]. This property is important because it ensures that $T_{Z,\boldsymbol{b}}$ induces a factor epimorphism of rings. Equivalence of (1) and (2) appears in M. G. Marinari, H. M. Möller and T. Mora [MMM, Proposition 2.4], and de Boor–Ron [BR$_2$, Proposition 6.1]. Interpolation defined by a projector with this property is sometimes called Hermite interpolation but it was suggested to use this word for interpolation with better properties (de Boor–Shekhtman [BS]).

Proof. Suppose that $Z_{\boldsymbol{b}}\downarrow$ is D-invariant. If $p \in Z_{\boldsymbol{b}}\downarrow$ and $f \in (Z_{\boldsymbol{b}}\downarrow)^{\perp_n}$, we have

$$S_{n,\boldsymbol{b}}\langle p \mid t_i f\rangle = S_{n,\boldsymbol{b}}\langle \partial p/\partial \tau_i \mid f\rangle = 0 \qquad (i = 1, \dots, n)$$

by Lemma 5.7. This implies that $t_i(Z_{\boldsymbol{b}}\downarrow)^{\perp_n} \subset (Z_{\boldsymbol{b}}\downarrow)^{\perp_n}$. Then for any $g(\boldsymbol{t}) \in \mathbb{C}[\boldsymbol{t}]$, we have $g(\boldsymbol{t})(Z_{\boldsymbol{b}}\downarrow)^{\perp_n} \subset (Z_{\boldsymbol{b}}\downarrow)^{\perp_n}$. Taking the weak limit shows that this holds for $g \in \mathcal{O}_{n,\boldsymbol{b}}$, which implies that $(Z_{\boldsymbol{b}}\downarrow)^{\perp_n}$ is an ideal of $\mathcal{O}_{n,\boldsymbol{b}}$ and completes the proof of (1)$\Rightarrow$(2). In a similar way, we see that if $(Z_{\boldsymbol{b}}\downarrow)^{\perp_n}$ is an ideal of $\mathcal{O}_{m,\boldsymbol{a}}$, then $(Z_{\boldsymbol{b}}\downarrow)^{\perp_n\perp_n}$ is D-invariant. Since every homogeneous part of $Z_{\boldsymbol{b}}\downarrow$ is finite-dimensional and belongs to $Z_{\boldsymbol{b}}\downarrow$, we see that $Z_{\boldsymbol{b}}\downarrow$ is weakly closed. Then $(Z_{\boldsymbol{b}}\downarrow)^{\perp_n\perp_n} = Z_{\boldsymbol{b}}\downarrow$ and (2)$\Rightarrow$(1) follows. ∎

REMARK 7.5. If (2) holds and $Z_{\boldsymbol{b}} \neq \emptyset$, the vector subspace $Z_{\boldsymbol{b}}$ has the structure of a $\mathbb{C}$-algebra such that the linear mapping $T_{Z,\boldsymbol{b}} : \mathcal{O}_{n,\boldsymbol{b}} \to Z_{\boldsymbol{b}}$ is a factor epimorphism of a $\mathbb{C}$-algebra [BC$_2$, Corollary 3.6]. Since $\dim_{\mathbb{C}} Z_{\boldsymbol{b}} < \infty$, it is a local analytic algebra of Krull dimension 0. Then it is Artinian in the sense that it satisfies the descending chain condition of ideals (cf. [Mat]). Thus we have associated to each point of U_Z^{inv} an Artinian local algebra. Since all the elements of the maximal ideal are nilpotent, $Z_{\boldsymbol{b}}$ is not a subalgebra of $\mathcal{O}_{n,\boldsymbol{a}}$ in general.

EXAMPLE 7.6. We show a common example. Let us take the vector space

$$Z := \operatorname{Span}_{\mathbb{C}}(1, s, t, t^2 + st^2, t^3) \subset \mathcal{O}_2(\mathbb{R}^2).$$

Paying attention to the normalizing factor $1/\boldsymbol{\nu}!$ of fibre coordinates in §3, we have Diagram 3 below for the jets of basis of Z. For example, the bottom of the diagram implies $j^3(t^3)(\boldsymbol{b}) = (\boldsymbol{b}; b^3,\ 0,\ 3b^2,\ 0,\ 0,\ 3b,\ 0,\ 0,\ 0,\ 1)$ at $\boldsymbol{b} = (a, b)$. Then we have

$$\dim_{\mathbb{C}} R_k(\boldsymbol{b}) = \begin{cases} 1 & (k = 0), \\ 3 & (k = 1), \\ 3 & (a = -1,\ b = 0,\ k = 2), \\ 4 & (a \neq -1,\ b = 0,\ k = 2), \\ 5 & (b \neq 0,\ k = 2), \\ 5 & (k \geq 3). \end{cases}$$

Then, denoting $s' = s - a,\ t' = t - b$, we have the following.

Diagram 3. Bases of jet spaces of Z at (a, b)

<table>
<tr><td rowspan="4">jet space</td><td>R_0</td><td colspan="6"></td></tr>
<tr><td colspan="3">R_1</td><td colspan="4"></td></tr>
<tr><td colspan="5">R_2</td><td colspan="2"></td></tr>
<tr><td colspan="7">R_3</td></tr>
<tr><td>fibre coordinates</td><td>$u_{(0,0)}$</td><td>$u_{(1,0)}$</td><td>$u_{(0,1)}$</td><td>$u_{(1,1)}$</td><td>$u_{(0,2)}$</td><td>$u_{(1,2)}$</td><td>$u_{(0,3)}$</td></tr>
<tr><td>1</td><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>s</td><td>a</td><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>t</td><td>b</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>$t^2 + st^2$</td><td>$b^2 + ab^2$</td><td>b^2</td><td>$2(1 + a)b$</td><td>$2b$</td><td>$1 + a$</td><td>1</td><td>0</td></tr>
<tr><td>t^3</td><td>b^3</td><td>0</td><td>$3b^2$</td><td>0</td><td>$3b$</td><td>0</td><td>1</td></tr>
</table>

(1) If $b \neq 0$, then

$$Z_{(a,\,b)}\!\downarrow\, = \mathrm{Span}(1, \sigma, \tau, \sigma\tau, \tau^2), \qquad (Z_{(a,\,b)}\!\downarrow)^{\perp_2} = (s'^2, s't'^2, t'^3)\mathbb{C}\{s', t'\}.$$

These points form U^{bdl} of Z and, since the kernels $(Z_{(a,\,b)}\!\downarrow)^{\perp_2}$ are of the same form, the associated Artinian algebras are all isomorphic at these points.

(2) If $a \neq -1,\ b = 0$, then

$$Z_{(a,b)}\!\downarrow\, = \mathrm{Span}(1, \sigma, \tau, \tau^2, \tau^3), \qquad (Z_{(a,b)}\!\downarrow)^{\perp_2} = (s'^2, s't', t'^4)\mathbb{C}\{s', t'\}.$$

These points form $U^{\mathrm{inv}} \setminus U^{\mathrm{bdl}}$ of Z. The associated Artinian algebras are all isomorphic at these points but not isomorphic to those associated to bundle points.

(3) If $a = -1,\ b = 0$, then

$$Z_{(a,b)}\!\downarrow\, = \mathrm{Span}(1, \sigma, \tau, \sigma\tau^2, \tau^3),$$
$$(Z_{(a,b)}\!\downarrow)^{\perp_2} = (s'^3, s't'^3, t'^4)\mathbb{C}\{s', t'\} + \mathbb{C}s't' + \mathbb{C}t'^2.$$

The least space of Z is not D-invariant here.

We will see this example again in Example 10.8.

8. Intrinsic treatment of least spaces. Here we show an intrinsic treatment of the least space, D-invariance property and the Artinian algebra associated to a D-invariant point. These are defined for complex manifolds independently of local coordinates $\boldsymbol{t} = (t_1, \ldots, t_n)$. To explain them we distinguish the coordinate expression $f_{\mathbf{0}}^{\boldsymbol{t}}\downarrow := (f \circ \boldsymbol{\Phi})_{\mathbf{0}}\downarrow$ of the least part (resp. the least space $Z_{\boldsymbol{b}}^{\boldsymbol{t}}\downarrow$) and the intrinsic one $f_{\boldsymbol{b}}\downarrow$ (resp. $Z_{\boldsymbol{b}}$) of $f \in \mathcal{O}_{M,\boldsymbol{b}}$ in this section.

Let M be an n-dimensional complex manifold and φ a local parametrisation of M centred at $\boldsymbol{b}$: $\varphi(0) = \boldsymbol{b}$. Let $\boldsymbol{s}$ be the local coordinates for φ. Then $f_{\mathbf{0}}^{\boldsymbol{s}} := (f \circ \boldsymbol{\Phi})_{\mathbf{0}}\downarrow$ is expressed as a homogeneous polynomial, say of degree k, in $\boldsymbol{\sigma} := \boldsymbol{s}_{\mathbf{0}}\downarrow$. If $\boldsymbol{\Psi}$ is another local parametrisation of M centred at $\boldsymbol{b}$ with local coordinates $\boldsymbol{t}$ and dual coordinates $\boldsymbol{\tau} := \boldsymbol{t}_{\mathbf{0}}\downarrow$, then there exists a biholomorphic germ $\boldsymbol{\Theta}$ with $\boldsymbol{\Phi} = \boldsymbol{\Psi} \circ \boldsymbol{\Theta}$. Let

$$\tilde{\jmath}_\theta : \mathbb{C}[\boldsymbol{\tau}] \to \mathbb{C}[\boldsymbol{\sigma}], \qquad p(\boldsymbol{\tau}) \mapsto p(J_\theta \boldsymbol{\sigma}),$$

denote the isomorphism induced by the linear coordinate transformation expressed by the Jacobian matrix $J_\theta := \partial \boldsymbol{t}/\partial \boldsymbol{s}$ evaluated at $\boldsymbol{s} = \mathbf{0}$, where $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are treated as column vectors. By the definition of the least part, we have

$$\begin{aligned}\tilde{\jmath}_\theta(f_{\mathbf{0}}^{\boldsymbol{t}}\downarrow) &= f_{\mathbf{0}}^{\boldsymbol{t}}\downarrow(J_\theta\boldsymbol{\sigma}) = (f \circ \boldsymbol{\Psi})_{\mathbf{0}}\downarrow(J_\theta\boldsymbol{\sigma}) = ((f \circ \boldsymbol{\Psi})(J_\theta \boldsymbol{s}))_{\mathbf{0}}\downarrow \\ &= (f \circ \boldsymbol{\Psi} \circ \Theta \circ \boldsymbol{s})_{\mathbf{0}}\downarrow = (f \circ \boldsymbol{\Phi})_{\mathbf{0}}\downarrow = f_{\mathbf{0}}^{\boldsymbol{s}}\downarrow.\end{aligned}$$

Then the collection

$$f_{\boldsymbol{b}}\downarrow := \{(\boldsymbol{\Phi}, f_{\mathbf{0}}^{\boldsymbol{s}}\downarrow) : \boldsymbol{\Phi} = \boldsymbol{\Phi}(\boldsymbol{s}) \text{ is a local parametrisation of } M \text{ at } \boldsymbol{b}\},$$

whose elements are related by $\tilde{\jmath}_\theta$ as above, can be seen as an evaluation at $\boldsymbol{b}$ of the k-fold symmetric tensor product, over the sheaf $\mathcal{O}_M$ of holomorphic functions on M, of the usual cotangent sheaf $\mathcal{T}^*$ of $\mathbb{C}^n$. The least part $f_{\mathbf{0}}^{\boldsymbol{s}}\downarrow \in \mathbb{C}[\boldsymbol{\sigma}]$ is a coordinate expression of $f_{\boldsymbol{b}}\downarrow$. Hence, we have the intrinsic form of the least space:

$$\mathcal{O}_{M,\boldsymbol{b}}\downarrow := \{f_{\boldsymbol{b}}\downarrow : f \in \mathcal{O}_{M,\boldsymbol{b}}\} \cong \bigoplus_k \mathcal{T}^*(\boldsymbol{b})^{\odot k},$$

where the symmetric tensor product $\odot$ is taken over $\mathbb{C}$ (cf. Quillen [Qu, p. 2]).

Definition 8.1. Let M be an n-dimensional complex manifold with a local parametrisation $\boldsymbol{\Phi} = \boldsymbol{\Phi}(\boldsymbol{s})$ at $\boldsymbol{b}$, and $Z \subset \mathcal{O}_{M,\boldsymbol{b}}$ a finite-dimensional vector subspace. We denote by M_Z^{bdl} the set of bundle points of all the jet spaces of

$$Z_{\boldsymbol{b}}^{\boldsymbol{s}} := \{(f \circ \boldsymbol{\Phi})_{\mathbf{0}} : f \in Z\} \subset \mathbb{C}\{\boldsymbol{s}\},$$

and by M_Z^{inv} the set of D-invariant points of

$$Z_{\boldsymbol{b}}^{\boldsymbol{s}}\downarrow := \mathrm{Span}_{\mathbb{C}}(f_{\mathbf{0}}^{\boldsymbol{s}}\downarrow : f \in Z) \subset \mathbb{C}[\boldsymbol{\sigma}].$$

THEOREM 8.2. *The sets M_Z^{bdl} and M_Z^{inv} are well-defined independently of the local parametrisation φ, and $M_Z^{\mathrm{bdl}} \subset M_Z^{\mathrm{inv}}$.*

Proof. Take two local parametrisations $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ and the isomorphism $\tilde{j}_\theta$ as above. Since $\tilde{j}_\theta$ is a result of a linear change of variables, it does not alter the rank of R_k in §3. Then the set M_Z^{bdl} of bundle points of all jet spaces of Z is well-defined independently of φ. The transformation $\tilde{j}_\theta$ changes a partial differentiation to a linear combination of partial differentiations. Hence the set M_Z^{inv} of D-invariant points for Z is also well-defined. By Theorem 4.5, we have $M_Z^{\mathrm{bdl}} \subset M_Z^{\mathrm{inv}}$. ∎

The collection of $Z_{\boldsymbol{b}}^{\boldsymbol{s}}{\downarrow}$ for all φ forms the intrinsic least space $Z_{\boldsymbol{b}}{\downarrow} \subset \mathcal{O}_{M,\boldsymbol{b}}{\downarrow}$ at $\boldsymbol{b}$.

THEOREM 8.3. *Let M be an n-dimensional complex manifold and $Z \subset \mathcal{O}(M)$ be a finite-dimensional vector subspace. If $\boldsymbol{b} \in M_Z^{\mathrm{inv}}$, then the vector subspace $Z_{\boldsymbol{b}}$ has the structure $\mathcal{O}_{n,\boldsymbol{b}}/(Z_{\boldsymbol{b}}{\downarrow})^{\perp_n}$ of an Artinian algebra, which is unique up to a canonical isomorphism* (*induced by ${}^s\tilde{j}_\theta$ in the proof*) *as a contravariant tensor.*

Proof. Existence of the structure of an Artinian algebra is explained in Remark 7.5. Take two local parametrisations around $\boldsymbol{b} \in M$ which induce isomorphisms $\varphi : \mathcal{O}_{n,\boldsymbol{b}} \to \mathbb{C}\{\boldsymbol{s}\}$ and $\psi : \mathcal{O}_{n,\boldsymbol{b}} \to \mathbb{C}\{\boldsymbol{t}\}$ such that $\varphi = \theta \circ \psi$ for the third isomorphism θ of algebras. We have the homomorphism $\tilde{j}_\theta : \mathbb{C}[\boldsymbol{\tau}] \to \mathbb{C}[\boldsymbol{\sigma}]$ defined above and its adjoint ${}^s\tilde{j}_\theta : \mathbb{C}\{\boldsymbol{s}\} \to \mathbb{C}\{\boldsymbol{t}\}$, which is nothing but the homomorphism corresponding to the coordinate transformation defined by the linear part of θ. We have the following implications:

$$\begin{aligned} f \in (Z_{\boldsymbol{b}}^{\boldsymbol{t}}{\downarrow})^{\perp_n} \;&\Leftrightarrow\; f \in (\tilde{j}_\theta(Z_{\boldsymbol{b}}^{\boldsymbol{s}}{\downarrow}))^{\perp_n} \;\Leftrightarrow\; \forall p \in \tilde{j}_\theta(Z_{\boldsymbol{b}}^{\boldsymbol{s}}{\downarrow}) : S_{n,\mathbf{0}}\langle p \mid f\rangle = 0 \\ &\Leftrightarrow\; \forall q \in Z_{\boldsymbol{b}}^{\boldsymbol{s}}{\downarrow} : S_{n,\mathbf{0}}\langle \tilde{j}_\theta(q) \mid f\rangle = 0 \;\Leftrightarrow\; \forall q \in Z_{\boldsymbol{b}}^{\boldsymbol{s}}{\downarrow} : S_{n,\mathbf{0}}\langle q \mid {}^s\tilde{j}_\theta(f)\rangle = 0 \\ &\Leftrightarrow\; {}^s\tilde{j}_\theta(f) \in (Z_{\boldsymbol{b}}^{\boldsymbol{s}}{\downarrow})^{\perp_n} \;\Leftrightarrow\; f \in ({}^s\tilde{j}_\theta)^{-1}((Z_{\boldsymbol{b}}^{\boldsymbol{s}}{\downarrow})^{\perp_n}). \end{aligned}$$

This proves

$$(Z_{\boldsymbol{b}}^{\boldsymbol{t}}{\downarrow})^{\perp_n} = ({}^s\tilde{j}_\theta)^{-1}((Z_{\boldsymbol{b}}^{\boldsymbol{s}}{\downarrow})^{\perp_n})$$

and $(Z_{\boldsymbol{b}}^{\boldsymbol{s}}{\downarrow})^{\perp_n}$ is the image of $(Z_{\boldsymbol{b}}^{\boldsymbol{t}}{\downarrow})^{\perp_n}$ under the isomorphism ${}^s\tilde{j}_\theta$. This isomorphism is obtained by replacing $\boldsymbol{s}$ by ${}^sJ_\theta\boldsymbol{t}$ and hence it is even an algebra isomorphism. Then $(Z_{\boldsymbol{b}}^{\boldsymbol{s}}{\downarrow})^{\perp_n}$ is an ideal if and only if $(Z_{\boldsymbol{b}}^{\boldsymbol{t}}{\downarrow})^{\perp_n}$ is so, and hence $Z_{\boldsymbol{b}}^{\boldsymbol{s}}{\downarrow}$ is D-invariant if and only if $Z_{\boldsymbol{b}}^{\boldsymbol{t}}{\downarrow}$ is so by Lemma 7.4. When this is the case, we have an isomorphism

$$\mathbb{C}\{\boldsymbol{s}\}/(Z_{\boldsymbol{b}}^{\boldsymbol{s}}{\downarrow})^{\perp_n} \to \mathbb{C}\{\boldsymbol{t}\}/(Z_{\boldsymbol{b}}^{\boldsymbol{t}}{\downarrow})^{\perp_n}$$

of algebras induced by ${}^s\tilde{j}_\theta$. ∎

THEOREM 8.4. *Let $\varphi : \mathcal{O}_{m,\boldsymbol{a}} := \mathbb{C}\{\boldsymbol{x}\} \to \mathcal{O}_{n,\boldsymbol{b}} := \mathbb{C}\{\boldsymbol{t}\}$ be a homomorphism of analytic algebras and let ${}^s\varphi : \mathbb{C}[\boldsymbol{\tau}] \to \mathbb{C}[\boldsymbol{\xi}]$ denote its adjoint mapping with respect to the sesquilinear forms $S_{m,\boldsymbol{a}}$ and $S_{n,\boldsymbol{b}}$. If $Q \subset \mathbb{C}[\boldsymbol{\tau}]$*

is a finite-dimensional vector subspace, then φ induces a monomorphism ψ of the factor vector spaces

$$\psi : \mathcal{O}_{m,\boldsymbol{a}}/({}^s\varphi(Q))^{\perp_m} \to \mathcal{O}_{n,\boldsymbol{b}}/Q^{\perp_n}.$$

Furthermore:

(1) *If Q is D-invariant, so is ${}^s\varphi(Q)$ and ψ is a monomorphism of Artinian algebras.*
(2) *If φ is an epimorphism, then ψ is an isomorphism and ${}^s\varphi(Q)$ is D-invariant if and only if Q is so. If this is the case, then ψ is an isomorphism of Artinian algebras.*

Proof. To prove that ψ exists and is a monomorphism, we have only to prove the equality $({}^s\varphi(Q))^{\perp_m} = \varphi^{-1}(Q^{\perp_n})$. This is obvious from

$$\begin{aligned} f \in ({}^s\varphi(Q))^{\perp_m} &\Leftrightarrow \ \forall q \in Q : S_{m,\boldsymbol{a}}\langle {}^s\varphi(q) \mid f\rangle = 0 \\ &\Leftrightarrow \ \forall q \in Q : \ S_{n,\boldsymbol{b}}\langle q \mid \varphi(f)\rangle = 0 \ \Leftrightarrow \ f \in \varphi^{-1}(Q^{\perp_n}). \end{aligned}$$

(1) The algebra $\mathcal{O}_{n,\boldsymbol{b}}/Q^{\perp_n}$ is Artinian because $\dim_{\mathbb{C}} \mathcal{O}_{n,\boldsymbol{b}}/Q^{\perp_n} = \dim_{\mathbb{C}} Q < \infty$ by [Bo$_2$, 4, §1, n°5, Prop. 5]. If Q is D-invariant, then $Q^{\perp_n}$ is a proper ideal by Theorem 7.4. Then, by the implication above, $({}^s\varphi(Q))^{\perp_m}$ is also an ideal and ψ is a monomorphism of algebras. Then $\mathcal{O}_{m,\boldsymbol{a}}/({}^s\varphi(Q))^{\perp_m}$ is also Artinian.

(2) If φ is an epimorphism, it is obvious that ψ is an isomorphism and that the equality $({}^s\varphi(Q))^{\perp_m} = \varphi^{-1}(Q^{\perp_n})$ is equivalent to $\varphi(({}^s\varphi(Q))^{\perp_m}) = Q^{\perp_n}$. Thus, $({}^s\varphi(Q))^{\perp_m}$ is an ideal if and only if $Q^{\perp_n}$ is so. This proves equivalence of D-invariance of ${}^s\varphi(Q)$ and of Q. The last assertion is trivial by (1). ■

9. Higher order tangents of Bos and Calvi. Following the method of Bos–Calvi, we introduce the space $D_{\boldsymbol{a}}^{\varphi,d}$ of higher order tangents of a complex analytic submanifold X of an open subset $\Omega \subset \mathbb{C}^m$ at $\boldsymbol{a} \in X$. It is not an intrinsic object associated to $X_{\boldsymbol{a}}$ as a germ of a complex space but it reflects the properties of the embedding germ $X_{\boldsymbol{a}} \subset \mathbb{C}^m_{\boldsymbol{a}}$. It also depends upon the choice of the local parametrisation φ of $X_{\boldsymbol{a}}$ in general. It is a dual space of the space $P^d(X_{\boldsymbol{a}})$ of polynomial functions of degrees at most d. We will often skip the modifier "higher order" for tangents.

Let $X_{\boldsymbol{a}}$ be the germ of a regular complex submanifold X of an open subset $\Omega \subset \mathbb{C}^m$ at $\boldsymbol{a}$. The algebra $\mathcal{O}_{X,\boldsymbol{a}}$ of germs at $\boldsymbol{a}$ of holomorphic functions on respective neighbourhoods (in X) of $\boldsymbol{a}$ is isomorphic to the factor algebra of $\mathcal{O}_{m,\boldsymbol{a}} = \mathbb{C}\{\boldsymbol{x}\}$ ($\boldsymbol{x} := (x_1, \ldots, x_m)$) by the ideal $I_{\boldsymbol{a}}$ of convergent power series vanishing on the germ $X_{\boldsymbol{a}}$: $\mathcal{O}_{X,\boldsymbol{a}} \cong \mathcal{O}_{m,\boldsymbol{a}}/I_{\boldsymbol{a}}$. Hence $\mathcal{O}_{X,\boldsymbol{a}}$ is an analytic local algebra. Let

$$\pi : \mathcal{O}_{m,\boldsymbol{a}} \to \mathcal{O}_{X,\boldsymbol{a}}$$

denote the factor epimorphism. By the assumption that X is a submanifold, there is an isomorphism

$$\psi : \mathcal{O}_{X,\boldsymbol{a}} \to \mathcal{O}_{n,\boldsymbol{b}} = \mathbb{C}\{\boldsymbol{t} - \boldsymbol{b}\} \quad (\boldsymbol{t} := (t_1, \dots, t_n),\ \dim X = n).$$

Then we have the epimorphism

$$\varphi := \psi \circ \pi : \mathcal{O}_{m,\boldsymbol{a}} \to \mathcal{O}_{n,\boldsymbol{b}}.$$

This is just the epimorphism defined by the pullback by the germ of the embedding

$$\boldsymbol{\Phi} = (\Phi_1, \dots, \Phi_m) : \mathbb{C}^n_{\boldsymbol{b}} \to \mathbb{C}^m_{\boldsymbol{a}},$$

namely $\varphi(f) = f \circ \boldsymbol{\Phi}$. We call this φ or $\boldsymbol{\Phi}$ a *local parametrisation* of X at $\boldsymbol{a}$. Let $\mathbb{C}[\boldsymbol{\Phi}] \subset \mathcal{O}_{n,\boldsymbol{b}}$ denote the algebra of pullbacks of $\mathbb{C}[\boldsymbol{x}]$ by φ. Let

$$P(X_{\boldsymbol{a}}) = \mathbb{C}[\boldsymbol{x}]|_{X_{\boldsymbol{a}}} = \pi(\mathbb{C}[\boldsymbol{x}]) \subset \mathcal{O}_{X,\boldsymbol{a}}$$

denote the ring of germs of polynomial functions on $X_{\boldsymbol{a}}$. It is easy to see the following.

Lemma 9.1. *We have the algebra isomorphism*

$$\mathbb{C}[\boldsymbol{\Phi}] := \varphi(\mathbb{C}[\boldsymbol{x}]) = \psi(P(X_{\boldsymbol{a}})) \cong P(X_{\boldsymbol{a}}).$$

By the general property of the transpose mapping of a surjective homomorphism, ${}^t\varphi$ is injective and hence so is ${}^s\varphi : \mathbb{C}[\boldsymbol{\tau}] \to \mathbb{C}[\boldsymbol{\xi}]$. The image $D^{\varphi}_{\boldsymbol{a}} := {}^s\varphi(\mathbb{C}[\boldsymbol{\tau}])$ is geometrically the space of higher order tangents of X at $\boldsymbol{a}$. The property $D^{\varphi}_{\boldsymbol{a}} = I^{\perp_m}_{\boldsymbol{a}}$ shown below means that higher order tangents of $X_{\boldsymbol{a}}$ are just the higher order tangents of $\mathbb{C}^m_{\boldsymbol{a}}$ which annihilate all the functions vanishing on $X_{\boldsymbol{a}}$. The sesquilinear form $S_{n,\boldsymbol{b}}$ induces a non-degenerate sesquilinear form

$$S^{\varphi}_{X,\boldsymbol{a}} : D^{\varphi}_{\boldsymbol{a}} \times \mathcal{O}_{X,\boldsymbol{a}} \to \mathbb{C}$$

through ψ and ${}^s\psi$ (see Diagram 4 in §10). Let $I^{\perp_m}_{\boldsymbol{a}}$ and $I^{\perp_X}_{\boldsymbol{a}}$ denote the spaces of annihilators of $I_{\boldsymbol{a}}$ with respect to $S^{\varphi}_{m,\boldsymbol{a}}$ and $S^{\varphi}_{X,\boldsymbol{a}}$ respectively.

Proposition 9.2. *We have $(D^{\varphi}_{\boldsymbol{a}})^{\perp_m} = I_{\boldsymbol{a}}$ and $I^{\perp_m}_{\boldsymbol{a}} = D^{\varphi}_{\boldsymbol{a}}$. Hence $D^{\varphi}_{\boldsymbol{a}}$ is independent of the local parametrisation φ.*

Thus we may omit the superscript φ of $D^{\varphi}_{\boldsymbol{a}}$.

Proof. The first equality follows from the implications

$$\begin{aligned} &f \in I_{\boldsymbol{a}} \;\Leftrightarrow\; \varphi(f) = 0 \;\Leftrightarrow\; \forall p \in \mathbb{C}[\boldsymbol{\tau}] : S_{n,\boldsymbol{b}}\langle p \mid \varphi(f)\rangle = 0 \\ &\;\Leftrightarrow\; \forall p \in \mathbb{C}[\boldsymbol{\tau}] : S_{m,\boldsymbol{a}}\langle {}^s\varphi(p) \mid f\rangle = 0 \Leftrightarrow\; f \in ({}^s\varphi(\mathbb{C}[\boldsymbol{\tau}]))^{\perp_m} \Leftrightarrow\; f \in (D^{\varphi}_{\boldsymbol{a}})^{\perp_m}. \end{aligned}$$

Then we have $I^{\perp_m}_{\boldsymbol{a}} = (D^{\varphi}_{\boldsymbol{a}})^{\perp_m \perp_m}$. To see the equality $I^{\perp_m}_{\boldsymbol{a}} = D^{\varphi}_{\boldsymbol{a}}$, we have only to prove that $(D^{\varphi}_{\boldsymbol{a}})^{\perp_m \perp_m} = D^{\varphi}_{\boldsymbol{a}}$, or that $D^{\varphi}_{\boldsymbol{a}} = {}^s\varphi(\mathbb{C}[\boldsymbol{\tau}])$ is weakly closed in $\mathbb{C}[\boldsymbol{\xi}]$. Since φ is an open continuous epimorphism by Lemma 6.1,

${}^{t}\varphi(\mathbb{C}[\boldsymbol{\tau}])$ is weakly closed by [Bo$_2$, 4, §4, n°1, Proposition 4]. Since the complex conjugations $u : \mathbb{C}[\boldsymbol{\tau}] \to \mathbb{C}[\boldsymbol{\tau}]$ and $u : \mathbb{C}[\boldsymbol{\xi}] \to \mathbb{C}[\boldsymbol{\xi}]$ are homeomorphisms, ${}^{s}\varphi(\mathbb{C}[\boldsymbol{\tau}]) = u \circ {}^{t}\varphi \circ u(\mathbb{C}[\boldsymbol{\tau}])$ is also weakly closed. ∎

Let

$$P^d(X_{\boldsymbol{a}}) := \{p \bmod I_{\boldsymbol{a}} : p \in \mathbb{C}[\boldsymbol{x}],\ \deg p \leq d\} \subset P(X_{\boldsymbol{a}}) \subset \mathcal{O}_{X,\boldsymbol{a}}$$

denote the vector space of *polynomial functions* on X of degree at most d at $\boldsymbol{a}$.

Remark 9.3. If $X_{\boldsymbol{a}}$ is defined by an ideal $I_{\boldsymbol{a}} \subset \mathcal{O}_{m,\boldsymbol{a}}$, the algebra

$$\mathbb{C}[\boldsymbol{x}]/((I_{\boldsymbol{a}} + \mathfrak{m}_{\boldsymbol{a}}^{d+1}) \cap \mathbb{C}[\boldsymbol{x}]) \cong \mathbb{C}\{\boldsymbol{x}\}/(I_{\boldsymbol{a}} + \mathfrak{m}_{\boldsymbol{a}}^{d+1})$$

is different from the vector space $P^d(X_{\boldsymbol{a}})$. The canonical mapping

$$\pi_d : P^d(X_{\boldsymbol{a}}) \to \mathbb{C}[\boldsymbol{x}]/((I_{\boldsymbol{a}} + \mathfrak{m}_{\boldsymbol{a}}^{d+1}) \cap \mathbb{C}[\boldsymbol{x}])$$

is surjective but not always injective.

Let

$$\mathbb{C}[\boldsymbol{x}]^d := \{f(\boldsymbol{x}) : f \in \mathbb{C}[\boldsymbol{x}],\ \deg f \leq d\} \subset \mathbb{C}[\boldsymbol{x}]$$

denote the vector space of polynomials of degree at most d. If we put

$$\mathbb{C}[\boldsymbol{\Phi}]^d := \varphi(\mathbb{C}[\boldsymbol{x}]^d) = \psi(P^d(X_{\boldsymbol{a}})) \subset \mathcal{O}_{n,\boldsymbol{b}}$$

using a local parametrisation $\boldsymbol{\Phi}$, we have an increasing sequence of finite-dimensional vector subspaces

$$\mathbb{C} = \mathbb{C}[\boldsymbol{\Phi}]^0 \subset \mathbb{C}[\boldsymbol{\Phi}]^1 \subset \cdots$$

of the $\mathbb{C}$-algebra $\mathbb{C}[\boldsymbol{\Phi}] \subset \mathcal{O}_{n,\boldsymbol{b}}$. Then we have a sequence

$$\mathbb{C} = \mathbb{C}[\boldsymbol{\Phi}]^0_{\boldsymbol{b}}{\downarrow} \subset \mathbb{C}[\boldsymbol{\Phi}]^1_{\boldsymbol{b}}{\downarrow} \subset \cdots$$

of finite-dimensional vector subspaces of $\mathbb{C}[\boldsymbol{\tau}] = \mathcal{O}_{n,\boldsymbol{b}}{\downarrow}$. Let us fix the degree d hereafter. Recall that $S_{n,\boldsymbol{b}}$ induces a non-degenerate sesquilinear form

$$S^{\varphi,d}_{n,\boldsymbol{b}} : \mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}{\downarrow} \times \mathbb{C}[\boldsymbol{\Phi}]^d \to \mathbb{C}$$

by Theorem 5.5.

Definition 9.4. Set

$$D^{\varphi,d}_{\boldsymbol{a}} := {}^{s}\varphi(\mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}{\downarrow}) \subset D_{\boldsymbol{a}}.$$

Since there is a natural isomorphism $\psi|_{P^d(X_{\boldsymbol{a}})} : P^d(X_{\boldsymbol{a}}) \to \mathbb{C}[\boldsymbol{\Phi}]^d$ by Lemma 9.1, we see that $S^{\varphi,d}_{n,\boldsymbol{b}}$ induces a non-degenerate sesquilinear form

$$S^{\varphi,d}_{X,\boldsymbol{a}} : D^{\varphi,d}_{\boldsymbol{a}} \times P^d(X_{\boldsymbol{a}}) \to \mathbb{C}.$$

We call the elements of $D^{\varphi,d}_{\boldsymbol{a}}$ *Bos–Calvi tangents of $X_{\boldsymbol{a}}$ of dual degree d* (see Diagram 4 in §10).

Usually, some element of $D_{\boldsymbol{a}}^{\varphi,d}$ has a degree higher than d, as we will see in the examples below.

LEMMA 9.5. *Let $X_{\boldsymbol{a}}$ be the germ of a regular complex submanifold X of an open subset $\Omega \subset \mathbb{C}^m$ at $\boldsymbol{a}$. Take two local parametrisations*

$$\boldsymbol{\Phi} : \mathbb{C}^n_{\boldsymbol{b}} \to \mathbb{C}^m_{\boldsymbol{a}}, \qquad \boldsymbol{\Psi} : \mathbb{C}^n_{\boldsymbol{b}'} \to \mathbb{C}^m_{\boldsymbol{a}}$$

of $X_{\boldsymbol{a}}$. Let $\boldsymbol{\Theta} : \mathbb{C}^n_{\boldsymbol{b}} \to \mathbb{C}^n_{\boldsymbol{b}'}$ denote the biholomorphic germ (coordinate change) such that $\boldsymbol{\Phi} = \boldsymbol{\Psi} \circ \boldsymbol{\Theta}$ and $\varphi = \theta \circ \psi$. Then:

(1) *The isomorphism θ induces an isomorphsm $\theta|_{\mathbb{C}[\boldsymbol{\Psi}]^d} : \mathbb{C}[\boldsymbol{\Psi}]^d \to \mathbb{C}[\boldsymbol{\Phi}]^d$.*
(2) *If $\boldsymbol{b}$ is a bundle point of all the jet spaces of $\mathbb{C}[\boldsymbol{\Phi}]^d$, it is so for $\mathbb{C}[\boldsymbol{\Psi}]^d$. If $\boldsymbol{b}$ is an D-invariant point of $\mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow$, it is so for $\mathbb{C}[\boldsymbol{\Psi}]^d_{\boldsymbol{b}}\!\downarrow$.*

Proof. There are isomorphisms

$$\mathbb{C}[\boldsymbol{\Phi}]^d \overset{\cong}{\to} P^d(X_{\boldsymbol{a}}) \overset{\cong}{\leftarrow} \mathbb{C}[\boldsymbol{\Psi}]^d$$

obtained as restrictions of the isomorphism in Lemma 9.1. Since

$$\theta|_{\mathbb{C}[\boldsymbol{\Psi}]^d} : \mathbb{C}[\boldsymbol{\Psi}]^d \to \mathbb{C}[\boldsymbol{\Phi}]^d$$

is compatible with these mappings, it is also an isomorphism. Assertion (2) follows from Theorem 8.2. ∎

EXAMPLE 9.6. If $\dim X \geq 2$, $D_{\boldsymbol{a}}^{\varphi,d}$ is very sensitive to a change of the local parametrisation even in the case $d = 1$. Let us consider the surface $X \subset \mathbb{C}^3$ defined by $x_3 = x_2^2$. Take two global parametrisations:

$$\varphi : x_1 = s_1,\ x_2 = s_2,\ x_3 = s_2^2;$$
$$\psi : x_1 = t_1 + t_2,\ x_2 = t_2,\ x_3 = t_2^2.$$

These are related by a simple linear transformation of local coordinates: $t_1 = s_1 - s_2$, $t_2 = s_2$. The dimensions of the spaces $R_k(\boldsymbol{b})$ ($\boldsymbol{a} = \varphi(\boldsymbol{b})$) for $\mathbb{C}[\boldsymbol{\Phi}]^1$ defined in §3 are independent of $\boldsymbol{b} \in \mathbb{R}^2$:

$$\dim_{\mathbb{C}} R_0(\boldsymbol{b}) = 1, \qquad \dim_{\mathbb{C}} R_1(\boldsymbol{b}) = 3, \qquad \dim_{\mathbb{C}} R_k(\boldsymbol{b}) = 4 \quad (k \geq 2).$$

This means that all points are bundle points of all the jet spaces of $\mathbb{C}[\boldsymbol{\Phi}]^1$. Then $\mathbb{C}[\boldsymbol{\Psi}]^1$ also has the bundle property everywhere by Lemma 9.5. Set

$$\sigma_i := s_{i,\boldsymbol{b}}\!\downarrow, \qquad \tau_i := t_{i,\boldsymbol{b}}\!\downarrow, \qquad \xi_i := x_{i,\boldsymbol{a}}\!\downarrow.$$

The least spaces of the pullbacks of polynomials of degree at most 1 with respect to them are

$$\mathbb{C}[\boldsymbol{\Phi}]^1_{\boldsymbol{b}}\!\downarrow = \mathrm{Span}_{\mathbb{C}}(1, \sigma_1, \sigma_2, \sigma_2^2), \qquad \mathbb{C}[\boldsymbol{\Psi}]^1_{\boldsymbol{b}}\!\downarrow = \mathrm{Span}_{\mathbb{C}}(1, \tau_1, \tau_2, \tau_2^2),$$

and they are D-invariant everywhere (which is also a consequence of Theorem 4.5). The pushforwards of these bases are computed as follows:

$$^s\varphi(1) = 1, \qquad ^s\varphi(\sigma_1) = \xi_1, \qquad ^s\varphi(\sigma_2) = \xi_2 + 2s_2\xi_3,$$
$$^s\varphi(\sigma_2^2) = 2\xi_3 + \xi_2^2 + 4s_2\xi_2\xi_3 + 4s_2^2\xi_3^2;$$

$$
{}^s\psi(1) = 1, \quad {}^s\psi(\tau_1) = \xi_1, \quad {}^s\psi(\tau_2) = \xi_1 + \xi_2 + 2t_2\xi_3,
$$
$$
{}^s\psi(\tau_2^2) = 2\xi_3 + \xi_1^2 + 2\xi_1\xi_2 + 4t_2\xi_1\xi_3 + \xi_2^2 + 4t_2\xi_2\xi_3 + 4t_2^2\xi_3^2.
$$

The monomial ξ_1^2 appears in the linear span of the latter but not in that of the former. Hence the two spaces of Bos–Calvi tangents $D_{\boldsymbol{a}}^{\varphi,1}$ and $D_{\boldsymbol{a}}^{\psi,1}$ at $\boldsymbol{a} = \boldsymbol{\Phi}(\boldsymbol{b}) = \boldsymbol{\Psi}(\boldsymbol{b}) \in X$ are different.

Remark 9.7. In this paper we use the word "contravariant" to mean that the objects are mapped in the opposite direction to the geometric mapping of the underlying complex spaces. This is an intrinsic usage and it is equivalent to "covariant in the classical sense" which refers to change of components (coefficients). The inconsistency in Example 9.6 originates from the treatment of the elements of the least space. An element of $\mathbb{C}[\boldsymbol{t}]_{\boldsymbol{b}}^1\!\downarrow$ is defined as a tensor of cotangents, a contravariant object. Then we identify it as a higher order tangent, a covariant one, using a positive sesquilinear form (see Remark 5.1) and send it to $\mathbb{C}[\boldsymbol{\xi}]$ as a higher order tangent. Thus our higher order tangents are not geometric objects.

10. Taylor projector. Now we can introduce the Taylor projector of degree d using Bos–Calvi tangents defined in the previous section. In general, this projector depends upon the local parametrisation but, in the case of a curve, it is independent at a general point.

Assume the same as in the previous section for $\boldsymbol{a} \in X \subset \mathbb{C}^m$ and its local parametrisation $\varphi : \mathcal{O}_{m,\boldsymbol{a}} \to \mathcal{O}_{n,\boldsymbol{b}}$.

Definition 10.1. Let $\iota : D_{\boldsymbol{a}}^{\varphi,d} \to D_{\boldsymbol{a}}$ denote the inclusion mapping. We call its adjoint linear mapping

$$
T_{\boldsymbol{a}}^{\varphi,d} := {}^s\iota : \mathcal{O}_{X,\boldsymbol{a}} \to P^d(X_{\boldsymbol{a}})
$$

the *φ-Taylor projector of degree d* at $\boldsymbol{a}$. This was introduced by Bos and Calvi [BC$_1$], [BC$_2$]. It is a little different from ours. Their projector is the composition of our $T_{\boldsymbol{a}}^{\varphi,d}$ and the factor epimorphism $\pi : \mathcal{O}_{m,\boldsymbol{a}} \to \mathcal{O}_{X,\boldsymbol{a}}$. The image $T_{\boldsymbol{a}}^{\varphi,d}(f)$ is called the *φ-Taylor polynomial* of f of degree d.

We know the following by Proposition 7.2:

(1) The φ-Taylor projector $T_{\boldsymbol{a}}^{\varphi,d}$ is a weakly continuous linear mapping.
(2) We have the equalities

$$
\operatorname{Ker} T_{\boldsymbol{a}}^{\varphi,d} = (D_{\boldsymbol{a}}^{\varphi,d})^{\perp_X}, \quad (\operatorname{Ker} T_{\boldsymbol{a}}^{\varphi,d})^{\perp_X} = D_{\boldsymbol{a}}^{\varphi,d},
$$
$$
\operatorname{Ker} T_{\boldsymbol{a}}^{\varphi,d} \circ \pi = (D_{\boldsymbol{a}}^{\varphi,d})^{\perp_m}, \quad (\operatorname{Ker} T_{\boldsymbol{a}}^{\varphi,d} \circ \pi)^{\perp_m} = D_{\boldsymbol{a}}^{\varphi,d},
$$

where $\perp_X$ (resp. $\perp_m$) denotes the space of annihilators with respect to $S_{X,\boldsymbol{a}}^{\boldsymbol{\varphi}}$ (resp. $S_{m,\boldsymbol{a}}^{\boldsymbol{\varphi}}$).

(3) The φ-Taylor projector $T^{\varphi,d}_{\boldsymbol{a}}$ is a retraction of a vector space, i.e. $T^{\varphi,d}_{\boldsymbol{a}} \circ \kappa : P^d(X_{\boldsymbol{a}}) \to P^d(X_{\boldsymbol{a}})$ is the identity, where $\kappa : P^d(X_{\boldsymbol{a}}) \to \mathcal{O}_{X,\boldsymbol{a}}$ denotes the inclusion.

$$
\begin{tikzcd}
\mathcal{O}_{n,\boldsymbol{b}} \arrow[ddd, no head, very thick, "S_{n,\boldsymbol{b}}"'] & & & \mathcal{O}_{X,\boldsymbol{a}} \arrow[lll, "\cong", "\psi"'] \arrow[dl, dashed, "T^{\varphi,d}_{\boldsymbol{a}}"'] \arrow[ddd, no head, dashed, "S^{\boldsymbol{\varphi}}_{X,\boldsymbol{a}}"] & \mathcal{O}_{m,\boldsymbol{a}} \arrow[l, "\pi"'] \arrow[ddd, no head, very thick, "S_{m,\boldsymbol{a}}"] \arrow[llll, bend right=40, "\varphi"'] \\
& \mathbb{C}[\boldsymbol{\Phi}]^d \arrow[ul] \arrow[d, no head, very thick, "S^{\varphi,d}_{n,\boldsymbol{b}}"'] & P^d(X_{\boldsymbol{a}}) \arrow[l, "\cong"'] \arrow[ur, "\kappa"'] \arrow[d, no head, dashed, "S^{\varphi,d}_{X,\boldsymbol{a}}"] & & \\
& \mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow \arrow[dl] \arrow[r, "\cong"] & D^{\varphi,d}_{\boldsymbol{a}} \arrow[dr, "\iota"] & & \\
\mathbb{C}[\boldsymbol{\tau}] \arrow[rrr, "\cong", "{}^s\psi"'] \arrow[rrrr, bend right=40, "{}^s\varphi"'] & & & D_{\boldsymbol{a}} \arrow[r, hook] & \mathbb{C}[\boldsymbol{\xi}]
\end{tikzcd}
$$

Diagram 4. Dualities by the sesquilinear forms $\downarrow$

Summing up we have Diagram 4, where the bold lines indicate the dual pairings $S_{m,\boldsymbol{a}}$ and $S_{n,\boldsymbol{b}}$ with respect to the affine coordinates $\boldsymbol{x}$ and $\boldsymbol{t}$ and the dotted ones indicate those induced by $S_{n,\boldsymbol{b}}$ through the isomorphisms ψ and ${}^s\psi$. The upper half and the lower half correspond mutually by taking adjoints, except the inclusions in the upper half.

Example 10.2. Take an analytic set

$$X := \{(x,y,z) \in \mathbb{C}^3 : z = x + y + x^2 + y^2\}$$

with a parametrisation $\boldsymbol{\Phi}$ defined by

$$x = s, \quad y = t, \quad z = s + t + s^2 + t^2.$$

Obviously we have

$$\mathbb{C}[\boldsymbol{\Phi}]^1_{\boldsymbol{0}}\!\downarrow = \mathrm{Span}_{\mathbb{C}}(1, \sigma, \tau, \sigma^2 + \tau^2).$$

The pushforwards of $\sigma, \sigma^2 \in \mathbb{C}[\boldsymbol{\Phi}]^1_{\boldsymbol{0}}\!\downarrow$ are calculated as follows:

$$\frac{\partial f(s,t,s+t+s^2+t^2)}{\partial s} = f_x + (1+2s) f_z, \qquad {}^s\varphi(\sigma) = \xi + \zeta;$$

$$\begin{aligned}
\frac{\partial^2 f(s,t,s+t+s^2+t^2)}{\partial s^2} &= \frac{\partial}{\partial s}(f_x + (1+2s) f_z) \\
&= f_{xx} + (1+2s) f_{xz} + 2 f_z + (1+2s) f_{zx} + (1+2s)^2 f_{zz},
\end{aligned}$$

$${}^s\varphi(\sigma^2) = 2\zeta + \xi^2 + 2\xi\zeta + \zeta^2.$$

By the symmetry of σ and τ, we have

$$^s\varphi(\tau) = \eta + \zeta, \qquad ^s\varphi(\tau^2) = 2\zeta + \eta^2 + 2\eta\zeta + \zeta^2.$$

Thus

$$\begin{aligned} D_{\mathbf{0}}^{\varphi,1} &= \mathrm{Span}_{\mathbb{C}}(^s\varphi(1), {}^s\varphi(\sigma), {}^s\varphi(\tau), {}^s\varphi(\sigma^2 + \tau^2)) \\ &= \mathrm{Span}_{\mathbb{C}}(1, \xi + \zeta, \eta + \zeta, 4\zeta + \xi^2 + \eta^2 + 2\xi\zeta + 2\eta\zeta + 2\zeta^2). \end{aligned}$$

Then the element $f := \sum_{i,j,k=0}^{\infty} a_{ijk} x^i y^j z^k \in \mathcal{O}_{3,\mathbf{0}}$ belongs to $(\mathbb{C}[\boldsymbol{\Phi}]_{\mathbf{0}}^1\downarrow)^{\perp_3}$ if and only if

$$a_{000} = a_{100} + a_{001} = a_{010} + a_{001} = 2a_{001} + a_{200} + a_{020} + a_{101} + a_{011} + 2a_{002} = 0.$$

All the functions $f(x, y, z) \in \mathcal{O}_{3,\mathbf{0}}$ of degree greater than 2 at $\mathbf{0}$ are contained in $(\mathbb{C}[\boldsymbol{\Phi}]_{\mathbf{0}}^1\downarrow)^{\perp_3}$ and hence the Taylor expansions of their restrictions to X are identically 0. The Taylor expansions of the restrictions to X of linear functions are the identity by property (3) above.

Let us calculate the Taylor expansion of $x^2|_X$. We have

$$S_{3,\mathbf{0}}\langle 1 \mid x^2\rangle = 0, \qquad S_{3,\mathbf{0}}\langle \xi + \zeta \mid x^2\rangle = 0, \qquad S_{3,\mathbf{0}}\langle \eta + \zeta \mid x^2\rangle = 0;$$
$$S_{3,\mathbf{0}}\langle 4\zeta + \xi^2 + \eta^2 + 2\xi\zeta + 2\eta\zeta + 2\zeta^2 \mid x^2\rangle = 2;$$

on the other hand,

$$\begin{gathered} S_{3,\mathbf{0}}\langle 1 \mid a + bx + cy + dz\rangle = a, \\ S_{3,\mathbf{0}}\langle \xi + \zeta \mid a + bx + cy + dz\rangle = b + d, \\ S_{3,\mathbf{0}}\langle \eta + \zeta \mid a + bx + cy + dz\rangle = c + d, \\ S_{3,\mathbf{0}}\langle 4\zeta + \xi^2 + \eta^2 + 2\xi\zeta + 2\eta\zeta + 2\zeta^2 \mid a + bx + cy + dz\rangle = 4d. \end{gathered}$$

Solving the equations $a = 0,\ b + d = 0,\ c + d = 0,\ 4d = 2$, we have $a = 0,\ b = c = -1/2,\ d = 1/2$. This implies that $\mathcal{T}_{X,\mathbf{0}}^{\varphi,1}(x^2|_X) = \frac{-x-y+z}{2}\big|_X$.

Let us recall that, even in the 1-dimensional case, there can exist a point with two different Taylor projectors.

Example 10.3 (Bos–Calvi [BC$_2$, Example 4.2]). Let $X \subset \mathbb{C}^2$ be the plane curve defined by $y - x^2 - x^6 = 0$. Take two local parametrisations of X at $\mathbf{0}$:

$$\begin{aligned} &\varphi : \mathbb{C}_{\mathbf{0}} \to \mathbb{C}^2_{\mathbf{0}}, \quad s \mapsto \boldsymbol{\Phi}(s) := (s, s^2 + s^6), \\ &\psi : \mathbb{C}_{\mathbf{0}} \to \mathbb{C}^2_{\mathbf{0}}, \quad t \mapsto \boldsymbol{\Psi}(t) := (t + t^2, (t + t^2)^2 + (t + t^2)^6). \end{aligned}$$

Then, setting $\sigma := s_{\mathbf{0}}\downarrow$, $\tau := t_{\mathbf{0}}\downarrow$, we have

$$\mathbb{C}[\boldsymbol{\Phi}]_{\mathbf{0}}^2\downarrow = \mathrm{Span}_{\mathbb{C}}(1, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^6), \qquad \mathbb{C}[\boldsymbol{\Psi}]_{\mathbf{0}}^2\downarrow = \mathrm{Span}_{\mathbb{C}}(1, \tau, \tau^2, \tau^3, \tau^4, \tau^6).$$

These lack the degree 5 term and are not D-invariant. We can see that

$$\mathcal{T}_{\mathbf{0}}^{\varphi,2}(x^5|_X) = 0, \qquad \mathcal{T}_{\mathbf{0}}^{\psi,2}(x^5|_X) = 5(y - x^2)|_X.$$

This means that two different local parametrisations sometimes define different Taylor projectors and that our Taylor projector does not necessarily coincide with the ordinary Taylor projector.

DEFINITION 10.4. Set

$$\lambda_{\boldsymbol{\Phi}}(d) := \max\left\{k : \bigoplus_{i=0}^{k} \frac{\mathfrak{m}_{n,\boldsymbol{b}}^{i}}{\mathfrak{m}_{n,\boldsymbol{b}}^{i+1}} \subset \mathbb{C}[\boldsymbol{\Phi}]_{\boldsymbol{b}}^{d}\!\downarrow\right\} = \max\{k : \mathbb{C}[\boldsymbol{\tau}]^{k} \subset \mathbb{C}[\boldsymbol{\Phi}]_{\boldsymbol{b}}^{d}\!\downarrow\}$$
$$= \max\{k : (\mathbb{C}[\boldsymbol{\Phi}]_{\boldsymbol{b}}^{d}\!\downarrow)^{\perp_n} \subset \mathfrak{m}_{n,\boldsymbol{b}}^{k+1}\} = \max\{k : \mathcal{O}_{X,\boldsymbol{a}} \subset P^{d}(X_{\boldsymbol{a}}) + \mathfrak{m}_{X,\boldsymbol{a}}^{k+1}\}.$$

This is independent of the parametrisation $\boldsymbol{\Phi}$.

PROPOSITION 10.5. *We have* $\lambda_{\boldsymbol{\Phi}}(d) \geq d$.

Proof. Since $\boldsymbol{\Phi}$ generates $\mathfrak{m}_{n,\boldsymbol{b}}$, all $\tau_1, \dots, \tau_n$ appear in the linear terms of $\mathbb{C}[\boldsymbol{\Phi}]^{1}$. Hence all terms of $\mathbb{C}[\boldsymbol{\tau}]^{d}$ appear in $\mathbb{C}[\boldsymbol{\Phi}]_{\boldsymbol{b}}^{d}\!\downarrow$. ■

From the inequality $\lambda_{\boldsymbol{\Phi}}(d) \geq d$, we have the following formal error bound for our φ-Taylor polynomial $T_{\boldsymbol{a}}^{\varphi,d}(f)$:

$$f - \kappa \circ T_{\boldsymbol{a}}^{\varphi,d}(f) \in (D_{\boldsymbol{a}}^{\varphi,d})^{\perp_X} \subset \mathfrak{m}_{X,\boldsymbol{a}}^{\lambda_{\boldsymbol{\Phi}}(d)+1} \quad (f \in \mathcal{O}_{X,\boldsymbol{a}}),$$

where $\kappa : P^{d}(X_{\boldsymbol{a}}) \to \mathcal{O}_{X,\boldsymbol{a}}$ denotes inclusion. This formal error bound is equal to or smaller than that of the ordinary Taylor polynomial $T_{\boldsymbol{a}}^{*d}(f)$:

$$f - \kappa(T_{\boldsymbol{a}}^{*d}(f) \bmod I_{\boldsymbol{a}}) \in \mathfrak{m}_{X,\boldsymbol{a}}^{d+1} \quad (f \in \mathcal{O}_{X,\boldsymbol{a}}).$$

The author does not know whether or not

$$\forall f \in \mathcal{O}_{X,\boldsymbol{a}} : \operatorname{ord}_{X,\boldsymbol{a}}(f - \kappa \circ T_{\boldsymbol{a}}^{\varphi,d}(f)) \geq \operatorname{ord}_{X,\boldsymbol{a}}\big(f - \kappa(T_{\boldsymbol{a}}^{*d}(f) \bmod I_{\boldsymbol{a}})\big)$$

where $\operatorname{ord}_{X,\boldsymbol{a}}$ is the order on $X_{\boldsymbol{a}}$ defined by

$$\operatorname{ord}_{X,\boldsymbol{a}}(f) = \max\{k : f \in \mathfrak{m}_{X,\boldsymbol{a}}^{k}\} \quad (f \in \mathcal{O}_{X,\boldsymbol{a}}).$$

DEFINITION 10.6. If $\mathbb{C}[\boldsymbol{\Phi}]_{\boldsymbol{b}}^{d}\!\downarrow$ is D-invariant for all (or some by Lemma 9.5) local parametrisation φ, we call $\boldsymbol{a} = \boldsymbol{\Phi}(\boldsymbol{b}) \in X$ *D-invariant of degree d.* If $\boldsymbol{a}$ is D-invariant of degree d for all $d \in \mathbb{N}$, we call it *D-invariant of degree ∞.* If the φ-Taylor projector of degree d at $\boldsymbol{a} \in X$ is independent of the local parametrisation, we call $\boldsymbol{a}$ *Taylorian of degree d* (following Bos and Calvi). If $\boldsymbol{a}$ is Taylorian of degree d for all $d \in \mathbb{N}$, we call it *Taylorian of degree ∞.*

PROPOSITION 10.7. *Let X be a regular complex submanifold of an open subset of $\mathbb{C}^m$ with* $\dim X \geq 1$. *For $\boldsymbol{a} \in X$, the following conditions are equivalent for each fixed $d \in \mathbb{N}$:*

(1) *$\boldsymbol{a}$ is a D-invariant point of degree d.*
(2) $(\mathbb{C}[\boldsymbol{\Phi}]_{\boldsymbol{b}}^{d}\!\downarrow)^{\perp_n}$ *is an ideal of $\mathcal{O}_{n,\boldsymbol{b}}$ for every* (*or some*) *local parametrisation φ of $X_{\boldsymbol{a}}$.*
(3) $\operatorname{Ker} T_{\boldsymbol{a}}^{\varphi,d} = (D_{\boldsymbol{a}}^{\varphi,d})^{\perp_X}$ *is an ideal of $\mathcal{O}_{X,\boldsymbol{a}}$ for every* (*or some*) *local parametrisation φ.*

(4) $D^{\varphi,d}_{\boldsymbol{a}}$ *is* D*-invariant in* $\mathbb{C}[\boldsymbol{\xi}]$ *for every* (*or some*) *local parametrisation* φ.

If $\boldsymbol{b}$ *is a bundle point of all the jet spaces of* $\mathbb{C}[\boldsymbol{\Phi}]^d$, *all of these conditions hold.*

Proof. Note that the condition of bundle point is independent of the local parametrisation (Theorem 8.2 or Lemma 9.5). The equivalence (1)⇔(2) is proved in Theorem 7.4. Since the image or the inverse image of an ideal under a ring epimorphism is an ideal, the equivalence (2)⇔(3) holds. The equivalence (1)⇔(4) follows from Theorem 8.4(2). The last assertion on a bundle point follows from Theorem 8.2. ■

Example 10.8. Let us take the global parametrisation $\boldsymbol{\Phi} := (s, t, t^2 + st^2, t^3)$ of the surface $X := \{(x, y, z, w) : z = y^2 + xy^2, y^2 = w\} \subset \mathbb{C}^4$. Then $\mathbb{C}[\boldsymbol{\Phi}]^1$ coincides with Z of Example 7.6. All points of the form $(a, 0)$ $(a \neq -1)$ are D-invariant of degree 1 but they are not bundle points of 1-jets of $\mathbb{C}[\boldsymbol{\Phi}]^1$.

For plane algebraic curves, Bos and Calvi proved that D-invariance (gap-free property) is equivalent to the condition that the φ-Taylor projector is independent of the local parametrisation [BC$_2$, Theorem 3.4, 4.10]. Unfortunately this cannot be generalised. The Taylorian property fails in the simplest 2-dimensional example:

Example 10.9. Let us recall the surface $X \subset \mathbb{C}^3$ defined by $x_3 = x_2^2$ in Example 9.6. We have seen two local parametrisations φ and ψ. Although they are related by a linear transformation, their sets of Bos–Calvi tangents of order 1 are different:

$$(D^{\varphi,1}_{\boldsymbol{a}})^{\perp_X \perp_X} = D^{\varphi,1}_{\boldsymbol{a}} \neq D^{\psi,1}_{\boldsymbol{a}} = (D^{\psi,1}_{\boldsymbol{a}})^{\perp_X \perp_X} \qquad (\boldsymbol{a} \in X).$$

Then the kernels $(D^{\varphi,1}_{\boldsymbol{a}})^{\perp_X}$ and $(D^{\varphi,1}_{\boldsymbol{a}})^{\perp_X}$ of the Taylor projectors are different. Hence, no point of X is Taylorian although they are all bundle points.

Remark 10.10. For a general D-invariant point, we can only say that our Taylor projector of order d defines the structure of an Artinian algebra on $P^d(X_{\boldsymbol{a}})$. This structure is isomorphic to $\mathbb{C}\{\boldsymbol{t}\}/(\mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow)^{\perp_n}$, which is transformed by the isomorphism described in Theorem 8.3 as a contravariant tensor by a change of the local parametrisation.

11. Taylorian property of points on embedded curves. If we restrict ourselves to the case of embedded analytic curves, the Taylor projector is well-defined at a general point, which generalises [BC$_2$, Theorem 3.4] on plane algebraic curves.

Theorem 11.1. *Let* X *be a* 1*-dimensional regular complex submanifold of a neighbourhood of* $\boldsymbol{a} \in \mathbb{C}^m$ *and let* $\boldsymbol{\Phi} = (\Phi_1, \ldots, \Phi_m) : \mathbb{C}_{\boldsymbol{b}} \to \mathbb{C}^m_{\boldsymbol{a}}$ *be* *its local parametrisation. Then, for any fixed $d \in \mathbb{N}$, the following three properties of $\boldsymbol{a} \in X$ are equivalent:*

(1) *$\boldsymbol{a}$ is a bundle point of all the jet spaces of $\mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow$.*
(2) *The powers of monomials appearing in $\mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow \subset \mathbb{C}[\tau]$ form a gap-free sequence, i.e. $\boldsymbol{a}$ is D-invariant of degree d.*
(3) *$\boldsymbol{a}$ is Taylorian of degree d (see Definition* 10.6).

Proof. The degree k part of $\mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{t}}\!\downarrow$ is 0-dimensional or 1-dimensional for any $k \in \mathbb{N}_0$. Suppose that $\boldsymbol{a}$ satisfies condition (1). Then the degree k part of $\mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{t}}\!\downarrow$ is constant-dimensional in a neighbourhood of $\boldsymbol{b}$ for each k. If the degree k part is 0-dimensional on a neighbourhood V, the parts with higher degrees are all 0 on V. Then (2) holds.

Since $\dim_{\mathbb{C}} \mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow = \dim_{\mathbb{C}} \mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{a}} = d+1$, condition (2) implies that, for each k with $0 \le k \le d+1$, there is at least one $f_d \in \mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}$ with $f_d^{(k)}(\boldsymbol{b}) \neq 0$. This situation does not change in a neighbourhood, which implies (1).

To prove (2)⇒(3), recall that every ideal of $\mathcal{O}_{X,\boldsymbol{a}} \cong \mathbb{C}\{x\}$ is of the form $\mathfrak{m}^k_{X,\boldsymbol{a}}$, a power of the maximal ideal. If $\boldsymbol{a}$ is a D-invariant point of degree d, $\operatorname{Ker} T^{\varphi,d}_{\boldsymbol{a}} = (D^{\varphi,d}_{\boldsymbol{a}})^{\perp_X}$ is an ideal and it is determined by $\dim_{\mathbb{C}} \mathcal{O}_{X,\boldsymbol{a}}/(D^{\varphi,d}_{\boldsymbol{a}})^{\perp_X} = \dim_{\mathbb{C}} P^d_{X,\boldsymbol{a}}$. Hence it is independent of φ and we have $\operatorname{Ker} T^{\varphi,d}_{\boldsymbol{a}} = \operatorname{Ker} T^{\psi,d}_{\boldsymbol{a}}$ for any other local parametrisation ψ. Suppose that $\boldsymbol{a}$ is a D-invariant point of degree d and take any $f \in \mathcal{O}_{X,\boldsymbol{a}}$. Set $p := T^{\varphi,d}_{\boldsymbol{a}}(f) \in P^d(X_{\boldsymbol{a}})$ and $q := T^{\psi,d}_{\boldsymbol{a}}(f) \in P^d(X_{\boldsymbol{a}})$. In view of the retraction property of the projectors, we have

$$\begin{aligned} p - q &= T^{\varphi,d}_{\boldsymbol{a}}(\kappa(p) - \kappa(q)) = T^{\varphi,d}_{\boldsymbol{a}}\big((\kappa(p) - f) - (\kappa(q) - f)\big) \\ &\in T^{\varphi,d}_{\boldsymbol{a}}\big(\operatorname{Ker} T^{\varphi,d}_{\boldsymbol{a}} - \operatorname{Ker} T^{\psi,d}_{\boldsymbol{a}}\big) = T^{\varphi,d}_{\boldsymbol{a}}\big(\operatorname{Ker} T^{\varphi,d}_{\boldsymbol{a}} - \operatorname{Ker} T^{\varphi,d}_{\boldsymbol{a}}\big) = \{\mathbf{0}\}, \end{aligned}$$

where $\kappa : P^d(X_{\boldsymbol{a}}) \to \mathcal{O}_{X,\boldsymbol{a}}$ denotes inclusion. This proves that $T^{\varphi,d}_{\boldsymbol{a}}(f) = T^{\psi,d}_{\boldsymbol{a}}(f)$. Thus D-invariance implies the Taylorian property.

To prove the converse (3)⇒(2), we follow closely the idea of Bos–Calvi. The least space $\mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow$ is a subspace of $\mathbb{C}[\tau]$. Suppose that $\boldsymbol{a}$ is not a D-invariant point of degree d. Then there exists $s \in \mathbb{N}$ such that $\tau^s \notin \mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow$ and $\tau^{s+1} \in \mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow$. Let s be the maximum of such numbers. There exists a coordinate $x_j \in \mathbb{C}[\boldsymbol{x}]$ such that $\varphi(x_j)_{\boldsymbol{b}}\!\downarrow = \alpha\tau$ with $\alpha \neq 0$, otherwise the image of φ does not include the elements of order 1, contradicting the retraction property. Let l denote the maximal number such that $\tau^{s+l} \in \mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow$. Then $l \geq 1$ and

$$\tau^{s+1}, \tau^{s+2}, \ldots, \tau^{s+l} \in \mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow, \qquad \tau^{s+l+1}, \tau^{s+l+2}, \ldots \notin \mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow$$

by the maximality of s. The least non-zero monomial appearing in $\varphi(x_j^s)$ is $\alpha^s t^s$. Taking $g_{s+i} \in \mathbb{C}[\boldsymbol{\Phi}]^d$ such that $\varphi(g_{s+i})_{\boldsymbol{b}}\!\downarrow = \tau^{s+i}$ $(i = 1, \ldots, l)$, we can eliminate the monomials of degrees $s+1, \ldots, s+l$ appearing in $\varphi(x_j^s)$

by subtracting a linear combination $c_1\varphi(g_{s+1})+\cdots+c_{s+l}\varphi(g_{s+l})$, beginning from g_{s+i} with smaller i. Then if we set

$$h := x_j^s - (c_1 g_{s+1} + \cdots + c_{s+l} g_{s+l}),$$

we have

$$\varphi(h) = \alpha^s t^s + k(t)\cdot t^{s+l+1} \in (\mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}{\downarrow})^{\perp_n}$$

for some $k(t) \in \mathcal{O}_n$. This proves that $T^{\varphi,d}_{\boldsymbol{a}}(h) = 0$.

Now take another local parametrisation

$$\boldsymbol{\Psi}(t) := \boldsymbol{\Phi}(t' + t'^2) = (\Phi_1(t' + t'^2), \ldots, \Phi_n(t' + t'^2)).$$

The function $\psi(h)$ is expressed as

$$\psi(h) = \alpha^s (t' + t'^2)^s + k(t' + t'^2)\cdot (t' + t'^2)^{s+l+1}.$$

Here the coefficient of t'^{s+1} is not 0 and it follows that $S_n\langle \tau'^{s+1} \mid \psi(h)\rangle \neq 0$. Since $\tau^{s+1} \in \mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}{\downarrow}$ implies $\tau'^{s+1} \in \mathbb{C}[\boldsymbol{\Psi}]^d_{\boldsymbol{b}}{\downarrow}$ (see the proof of Theorem 8.3), we see that $T^{\psi,d}_{\boldsymbol{a}}(h) \neq 0$. This is inconsistent with $T^{\varphi,d}_{\boldsymbol{a}}(h) = 0$ and proves that $\boldsymbol{a}$ is not a Taylorian point. ∎

12. Zero-estimate and transcendency index. In this final section we recall that the growth of the space of Bos–Calvi tangents of dual degree d measures the transcendence of the embedding of the manifold germ $X_{\boldsymbol{a}}$. In particular, we explain that the D-invariance property of $X_{\boldsymbol{a}}$ implies that the embedding of $X_{\boldsymbol{a}}$ does not have a high index of transcendency.

First let us recall some known facts on zero-estimates on local algebras. The following invariant $\theta_{\boldsymbol{\Phi}}(d)$ is called "d-order" by Bos and Calvi in [BC$_1$]. It is more important than $\lambda_{\boldsymbol{\Phi}}(d)$ ($\leq \theta_{\boldsymbol{\Phi}}(d)$) defined in §10. This invariant coincides with the one treated by the present author in [Iz$_2$, §1].

Definition 12.1 (Izumi [Iz$_1$], [Iz$_2$]). Let $X_{\boldsymbol{a}}$ be a germ of a complex submanifold of a neighbourhood of $\boldsymbol{a} \in \mathbb{C}^m$ defined by an ideal $I \subset \mathcal{O}_{X,\boldsymbol{a}}$, and $\boldsymbol{\Phi} := (\Phi_1, \ldots, \Phi_m) : \mathbb{C}^n_{\boldsymbol{b}} \to \mathbb{C}^m_{\boldsymbol{a}}$ a local parametrisation of the germ $X_{\boldsymbol{a}}$. Let us use the abbreviation $f|_X := f \bmod I$ (the restriction of $f \in \mathcal{O}_{n,\boldsymbol{a}}$ to $X_{\boldsymbol{a}}$), $\boldsymbol{x}|_X := \{x_1|_X, \ldots, x_m|_X\}$ and $A := \mathcal{O}_{X,\boldsymbol{a}}$. The *zero-estimate function* is defined by

$$\begin{aligned}\theta_{A,\boldsymbol{x}|_X}(d) = \theta_{\mathcal{O}_{n,\boldsymbol{b}},\boldsymbol{\Phi}}(d) &:= \max\{\deg p : p \in \mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}{\downarrow} \setminus \{0\}\}\\ &= \sup\{\operatorname{ord}_{\boldsymbol{b}} F\circ\boldsymbol{\Phi} : F \in \mathbb{C}[\boldsymbol{x}]^d,\ F\circ\boldsymbol{\Phi} \neq 0\}\\ &= \sup\{\operatorname{ord}_{X,\boldsymbol{a}} F|_X : F \in \mathbb{C}[\boldsymbol{x}]^d,\ F|_X \neq 0\}\end{aligned}$$

and the *transcendency index* of $\boldsymbol{\Phi}$ by

$$\alpha(\boldsymbol{\Phi}) := \limsup_{d\to\infty} \log_d \theta_{A,\boldsymbol{x}|_X}(d) = \limsup_{d\to\infty} \log_d \theta_{\mathcal{O}_{n,\boldsymbol{b}},\boldsymbol{\Phi}}(d),$$

where $\operatorname{ord}_{X,\boldsymbol{a}}$ is defined in the last part of §10.

Note that the values of $\theta_{A,\boldsymbol{x}|X}(d) = \theta_{\mathcal{O}_{n,\boldsymbol{b}},\boldsymbol{\Phi}}(d)$ are finite. The zero-estimate function $\theta_{\boldsymbol{\Phi}}(d)$ and the transcendency index $\alpha(\boldsymbol{\Phi})$ are dependent on the embedding $X_{\boldsymbol{a}} \subset \mathbb{C}^m_{\boldsymbol{a}}$ but they are independent of the local parametrisation for a fixed $X_{\boldsymbol{a}} \subset \mathbb{C}^m_{\boldsymbol{a}}$. We know the following.

THEOREM 12.2 (Izumi [Iz_1], [Iz_2]). *Let X be an n-dimensional regular complex submanifold ($n \geq 1$) of a neighbourhood of $\boldsymbol{a} \in \mathbb{C}^m$ and let $\boldsymbol{\Phi}$: $\mathbb{C}^n_{\boldsymbol{b}} \to \mathbb{C}^m_{\boldsymbol{a}}$ be a local parametrisation of X at $\boldsymbol{a}$. Then*

$$\theta_{\mathcal{O}_{X,\boldsymbol{a}},\boldsymbol{x}|X}(d) \geq d, \qquad \alpha(\boldsymbol{\Phi}) \geq 1,$$

and the following conditions are equivalent:

(1) *The germ $X_{\boldsymbol{a}}$ is an analytic irreducible component of the germ of an algebraic set at $\boldsymbol{a}$.*
(2) *There exist $a \geq 1$ and $b \geq 0$ such that*

$$\theta_{\mathcal{O}_{X,\boldsymbol{a}},\boldsymbol{x}|X}(d) = \theta_{\mathcal{O}_{n,\boldsymbol{b}},\boldsymbol{\Phi}}(d) \leq ad + b \qquad (d \in \mathbb{N}).$$

(3) *There exist $a \geq 1$ and $b \geq 0$ such that*

$$\mathbb{C}[\boldsymbol{\Phi}]^d \cap \mathfrak{m}_{n,\boldsymbol{b}}^{ad+b+1} = \{0\}, \quad i.e. \quad \mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}{\downarrow} \cap \frac{\mathfrak{m}^i_{n,\boldsymbol{b}}}{\mathfrak{m}^{i+1}_{n,\boldsymbol{b}}} = \{0\} \ (i > ad + b).$$

(4) $\alpha(\boldsymbol{\Phi}) = 1$.

The latter condition of (3) appears here for the first time. The equivalence (2)⇔(3) is clear from the definition of the least part. We give $\alpha(\boldsymbol{\Phi})$ the name "transcendency index" because of the equivalence (1)⇔(4). It is known that $\alpha(\boldsymbol{\Phi})$ is not necessarily an integer in transcendence theory (cf. [P_1], [P_2], [P-W], see also a beginner's note [Iz_2, Example 3.4]). In [Iz_1] we treat polynomial functions on an analytically irreducible germ of the analytic subset X of an open subset of $\mathbb{C}^m$ and X need not be a smooth manifold. The general inequalities $\theta_{\mathcal{O}_{X,\boldsymbol{a}},\boldsymbol{x}|X}(d) \geq d$ and $\alpha(\boldsymbol{\Phi}) \geq 1$ follow from Proposition 10.5. A complete proof is given in [Iz_2, Theorem 2.3], in a stronger form.

The situation of this theorem may be well illustrated by the following.

EXAMPLE 12.3 (Izumi [Iz_1]). Let C be the transcendental plane curve defined by $y = e^x - 1$. If we parametrise it by $\boldsymbol{\Phi} = (t, e^t - 1)$, we have

$$\mathbb{C}[\boldsymbol{\Phi}]^d = \mathrm{Span}_{\mathbb{C}}(\mathbb{C}[t]^d, \mathbb{C}[t]^{d-1}e^t, \mathbb{C}[t]^{d-2}e^{2t}, \ldots, \mathbb{C}[t]^1 e^{(d-1)t}, e^{dt}).$$

This is just the space of solutions of the differential equation

$$D_t^{d+1}(D_t - 1)^d (D_t - 2)^{(d-1)} \cdots (D_t - d + 1)^2 (D_t - d)^1 f = 0 \qquad (D_t := d/dt).$$

By the elementary theory of ordinary differential equations, for any $a \in \mathbb{C}$, there exists a unique solution f with

$$f^{(\nu)}(a) = \begin{cases} 0 & (0 \leq \nu \leq (d+1)(d+2)/2 - 2), \\ 1 & (\nu = (d+1)(d+2)/2 - 1), \end{cases}$$

and there exists no solution $f \neq 0$ with

$$f^{(\nu)}(a) = 0 \quad (0 \leq \nu \leq (d+1)(d+2)/2) - 1).$$

This proves that $\theta_{\mathcal{O}_{1,a},\{t-a,\exp(t-a)-1\}}(d) = (d+1)(d+2)/2 - 1$ and $\alpha = 2$ at all points $(a, e^a - 1) \in C$.

Remark 12.4. An example of a plane curve with an extremely transcendental point ($\alpha(\boldsymbol{\Phi}) = \infty$) was given by Tetsuo Ueda (see [Iz$_1$], Example 2), using a gap power series, a functional analogue of the Liouville constant.

For an embedding germ $X_{\boldsymbol{a}} \subset \mathbb{C}^m$, let $\overline{X}_{\boldsymbol{a}}$ denote the Zariski closure of $X_{\boldsymbol{a}}$ in $\mathbb{C}^m$, that is, the smallest algebraic subset of $\mathbb{C}^m$ that includes some representative of the germ $X_{\boldsymbol{a}}$ (germs are always taken with respect to the Euclidean topology). Then the Hilbert function of $\overline{X}_{\boldsymbol{a}}$ is defined by

$$\chi(\overline{X}_{\boldsymbol{a}}, d) := \dim_{\mathbb{C}} P^d(X_{\boldsymbol{a}}) - \dim_{\mathbb{C}} P^{d-1}(X_{\boldsymbol{a}}) = \dim_{\mathbb{C}} D_{\boldsymbol{a}}^{\varphi,d} - \dim_{\mathbb{C}} D_{\boldsymbol{a}}^{\varphi,d-1}$$

with $\dim_{\mathbb{C}} P^{-1}(X_{\boldsymbol{a}}) = \dim_{\mathbb{C}} D_{\boldsymbol{a}}^{\varphi,-1} = 0$ (cf. Definition 9.4). This is known to coincide with a polynomial of degree $\dim \overline{X}_{\boldsymbol{a}} - 1$ in d for sufficiently large d.

Theorem 12.5. *Let $\varphi : \mathcal{O}_{m,\boldsymbol{a}} \to \mathcal{O}_{n,\boldsymbol{b}}$ $(n \geq 1)$ be a local parametrisation of an embedded manifold germ $X_{\boldsymbol{a}} \subset \mathbb{C}^m$ with component functions $\boldsymbol{\Phi} = (\Phi_1, \ldots, \Phi_m)$. If $\boldsymbol{a}$ is a D-invariant point of degree d, we have a zero-estimate inequality:*

$$\binom{n+d}{n} + \theta_{\mathcal{O}_{n,\boldsymbol{b}},\boldsymbol{\Phi}}(d) - d \leq \dim_{\mathbb{C}} \mathbb{C}[\boldsymbol{\Phi}]^d = \sum_{i=0}^{d} \chi(\overline{X}_{\boldsymbol{a}}, i) \leq \binom{m+d}{m}.$$

Hence, if $\boldsymbol{a}$ is a D-invariant point of degree ∞, we have an estimate of the transcendency index:

$$1 \leq \alpha(\boldsymbol{\Phi}) \leq \dim \overline{X}_{\boldsymbol{a}} \leq m.$$

Proof. Note that

$$\dim_{\mathbb{C}} P^d(X_{\boldsymbol{a}}) = \dim_{\mathbb{C}} \mathbb{C}[\boldsymbol{\Phi}]^d$$

by the isomorphism stated in Definition 9.4. If we take $p \in \mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow$ with $\deg p = \theta_{A,\{\Phi_1,\ldots,\Phi_m\}}(d)$ (the maximal degree), this dimension $\dim_{\mathbb{C}} \mathbb{C}[\boldsymbol{\Phi}]^d$ majorises the sum of the dimensions of the following linear subspaces:

(1) the space $\mathbb{C}[\boldsymbol{\tau}]^d$ that appeared in Proposition 10.5;
(2) the linear span of

$$\left\{ \frac{\partial^{|\boldsymbol{\nu}|} p}{\partial \boldsymbol{\tau}^{\boldsymbol{\nu}}} : d < \operatorname{ord}_{\mathfrak{m}} \frac{\partial^{|\boldsymbol{\nu}|} p}{\partial \boldsymbol{\tau}^{\boldsymbol{\nu}}} < \infty,\ \boldsymbol{\nu} \in \mathbb{N}_0^n \right\}$$

(by D-invariance).

Since the intersection of these spaces is $\{\mathbf{0}\}$, we have the left inequality of the first assertion. The right inequality follows from

$$\dim_{\mathbb{C}} P^d(X_{\boldsymbol{a}}) \le \dim_{\mathbb{C}} \mathbb{C}[\boldsymbol{x}]^d = \binom{m+d}{m}.$$

The first inequality in the theorem implies that

$$\theta_{\mathcal{O}_{n,\boldsymbol{b}},\{\Phi_1,\dots,\Phi_m\}}(d) \le \dim_{\mathbb{C}} \mathbb{C}[\boldsymbol{\Phi}]^d = \dim_{\mathbb{C}} P^d(X_{\boldsymbol{a}}) = \sum_{i=0}^{d} \chi(\overline{X}_{\boldsymbol{a}}, i).$$

Since the Hilbert function of $\overline{X}_{\boldsymbol{a}}$ coincides with a polynomial of degree $\dim \overline{X}_{\boldsymbol{a}} - 1$ for sufficiently large d, the last term is comparable to $d^{\dim \overline{X}_{\boldsymbol{a}}}$, and the inequality $\alpha(\boldsymbol{\Phi}) \le \dim \overline{X}_{\boldsymbol{a}}$ follows. ∎

Let $\boldsymbol{\Phi} : U \to \mathbb{C}^m$ be an embedding of an open subset of $\mathbb{C}^n$ onto a submanifold $X \subset \mathbb{C}^m$, i.e. $\boldsymbol{\Phi}$ induces a biholomorphic homeomorphism onto the image. We have seen that the complement of the set of bundle points of all the jet spaces of $\mathbb{C}[\boldsymbol{\Phi}]^d$ for all $d \in \mathbb{N}$ is contained in a countable union of thin closed analytic subsets of X. Then the zero-estimate inequality in Theorem 12.5 implies the following global result.

COROLLARY 12.6. *Let X be an n-dimensional regular complex submanifold of an open subset $\Omega \subset \mathbb{C}^m$ $(n \ge 1)$. Then there exists a countable union A of thin closed analytic subsets of X such that, for any local parametrisation $\boldsymbol{\Phi}$ at $\boldsymbol{a} \in \Omega \setminus A$, we have $\alpha(\boldsymbol{\Phi}) \le \dim \overline{X}_{\boldsymbol{a}} \le m$. Note that the set A is of first category in Baire's sense and with Lebesgue measure 0 in X.*

REMARK 12.7. Gabrielov gives a zero-estimate [Ga, Theorem 5] of Noetherian functions on an integral curve of a Noetherian vector field (see also [GK]). It immediately yields a zero-estimate of Noetherian functions on $\mathbb{C}^n$ as follows. Suppose that

$$\boldsymbol{\Psi} := \{\boldsymbol{x}, \boldsymbol{\Phi}\} \subset \mathcal{O}_{n,\boldsymbol{b}}$$

is a join of an affine coordinate system $\boldsymbol{x} := (x_1, \dots, x_n)$ and a Noetherian chain $\boldsymbol{\Phi} := \{\Phi_1, \dots, \Phi_m\}$ of order m, which means that

$$\frac{\partial \Phi_i}{\partial x_j} = P_{ij}(x_1, \dots, x_n, \Phi_1, \dots, \Phi_m) \quad (i = 1, \dots, m;\ j = 1, \dots, n)$$

for some polynomials P_{ij}. This $\boldsymbol{\Psi}$ is the set of mapping components of the embedding onto the graph $X \subset \mathbb{C}^{m+n}$ of the Noetherian chain $\{\Phi_1, \dots, \Phi_m\}$. Then we have

$$\alpha(\boldsymbol{\Psi}) \le 2(m+n)$$

(cf. [Iz$_4$, Corollary 12]).

Our Corollary 12.6 gives a slightly stronger estimate:

$$\alpha(\boldsymbol{\Psi}) \le \dim \overline{X}_{\boldsymbol{a}} \le m+n$$

for (only for) the complement of a small subset of X without the Noetherian condition. In view of Remark 12.4, exclusion of some point set is inevitable for our general analytic case.

PROPOSITION 12.8. *Let $X_{\boldsymbol{a}} \subset \mathbb{C}^m$ and $X'_{\boldsymbol{a}'} \subset \mathbb{C}^m$ be affine equivalent germs of embedded manifolds, i.e. there exists an affine transformation $\Theta : \mathbb{C}^m \to \mathbb{C}^m$ which maps $X_{\boldsymbol{a}} \subset \mathbb{C}^m$ to $X'_{\boldsymbol{a}'} \subset \mathbb{C}^m$ biholomorphically. Then $X_{\boldsymbol{a}} \subset \mathbb{C}^m$ and $X'_{\boldsymbol{a}'} \subset \mathbb{C}^m$ simultaneously have (or do not have) the properties of bundle point, D-invariance, and being Taylorian, and they have the same $\theta(d)$ and α.*

Proof. Let $\boldsymbol{\Phi}$ be a local parametrisation of $X_{\boldsymbol{a}}$. Then $\boldsymbol{\Phi}' := \Theta \circ \boldsymbol{\Phi}$ is a local parametrisation of $X'_{\boldsymbol{a}'}$. Since Θ is affine, $\mathbb{C}[\boldsymbol{\Phi}]^d = \mathbb{C}[\boldsymbol{\Phi}']^d$ and hence $\mathbb{C}[\boldsymbol{\Phi}]^d_{\boldsymbol{b}}\!\downarrow = \mathbb{C}[\boldsymbol{\Phi}']^d_{\boldsymbol{b}}\!\downarrow$. This implies everything. ∎

It is easy to see that germs of a quadratic curve in $\mathbb{C}^2$ at any pair of points are affine equivalent. Of course $\alpha = 1$ in this case because they are algebraic. It is interesting that the germs at points on the transcendental curve $y = \exp x - 1$ are also affine equivalent. In this case $\theta(d)$ and α are already given in Example 12.3.

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