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On the growth of an algebroid function with radially distributed values

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Abstract. We investigate how the growth of an algebroid function could be affected by the distribution of the arguments of its a-points in the complex plane. We give estimates of the growth order of an algebroid function with radially distributed values, which are counterparts of results for meromorphic functions with radially distributed values.

1. Introduction and the results. We assume that the reader is familiar with the fundamental results and standard notations of Nevanlinna theory in the unit disk $\varDelta = \{z : |z| < 1\}$ and in the complex plane $\mathbb{C}$ (see [5, 10, 14, 22]). A value on the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called a radially distributed value of a transcendental meromorphic function if most of the points at which the value is assumed distribute closely along a finite number of rays from the origin. The growth of meromorphic functions with radially distributed values has been thoroughly studied (see [1], [3], [4], [12, 13], [17, 18, 19] and [20, 21, 22]).

However, similar results concerning algebroid functions are rather few. This motivated us to investigate this case. Indeed in [23], we suggested that some aspects of algebroid functions are worthy of consideration, the first one being:

Problem. *How does an algebroid function grow when some restriction is imposed on arguments of certain a-points?*

The purpose of this paper is to discuss this problem. Before stating our results, we give some notations and definitions. Let $f = f(z)$ be the ν-valued algebroid function determined by an irreducible equation

$$F(z,w) := A_0(z)w^{\nu} + A_1(z)w^{\nu-1} + \cdots + A_{\nu}(z) = 0, \tag{1.1}$$

where $A_{\nu}(z), \ldots, A_0(z)$ are entire functions, at least one of which is transcen-

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dental, without any common zeros. Let $\vec{A} = (A_0, \ldots, A_\nu)$, $\vec{\infty} = (1, 0, \ldots, 0)$. For any $a \in \mathbb{C}$, put $\vec{a} = (a^\nu, a^{\nu-1}, \ldots, 1)$. Then we set

$$\|\vec{A}(z)\| = (|A_0|^2 + |A_1|^2 + \cdots + |A_\nu|^2)^{1/2},$$
$$\|\vec{a}\| = \begin{cases} (|a|^{2\nu} + |a|^{2\nu-2} + \cdots + |a|^2 + 1)^{1/2}, & a \neq \infty, \\ 1, & a = \infty. \end{cases}$$

Since $F(z, w)$ is irreducible, one can have $F(z, a) = \vec{A}(z) \cdot \vec{a} \not\equiv 0$, where $F(z, \infty) = A_0(z)$. Set $\log^+ x = \max\{0, \log x\}$ and define

$$\begin{aligned} m(r, \vec{a}, \vec{A}) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{\|\vec{A}(re^{i\theta})\| \, \|\vec{a}\|}{|F(re^{i\theta}, a)|} \, d\theta, \\ N(r, \vec{a}, \vec{A}) &= N(r, 0, F(z, a)) \\ &= \int_0^r \frac{n(t, 0, F(z, a)) - n(0, 0, F(z, a))}{t} \, dt + n(0, 0, F(z, a)) \log r, \\ T(r, \vec{a}, \vec{A}) &= m(r, \vec{a}, \vec{A}) + N(r, \vec{a}, \vec{A}), \end{aligned}$$

where $n(t, 0, F(z, a))$ is the number of roots of the equation $F(z, a) = 0$ in the disk $\{|z| \leq t\}$, counting multiplicities. More generally, $n(t, a, f(z))$ denotes the number of roots of $f(z) = a$ in $\{|z| \leq t\}$, counting multiplicities.

Following G. Valiron, we define the characteristic function of $f(z)$ as

$$T(r, f) = \frac{1}{2\nu\pi} \int_0^{2\pi} \log \max_{0 \leq j \leq \nu} |A_j(re^{i\theta})| \, d\theta.$$

By using Valiron's result (cf. [16]), we get a relation between $T(r, f)$ and $T(r, \vec{a}, \vec{A})$:

$$|T(r, \vec{a}, \vec{A}) - \nu T(r, f)| = O(1).$$

The counting function of a-points of $f(z)$ is defined as

$$N(r, a, f) = \frac{1}{\nu} N(r, 0, F(z, a)).$$

Put

$$\delta(a, f) = 1 - \limsup_{r \to \infty} \frac{N(r, a, f)}{T(r, f)} = 1 - \limsup_{r \to \infty} \frac{N(r, 0, F(z, a))}{T(r, \vec{a}, \vec{A})}.$$

The value a is called a *Nevanlinna deficient value* of f if $\delta(a, f) > 0$. The *order* and *lower order* of $f(z)$ are defined as

$$\lambda(f) := \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}, \qquad \mu(f) := \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

Given an angular domain $X = \{z : \alpha < \arg z < \beta\}$ $(0 < \beta - \alpha < 2\pi)$, define the counting function of a-points of $f(z)$ in X as

$$N(r,X,f=a)=\frac{1}{\nu}\int_1^r\frac{n(t,X,f=a)}{t}\,dt,$$

where $n(t,X,f=a)$ is the number of roots of $f(z)=a$ in $X\cap\{z:|z|<t\}$, counting multiplicities.

For algebroid functions, Niino [8] obtained the following theorem.

THEOREM A ([8]). *Let $f(z)$ be a ν-valued algebroid function of lower order μ satisfying $1\le\mu<\infty$ in $\mathbb{C}$ and with $\delta(a,f)>0$ for some $a\in\widehat{\mathbb{C}}$. Let Ω be an angular domain defined by*

$$\Omega=\left\{z:|\arg z-\theta|<\pi-\frac{2}{\mu}\arcsin\sqrt{\frac{\delta}{2}+\eta}\right\},\qquad 0\le\theta<2\pi,$$

where $\eta>0$ is a small real number. Suppose that the solutions in Ω of $f(z)=a$ are finite in number. Then the equation $f(z)=c$ has an infinite number of solutions in Ω except for at most $2\nu-1$ values of $c\ne a$.

We consider q pairs of real numbers $\{\alpha_j,\beta_j\}$ such that

$$-\pi\le\alpha_1<\beta_1\le\alpha_2<\beta_2\le\cdots\le\alpha_q<\beta_q\le\pi, \tag{1.2}$$

and define $\omega=\max_{1\le i\le q}\left\{\frac{\pi}{\beta_i-\alpha_i}\right\}$. We will establish the following results.

THEOREM 1.1. *Let $f(z)$ be the ν-valued algebroid function of finite lower order $\mu<\infty$ in $\mathbb{C}$ determined by* (1.1) *and with $\delta=\delta(a,f)>0$ for some $a\in\widehat{\mathbb{C}}$. If for q pairs of real numbers $\{\alpha_i,\beta_i\}$ satisfying* (1.2) *and 2ν distinct complex values $a_i\ne a$ $(i=1,\dots,2\nu)$, we have*

$$\begin{gathered}\sum_{i=1}^{2\nu}n(r,Y,f=a_i)=o(T(dr,f)),\\ n(r,Y,f=a)=o\left(\frac{T(dr,f)}{\log r}\right)\qquad(d\ge1),\end{gathered} \tag{1.3}$$

for $Y=\bigcup_{j=1}^{q}\{z:\alpha_j<\arg z<\beta_j\}$ and

$$\sum_{i=1}^{q}(\alpha_{i+1}-\beta_i)<\frac{4}{\beta}\arcsin\sqrt{\frac{\delta}{2}},\qquad\alpha_{q+1}=2\pi+\alpha_1, \tag{1.4}$$

where $\beta=\max\{\omega,\mu\}$, then $\lambda(f)\le\omega$.

THEOREM 1.2. *Let $f(z)$ and a be as in Theorem* 1.1. *If for q pairs of real numbers $\{\alpha_i,\beta_i\}$ satisfying* (1.2) *and 2ν distinct complex values $a_i\ne a$ $(i=1,\dots,2\nu)$, we have*

$$\limsup_{r\to\infty}\frac{\log^+\left(\sum_{i=1}^{2\nu}n(r,Y,f=a_i)+n(r,Y,f=a)\right)}{\log r}\le\rho \tag{1.5}$$

for $Y = \bigcup_{j=1}^{q}\{z : \alpha_j < \arg z < \beta_j\}$ *and* (1.4) *for* $\beta = \max\{\omega, \rho, \mu\}$*, then* $\lambda(f) \le \max\{\omega, \rho\}$.

From Theorem 1.1, we have the following corollary which improves Theorem A.

COROLLARY 1.3. *Let* $f(z)$ *be the* ν*-valued algebroid function of lower order* μ *satisfying* $0 < \mu < \infty$ *in* $\mathbb{C}$ *determined by* (1.1) *and with* $\delta = \delta(a, f) > 0$ *for some* $a \in \widehat{\mathbb{C}}$*. If for any angular domain* $X = \{z : \alpha < \arg z < \beta\}$ $(0 < \beta - \alpha \le 2\pi)$ *satisfying*

$$\beta - \alpha > \max\left\{\frac{\pi}{\mu}, 2\pi - \frac{4}{\mu}\arcsin\sqrt{\frac{\delta}{2}}\right\},$$

we have

$$n(r, X, f = a) = o\left(\frac{T(dr, f)}{\log r}\right),$$

then there exists a ray $\arg z = \theta \in (\alpha, \beta)$ *such that for any small* $\varepsilon > 0$*, and any* 2ν *distinct complex values* $a_i \ne a$ $(i = 1, \ldots, 2\nu)$,

$$\limsup_{r\to\infty} \frac{\sum_{i=1}^{2\nu} n(r, Z_\varepsilon(\theta), f = a_i)}{T(dr, f)} > 0, \tag{1.6}$$

where $Z_\varepsilon(\theta) = \{z : \theta - \varepsilon < \arg z < \theta + \varepsilon\}$ *and* $d \ge 1$.

REMARK. Let us show that Theorem A follows from Corollary 1.3. We take into account the amplitude of the angular domain Ω in Theorem A. Since

$$2\pi - \frac{4}{\mu}\arcsin\sqrt{\frac{\delta}{2}} + 2\eta \ge 2\pi - \frac{\pi}{\mu} + 2\eta \ge \frac{\pi}{\mu} + 2\eta,$$

by noting $\mu \ge 1$ in Theorem A, the amplitude of Ω is greater than

$$\max\left\{\frac{\pi}{\mu}, 2\pi - \frac{4}{\mu}\arcsin\sqrt{\frac{\delta}{2}}\right\}.$$

Since $f(z)$ in Corollary 1.3 is an algebroid function and not an algebraic function, we have

$$\frac{T(r, f)}{\log r} \to \infty \quad \text{as } r \to \infty.$$

If the equation $f(z) = a$ has finite roots in X, then we must have

$$n(r, X, f = a) = o\left(\frac{T(r, f)}{\log r}\right).$$

Therefore, in terms of Corollary 1.3, Ω contains a ray $\arg z = \phi$ such that (1.6) holds. We obtain immediately the result of Theorem A.

2. Some lemmas. First we need some auxiliary results for the proof of the theorems. The following result can be found in [22] for meromorphic functions.

LEMMA 2.1 ([22]). *Let $f(z)$ be an algebroid function in $\mathbb{C}$ of finite lower order $0 \le \mu < \infty$ and order $0 < \lambda \le \infty$. Then for any finite and positive number β satisfying $\mu \le \beta \le \lambda$ and a set E of finite logarithmic measure, i.e., $\int_E t^{-1}\,dt < \infty$, there exists a sequence $\{r_n\}$ of positive numbers such that*

(1) $r_n \notin E$ *and* $\lim_{n\to\infty} \frac{r_n}{n} = \infty$*;*

(2) $\liminf_{n\to\infty} \frac{\log T(r_n, f)}{\log r_n} \ge \beta$*;*

(3) $T(t,f) < (1+o(1))(2t/r_n)^{\beta} T(r_n/2, f)$ *for* $t \in [r_n/n, nr_n]$*;*

(4) $T(t,f)/t^{\beta-\varepsilon_n} \le 2^{\beta+1} T(r_n, f)/r_n^{\beta-\varepsilon_n}$ *for* $1 \le t \le nr_n$*, where* $\varepsilon_n = [\log n]^{-2}$.

Since the characteristic function $T(r,f)$ of an algebroid function $f(z)$ is also a non-decreasing, positive and continuous function defined in $(0,\infty)$, one can derive Lemma 2.1 directly from [22]. A sequence $\{r_n\}$ satisfying (1)–(4) of Lemma 2.1 is called a sequence of *Pólya peaks* of order β outside E. Given a positive function $\Lambda(r)$ satisfying $\lim_{r\to\infty} \Lambda(r) = 0$, for $r > 0$ and $a \in \widehat{\mathbb{C}}$, define

$$E_\Lambda(r,a) = \left\{\theta : \log^+ \frac{\|\vec{A}(re^{i\theta})\|\,\|\vec{a}\|}{|F(re^{i\theta}, a)|} > \Lambda(r) T(r,f)\right\}.$$

By meas(E) we denote the Lebesgue measure of the set E. The following result is a comprehensive version of the main result of Krytov [7] and Yang [16].

LEMMA 2.2 ([7, 16]). *Let $f(z)$ be a ν-valued algebroid function in $\mathbb{C}$ of finite lower order μ and order $0 < \lambda \le \infty$ with $\delta = \delta(a,f) > 0$ for some $a \in \widehat{\mathbb{C}}$. Then for any sequence $\{r_n\}$ of Pólya peaks of order $\beta > 0$ where $\mu \le \beta \le \lambda$, and for any positive function $\Lambda(r)$ with $\Lambda(r) \to 0$ as $r \to \infty$,*

$$\liminf_{n\to\infty} \operatorname{meas}(E_\Lambda(r_n, a)) \ge \min\left\{2\pi, \frac{4}{\beta} \arcsin\sqrt{\frac{\delta}{2}}\right\}. \tag{2.1}$$

Lemma 2.2 is called the spread relation of algebroid functions; it was proved in [7, 16] for Pólya peaks of order μ. By the same argument, one can derive Lemma 2.2 for Pólya peaks of order β $(\mu \le \beta \le \lambda)$.

LEMMA 2.3. *Let $f(z)$ be the ν-valued algebroid function determined by* (1.1) *in the complex plane. Assume that $u = u(z)$ is a conformal mapping* *from the angular domain* $X = \{z : \alpha < \arg z < \beta\}$ $(0 < \beta - \alpha \le 2\pi)$ *onto the unit disk. Let* $z(u)$ *be the inverse mapping of* $u(z)$*. Then* $f(z(u))$ *is a* ν*-valued algebroid function defined in the unit disk.*

Proof. It is obvious that $f(z(u))$ is a ν-valued algebroid function determined by the equation

$$F(z(u), w) := A_0(z(u))w^\nu + A_1(z(u))w^{\nu-1} + \cdots + A_{\nu-1}(z(u))w + A_\nu(z(u)) = 0.$$

The fact that $A_i(z)$ $(i = 1, \ldots, \nu)$ are entire functions implies that the composite functions $A_0(z(u)), \ldots, A_\nu(z(u))$ are also analytic. As $A_0(z), \ldots, A_\nu(z)$ have no common zeros, also $A_0(z(u)), \ldots, A_\nu(z(u))$ have no common zeros, because if $u = u_0$ is a common zero, then $z(u_0)$ is a common zero of $A_0(z), \ldots, A_\nu(z)$. Since $F(z, w)$ is irreducible, so is $F(z(u), w)$ because if

$$F(z(u), w) = F_1(u, w)F_2(u, w),$$

then

$$F(z, w) = F_1(u(z), w)F_2(u(z), w),$$

which is a contradiction. Hence, the proof is complete. ∎

Lemma 2.4 ([17]). *The transformation*

$$\zeta(z) = \frac{(ze^{-i\theta_0})^{\pi/(\beta-\alpha)} - 1}{(ze^{-i\theta_0})^{\pi/(\beta-\alpha)} + 1} \qquad (\theta_0 = (\alpha + \beta)/2) \tag{2.2}$$

maps the angular domain $X = \{z : \alpha < \arg z < \beta\}$ $(0 < \beta - \alpha \le 2\pi)$ *conformally onto the unit disk* $\{\zeta : |\zeta| < 1\}$ *in the* ζ*-plane, and maps* $z = e^{i\theta_0}$ *to* $\zeta = 0$*. The image of* $X_\varepsilon(r) = \{z : 1 \le |z| \le r, \alpha + \varepsilon \le \arg z \le \beta - \varepsilon\}$ $(0 < \varepsilon < (\beta - \alpha)/2)$ *in the* ζ*-plane is contained in the disk* $\{\zeta : |\zeta| \le h\}$*, where*

$$h = 1 - \frac{\varepsilon}{\beta - \alpha} r^{-\pi/(\beta-\alpha)}.$$

On the other hand, the inverse image of the disk $\{\zeta : |\zeta| \le h\}$ $(h < 1)$ *in the* z*-plane is contained in* $X \cap \{z : |z| \le r\}$*, where*

$$r = \left(\frac{2}{1-h}\right)^{(\beta-\alpha)/\pi}.$$

Moreover, for $|\zeta| \le h$*, we have*

$$\frac{\beta - \alpha}{\pi}\left(\frac{1-h}{2}\right)^{(\beta-\alpha)/\pi} \le |z'(\zeta)| \le \frac{\beta - \alpha}{\pi}\left(\frac{2}{1-h}\right)^{1+(\beta-\alpha)/\pi}, \tag{2.3}$$

where $z(\zeta)$ *is the inverse of the transformation* (2.2).

The proof of Lemma 2.4 can be found in [17].

LEMMA 2.5 ([6, 9]). *Suppose that $f(z)$ is a ν-valued algebroid function defined in the unit disk, and $a_i \in \widehat{\mathbb{C}}$ $(i = 1, \ldots, q)$ are q $(> 2\nu)$ distinct complex values. Then*

$$(q - 2\nu)T(r, f) \leq \sum_{i=1}^{q} N(r, a_i, f) + O(\log(1 - r)^{-1} + \log T(r, f))$$

for all r possibly outside a set $F \subset (0, 1)$ with $\int_F dr/(1 - r) < \infty$.

Lemma 2.5 is called the second fundamental theorem for algebroid functions in the unit disk; its proof can be found in [6, 9].

Now, we use by the Poisson–Jensen formula for meromorphic functions in order to estimate the logarithmic module $\log^+ \frac{\|\vec{A}(z)\| \, \|\vec{a}\|}{|F(z,a)|}$.

LEMMA 2.6. *Let $f(\xi)$ be the ν-valued algebroid function determined by (1.1) in the unit disk. Then, for any $z = re^{i\theta}$ such that $0 < r < R < 1$, we have*

$$\log^+ \frac{\|\vec{A}(z)\| \, \|\vec{a}\|}{|F(z, a)|} \leq \log^+(\nu + 1)^{1/2} + \frac{R + r}{R - r} m(R, \vec{a}, \vec{A}) + \sum_{t=1}^{M} \log \left| \frac{2R}{z - b_t} \right|, \tag{2.4}$$

where $b_1, \ldots, b_M$ are all the roots of $f(\xi) = a$ in $|\xi| < R$ appearing according to their multiplicities and $M = n(R, a, f)$.

Proof. We will prove that (2.4) holds for every point z. For any $z = re^{i\theta}$, $0 < r < R < 1$, there exists an integer $0 \leq k = k_z \leq \nu$ such that

$$\max_{0 \leq l \leq \nu} |A_l(z)| = |A_k(z)|.$$

Then

$$\begin{aligned} \log^+ \frac{\|\vec{A}(z)\| \, \|\vec{a}\|}{|F(z, a)|} &\leq \log^+ \frac{(\nu + 1)^{1/2} |A_k(z)| \, \|\vec{a}\|}{|F(z, a)|} \\ &\leq \log^+(\nu + 1)^{1/2} + \log^+ \frac{|A_k(z)| \, \|\vec{a}\|}{|F(z, a)|} \\ &= \log^+(\nu + 1)^{1/2} + \log^+ \left| \frac{A_k(z)}{F(z, a)} \|\vec{a}\| \right|. \end{aligned}$$

Since both $A_k(\xi)$ and $F(\xi, a)$ are analytic functions in the unit disk and $\|\vec{a}\|$ is a constant number, we deduce that $A_k(\xi)\|\vec{a}\|/F(\xi, a)$ is a meromorphic function in the unit disk. Now we apply the Poisson–Jensen formula to this function:

$$\log^+\left|\frac{A_k(z)\|\vec{a}\|}{F(z,a)}\right| \le \frac{1}{2\pi}\int\limits_0^{2\pi}\log^+\left|\frac{A_k(Re^{i\phi})\|\vec{a}\|}{F(Re^{i\phi},a)}\right|\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}\,d\phi + \sum_{t=1}^{M}\log\left|\frac{R^2-\overline{b}_t z}{R(z-b_t)}\right|.$$

Using the inequality $\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2} \le \frac{R+r}{R-r}$, we derive

$$\begin{aligned}\log^+\left|\frac{A_k(z)\|\vec{a}\|}{F(z,a)}\right| &\le \frac{1}{2\pi}\,\frac{R+r}{R-r}\int\limits_0^{2\pi}\log^+\left|\frac{A_k(Re^{i\phi})\|\vec{a}\|}{F(Re^{i\phi},a)}\right|d\phi + \sum_{t=1}^{M}\log\left|\frac{2R}{z-b_t}\right| \\ &= \frac{1}{2\pi}\,\frac{R+r}{R-r}\int\limits_0^{2\pi}\log^+\frac{|A_k(Re^{i\phi})|\,\|\vec{a}\|}{|F(Re^{i\phi},a)|}\,d\phi + \sum_{t=1}^{M}\log\left|\frac{2R}{z-b_t}\right|.\end{aligned}$$

In view of $|A_k(Re^{i\phi})| \le \|\vec{A}(Re^{i\phi})\|$, we have

$$\log^+\left|\frac{A_k(z)\|\vec{a}\|}{F(z,a)}\right| \le \frac{R+r}{R-r}m(R,\vec{a},\vec{A}) + \sum_{t=1}^{M}\log\left|\frac{2R}{z-b_t}\right|.$$

This completes the proof of (2.4). ∎

Now, we establish a lemma of independent interest. Define, for $0 \le \alpha < \beta < 2\pi$,

$$m_{\alpha,\beta}(r,\vec{a},\vec{A}) = \frac{1}{2\pi}\int\limits_\alpha^\beta \log^+\frac{\|\vec{A}(re^{i\theta})\|\,\|\vec{a}\|}{|F(re^{i\theta},a)|}\,d\theta.$$

Lemma 2.7. *Let $f(z)$ be the ν-valued algebroid function determined by (1.1) in $\mathbb{C}$. Set $X = \{z : \alpha < \arg z < \beta\}$ $(0 < \beta-\alpha < 2\pi)$. Then, for any $0 < \varepsilon < (\beta-\alpha)/2$ and for all r except a set E of finite logarithmic measure,*

$$\begin{aligned}(2.5)\qquad & m_{\alpha+\varepsilon,\beta-\varepsilon}(r,\vec{a},\vec{A}) \\ & \le C_\varepsilon\left(r^\omega\int\limits_1^{kr}\frac{\sum_{i=1}^{2\nu}n(x,X,f=a_i)}{x^{1+\omega}}\,dx + n(kr,X,f=a)\log r + 1\right),\end{aligned}$$

where $\omega = \pi/(\beta-\alpha)$, $k = (8(\beta-\alpha)/\varepsilon)^{1/\omega}$ and C_ε is a positive number depending on ε and $\beta-\alpha$.

Proof. By Lemmas 2.3 and 2.4, $f(z(\zeta))$ is the ν-valued algebroid function in the unit disk determined by the irreducible equation

$$F(z(\zeta),w) = A_0(z(\zeta))w^\nu + A_1(z(\zeta))w^{\nu-1} + \cdots + A_\nu(z(\zeta)) = 0,$$

where $z(\zeta)$ is the inverse of the transformation (2.2). Using Lemma 2.4 and noticing that the number of roots of an equation in a region is a conformal invariant, we have

$$n(t,\tau,f(z(\zeta)))-n(0,\tau,f(z(\zeta)))\le n\biggl(\Bigl(\frac{2}{1-t}\Bigr)^{1/\omega},X,f(z)=\tau\biggr)$$

for $\tau=a_i$ $(i=1,\ldots,2\nu)$ and for $\tau=a$. Therefore,

$$\begin{aligned}N(h',\tau,f(z(\zeta)))&=\frac{1}{\nu}\int_0^{h'}\frac{n(t,\tau,f(z(\zeta)))-n(0,\tau,f(z(\zeta)))}{t}\,dt\\&\quad+\frac{n(0,\tau,f(z(\zeta)))}{\nu}\log h'\\&\le\frac{1}{\nu}\int_{1/2}^{h'}\frac{n\bigl((\frac{2}{1-t})^{1/\omega},X,f(z)=\tau\bigr)}{t}\,dt+O(1)\\&\le\frac{2}{\nu}\int_{1/2}^{h'}n\biggl(\Bigl(\frac{2}{1-t}\Bigr)^{1/\omega},X,f(z)=\tau\biggr)\,dt+O(1)\\&\le\frac{4\omega}{\nu}\int_1^{(\frac{2}{1-h'})^{1/\omega}}\frac{n(x,X,f(z)=\tau)}{x^{\omega+1}}\,dx+O(1).\end{aligned}$$

By the first fundamental theorem,

$$\begin{aligned}m(h',\vec{a},\vec{A}(z(\zeta)))&=T(h',\vec{a},\vec{A}(z(\zeta)))-N(h',\vec{a},\vec{A}(z(\zeta)))\\&=\nu\bigl(T(h',f(z(\zeta)))-N(h',a,f)\bigr)+O(1).\end{aligned}\tag{2.6}$$

By applying Lemma 2.5 to $f(z(\zeta))$, we have

$$\begin{aligned}&T(h',f(z(\zeta)))-N(h',a,f)\\&\le\sum_{\tau=a_i}N(h',\tau,f(z(\zeta)))+O\biggl(\log\frac{1}{1-h'}+\log T(h',f(z(\zeta)))\biggr)\\&\le\frac{4\omega}{\nu}\int_1^{(\frac{2}{1-h'})^{1/\omega}}\frac{n(x,X)}{x^{\omega+1}}\,dx+O\biggl(\log\frac{1}{1-h'}+\log T(h',f(z(\zeta)))\biggr)\\&\le\frac{4\omega}{\nu}\int_1^{(\frac{2}{1-h'})^{1/\omega}}\frac{n(x,X)}{x^{\omega+1}}\,dx+O\biggl(\log\frac{1}{1-h'}\biggr)+O(\log T(h',f(z(\zeta)))),\end{aligned}\tag{2.7}$$

where $n(x,X)=\sum_{i=1}^{2\nu}n(x,X,f=a_i)$.

If $m(h',\vec{a},\vec{A}(z(\zeta)))\le 2\nu O(\log T(h',f(z(\zeta))))$, then we can obtain (2.5) by the similar method as the following implication.

Now we assume

$$\tfrac{1}{2}m(h',\vec{a},\vec{A}(z(\zeta)))\ge\nu O(\log T(h',f(z(\zeta)))).$$

Combining (2.6) with (2.7), we obtain

$$
\begin{aligned}
(2.8)\qquad & m(h',\vec{a},\vec{A}(z(\zeta))) \\
& \le 8\omega \int_1^{(\frac{2}{1-h'})^{1/\omega}} \frac{n(x,X)}{x^{\omega+1}}\,dx + O\left(\log\frac{1}{1-h'}\right), \qquad h'\notin F,
\end{aligned}
$$

where F is the set described in Lemma 2.5 satisfying $\int_F dt/(1-t)<\infty$. We set

$$
E=\left\{r: t=1-\frac{\varepsilon}{4(\beta-\alpha)}r^{-\omega},\ t\in F\right\}, \tag{2.9}
$$

where $\varepsilon>0$ is a real number small enough. Put

$$
\begin{aligned}
&\zeta=\zeta(re^{i\phi}) \qquad (\alpha+\varepsilon\le\phi\le\beta-\varepsilon),\\
&h=1-\frac{\varepsilon}{(\beta-\alpha)}r^{-\omega},\\
&h'=\frac{3+h}{4}=1-\frac{\varepsilon}{4(\beta-\alpha)}r^{-\omega}\notin F,
\end{aligned} \tag{2.10}
$$

where $\zeta=\zeta(z)$ is the mapping described in Lemma 2.4. Combining (2.9) with (2.10), we can see that if $h'\notin F$, then $r\notin E$ and E is a set of finite logarithmic measure, because

$$
\int_E\frac{dr}{r}=\frac{1}{\omega}\int_F\frac{dh}{1-h}<\infty.
$$

Next we apply (2.4) to estimate the logarithmic module:

$$
\log^+\frac{\|\vec{A}(re^{i\phi})\|\,\|\vec{a}\|}{|F(re^{i\phi},a)|}=\log^+\frac{\|\vec{A}(z(\zeta))\|\,\|\vec{a}\|}{|F(z(\zeta),a)|}.
$$

Applying Lemma 2.6 gives

$$
\begin{aligned}
(2.11)\qquad & \log^+\frac{\|\vec{A}(re^{i\phi})\|\,\|\vec{a}\|}{|F(re^{i\phi},a)|}=\log^+\frac{\|\vec{A}(z(\zeta))\|\,\|\vec{a}\|}{|F(z(\zeta),a)|}\\
&\le \log(\nu+1)^{1/2}+\frac{h'+h}{h'-h}m(h',\vec{a},\vec{A}(z(\zeta)))+\sum_{l=1}^{N}\log\frac{2h'}{|\zeta(re^{i\phi})-\zeta(b_l)|},
\end{aligned}
$$

where $\zeta(b_l)$ $(l=1,\dots,N)$ are the roots of $f(z(\zeta))=a$ contained in the disk $|\zeta|\le h'$.

From (2.3) it follows that

$$
\begin{aligned}
(2.12)\qquad & \sum_{l=1}^{N}\log\frac{2h'}{|\zeta(z)-\zeta(b_l)|}=\sum_{l=1}^{N}\log\frac{2h'}{|\zeta'(\xi)(z-b_l)|}\\
&\le\sum_{l-1}^{N}\log\frac{2\hat{\kappa}r^{1+\omega}}{|z-b_l|}=N\log 2\hat{\kappa}r^{\omega}+\sum_{l=1}^{N}\log\frac{r}{|z-b_l|},
\end{aligned}
$$

where $\hat{\kappa} = \frac{1}{\omega}\big(\frac{2(\beta-\alpha)}{\varepsilon}\big)^{1+1/\omega}$. By using Lemma 1.2.2 of [22], we have

$$(2.13)\qquad \frac{1}{2\pi}\int\limits_{\alpha+\varepsilon}^{\beta-\varepsilon}\sum_{l=1}^{N}\log\frac{r}{|re^{i\phi}-b_l|}\,d\phi \le KN,$$

where K is a positive constant depending only on ε.

Combining (2.8) and (2.11)–(2.13), we obtain

$$(2.14)\qquad m_{\alpha+\varepsilon,\beta-\varepsilon}(r,\vec{a},\vec{A}) = \frac{1}{2\pi}\int\limits_{\alpha+\varepsilon}^{\beta-\varepsilon}\log^{+}\frac{\|\vec{A}(re^{i\phi})\|\,\|\vec{a}\|}{|F(re^{i\phi},a)|}\,d\phi$$
$$\le (\beta-\alpha)\log(\nu+1)^{1/2} + 8\pi\frac{h'+h}{h'-h}\int\limits_{1}^{kr}\frac{n(x,X)}{x^{1+\omega}}\,dx$$
$$+\pi n(kr,X,f=a)\log r + ((\beta-hh\alpha)\log 2\hat{\kappa}+K)n(kr,X,f=a),$$

where $k=(8(\beta-\alpha)/\varepsilon)^{1/\omega}$. Therefore, (2.5) follows by noticing that $h'-h = \frac{3\varepsilon}{4(\beta-\alpha)}r^{-\omega}$. ∎

By Lemma 2.7, we can establish Lemmas 2.8 and 2.9, which are used to further estimate $m_{\alpha+\varepsilon,\beta-\varepsilon}(r,\vec{a},\vec{A})$ in two different ways. These two lemmas are of significance for the study of the problem.

Lemma 2.8. *Let $f(z)$ be the ν-valued algebroid function of finite lower order $\mu<\infty$ in $\mathbb{C}$ determined by (1.1). Assume that there exist $2\nu+1$ distinct complex values $a_i \ne a$ $(i=1,\dots,2\nu)$ such that for $X=\{z:\alpha<\arg z<\beta\}$ $(0<\beta-\alpha\le 2\pi)$, we have*

$$\sum_{i=1}^{2\nu} n(r,X,f=a_i) = o(T(dr,f)),$$
$$n(r,X,f=a) = o\bigg(\frac{T(dr,f)}{\log r}\bigg)\qquad (d\ge 1).$$

Then, for any $0<\varepsilon<(\beta-\alpha)/2$ and for any sequence $\{r_n\}$ of Pólya peaks of order $\sigma>\omega=\pi/(\beta-\alpha)$ of $f(z)$ outside a set E of finite logarithmic measure,

$$m_{\alpha+\varepsilon,\beta-\varepsilon}(r_n,\vec{a},\vec{A}) = o(T(r_n,f(z))).$$

Proof. As $\{r_n\}$ is a sequence of Pólya peaks of order $\sigma>\omega$ for $f(z)$, we have

$$\int\limits_{1}^{kr_n}\frac{n(x,X)}{x^{\omega+1}}\,dx = o\bigg(\int\limits_{1}^{kr_n}\frac{T(dx,f(z))}{x^{\omega+1}}\,dx\bigg)$$
$$\le o\bigg(\int\limits_{1}^{kr_n}\frac{T(r_n,f(z))}{x^{\omega+1}}\bigg(\frac{dx}{r_n}\bigg)^{\sigma-\varepsilon_n}dx\bigg) = o\bigg(\frac{T(r_n,f(z))}{r_n^{\omega}}\bigg),$$

where $n(x,X)=\sum_{i=1}^{2\nu}n(x,X,f=a_i)$. By the property of Pólya peaks, $o(T(dr_n,f))=o(T(r_n,f))$. Using (2.5), we complete the proof. ∎

LEMMA 2.9. *Let $f(z)$ be the ν-valued algebroid function of finite lower order $\mu<\infty$ in $\mathbb{C}$ determined by (1.1). Assume that there exist $2\nu+1$ distinct complex values $a_i\neq a$ $(i=1,\dots,2\nu)$ such that*

$$\limsup_{r\to\infty}\frac{\log^+[\sum_{i=1}^{2\nu}n(r,X,f=a_i)+n(r,X,f=a)]}{\log r}\le\rho \tag{2.15}$$

for $X=\{z:\alpha<\arg z<\beta\}$ $(0<\beta-\alpha\le2\pi)$. Then, for any $0<\varepsilon<(\beta-\alpha)/2$ and for all r except a set E of finite logarithmic measure,

$$m_{\alpha+\varepsilon,\beta-\varepsilon}(r,\vec{a},\vec{A})=O(r^{\eta+\varepsilon}),$$

where $\eta=\max\{\rho,\omega\}$, $\omega=\pi/(\beta-\alpha)$.

Proof. As $n(x,X)=O(x^{\rho+\varepsilon})$, we have

$$\int_1^{kr}\frac{n(x,X)}{x^{\omega+1}}\,dx=O\Big(\int_1^{kr}\frac{x^{\rho+\varepsilon}}{x^{\omega+1}}\,dx\Big)=O\Big(\int_1^{kr}x^{\rho+\varepsilon-\omega-1}\,dx\Big)$$
$$=O(r^{\max\{\rho-\omega,0\}+\varepsilon/2})$$

and

$$n(dr,X,f=a)\log r=O(r^{\rho+\varepsilon/2}\log r)=O(r^{\rho+\varepsilon}),$$

where $n(x,X)=\sum_{i=1}^{2\nu}n(x,X,f=a_i)$. From (2.5), we complete the proof. ∎

3. Proof of Theorem 1.1. The idea of the proof comes from [21]. Suppose conversely that $\lambda(f)>\omega$. We consider the following two cases.

I. $\lambda(f)>\beta\ge\mu(f)$. By (1.4), we can choose $\varepsilon>0$ such that

$$\sum_{i=1}^{q}(\alpha_{i+1}-\beta_i+2\varepsilon)+2\varepsilon<\frac{4}{\beta+2\varepsilon}\arcsin\sqrt{\frac{\delta}{2}}, \tag{3.1}$$

where $\alpha_{q+1}=2\pi+\alpha_1$ and $\lambda(f)>\beta+2\varepsilon>\mu$. Applying Lemma 2.1 to $f(z)$ gives a sequence $\{r_n\}$ of Pólya peaks of order $\beta+2\varepsilon$ for $f(z)$ outside E, where E is the set of Lemma 2.8. Set $\Lambda(r)=\Gamma^{1/2}(r)$ and

$$\Gamma(r)=\max\left\{\frac{m_{\alpha_i+\varepsilon,\beta_i-\varepsilon}(r_n,\vec{a},\vec{A})}{T(r_n,f)}:1\le i\le q\right\},\qquad r_{n-1}<r\le r_n. \tag{3.2}$$

Applying Lemma 2.8 to the Pólya peaks $\{r_n\}$ of order $\beta+2\varepsilon$ for $f(z)$ and using $\beta+2\varepsilon>\omega_i=\pi/(\beta_i-\alpha_i)$, we can deduce that $\lim_{r\to\infty}\Lambda(r)=0$. Then from Lemma 2.2 for sufficiently large n we have

$$\operatorname{meas}E_\Lambda(r_n,a)>\frac{4}{\beta+2\varepsilon}\arcsin\sqrt{\frac{\delta}{2}}-\varepsilon, \tag{3.3}$$

since $\beta + 2\varepsilon > 1/2$. We can assume (3.3) holds for all n. Set

$$K := \operatorname{meas}\Big(E_\Lambda(r_n, a) \cap \bigcup_{i=1}^{q} (\alpha_i + \varepsilon, \beta_i - \varepsilon)\Big).$$

From (3.1) and (3.3), we derive

$$\begin{aligned} K &\geq \operatorname{meas}(E_\Lambda(r_n, a)) - \operatorname{meas}\Big([-\pi, \pi) \setminus \bigcup_{i=1}^{q} (\alpha_i + \varepsilon, \beta_i - \varepsilon)\Big) \\ &= \operatorname{meas}(E_\Lambda(r_n, a)) - \operatorname{meas}\Big(\bigcup_{i=1}^{q} (\beta_i - \varepsilon, \alpha_{i+1} + \varepsilon)\Big) \\ &= \operatorname{meas}(E_\Lambda(r_n, a)) - \sum_{i=1}^{q} (\alpha_{i+1} - \beta_i + 2\varepsilon) > \varepsilon > 0. \end{aligned}$$

It is easy to see that there exists i_0 such that for infinitely many n, we have

$$\operatorname{meas}(E_\Lambda(r_n, a) \cap (\alpha_{i_0} + \varepsilon, \beta_{i_0} - \varepsilon)) > K/q > \varepsilon/q. \tag{3.4}$$

We can assume (3.4) holds for all n. Set $E_n = E_\Lambda(r_n, a) \cap (\alpha_{i_0} + \varepsilon, \beta_{i_0} - \varepsilon)$. From the definition of $E_\Lambda(r_n, a)$ it follows that

$$\begin{aligned} &\frac{1}{2\pi} \int\limits_{E_n} \log^+ \frac{\|\vec{A}(r_n e^{i\theta})\| \, \|\vec{a}\|}{|F(r_n e^{i\theta}, a)|} \, d\theta \\ &\qquad > \Lambda(r_n) T(r_n, f) \operatorname{meas}(E_n) > \frac{\varepsilon}{q} \Lambda(r_n) T(r_n, f). \end{aligned} \tag{3.5}$$

On the other hand, by (3.2), we have

$$m_{\alpha_{i_0} + \varepsilon, \beta_{i_0} - \varepsilon}(r_n, \vec{a}, \vec{A}) \leq \Lambda^2(r_n) T(r_n, f). \tag{3.6}$$

Combining (3.5) with (3.6) gives $\varepsilon/q \leq \Lambda(r_n) \to 0$, which is impossible.

II. $\lambda(f) = \mu(f)$. Then $\beta = \mu = \lambda(f)$. By the same argument as in I with all the $\beta + 2\varepsilon$ replaced by $\beta = \mu$, we can derive a contradiction.

Theorem 1.1 follows.

4. Proof of Theorem 1.2. Suppose that $\lambda(f) > \max\{\omega, \rho\}$. We will derive a contradiction by making a minor modification of the proof of Theorem 1.1. We consider two cases.

I. $\lambda(f) > \mu$. Let $\{r_n\}$ be a sequence of Pólya peaks of order $\beta + 2\varepsilon$ for $f(z)$. Set $\Lambda(r) = [\log r]^{-1}$. From (3.5), we have

$$\frac{1}{2\pi} \int\limits_{E_n} \log^+ \frac{\|\vec{A}(r_n e^{i\theta})\| \, \|\vec{a}\|}{|F(r_n e^{i\theta}, a)|} \, d\theta > \frac{\varepsilon}{q} \frac{T(r_n, f)}{\log r_n}. \tag{4.1}$$

On the other hand, by Lemma 2.9 and noticing that $\eta \leq \beta$, we have

$$m_{\alpha + \varepsilon, \beta - \varepsilon}(r, \vec{a}, \vec{A}) \leq O(r^{\beta + \varepsilon}), \qquad r \notin E. \tag{4.2}$$

Combining (4.1) with (4.2) gives

$$T(r_n, f) \leq K r_n^{\beta+\varepsilon} \log r_n.$$

Therefore,

$$\beta + 2\varepsilon \leq \liminf_{n\to\infty} \frac{\log T(r_n, f)}{\log r_n} \leq \beta + \varepsilon.$$

This is impossible.

II. $\lambda(f) = \mu$. Then $\beta = \mu = \lambda(f)$. By the same arguments as in I with all the $\beta + 2\varepsilon$ replaced by $\beta = \mu$, we can derive

$$\mu = \beta \leq \max\{\rho, \omega\} + \varepsilon < \lambda(f).$$

This is also impossible.

Theorem 1.2 follows.

5. Proof of Corollary 1.3. Suppose that Corollary 1.3 does not hold. Then for any ray $\arg z = \theta \in (\alpha, \beta)$ there exist 2ν values $a_1(\theta), \ldots, a_{2\nu}(\theta)$ different from a in the extended complex plane such that

$$\sum_{j=1}^{2\nu} n(r, Z_{2\epsilon}(\theta), f = a_j(\theta)) = o(T(dr, f)),$$

for some $\epsilon = \epsilon(\theta)$ and $d = d(\theta) \geq 1$. In view of a Valiron-type theorem (cf. [22, Lemma 2.7.1]), we have

$$\begin{aligned} n(r, Z_\epsilon(\theta), f = b) &\leq C_\epsilon \Big(\sum_{j=1}^{2\nu} n(2r, Z_{2\epsilon}(\theta), f = a_j(\theta)) + n(2r, Z_{2\epsilon}(\theta), f = a) \Big) \\ &\quad + O((\log r) \log\log r) = o(T(2dr, f)) \end{aligned}$$

for all b possibly except a zero measure set of b.

Take an $\eta > 0$ such that

$$\beta - \alpha - 2\eta > \max\left\{ \frac{\pi}{\mu}, 2\pi - \frac{4}{\mu} \arcsin \sqrt{\frac{\delta}{2}} \right\}. \tag{5.1}$$

Since there exist finitely many θ_i such that $[\alpha + \eta, \beta - \eta] \subset \bigcup_{i=1}^{q} (\theta_i - \epsilon(\theta_i), \theta_i + \epsilon(\theta_i))$, we can find 2ν values a_j different from a and $d \geq 1$ such that

$$\sum_{j=1}^{2\nu} n(r, \Omega_\eta, f = a_j) = o(T(dr, f)).$$

It is easy to see from (5.1) that

$$(2\pi + \alpha + \eta) - (\beta - \eta) < \frac{4}{\mu} \arcsin \sqrt{\frac{\delta}{2}} \quad \text{and} \quad \omega_\eta = \frac{\pi}{\beta - \alpha - 2\eta} < \mu.$$

Therefore, by Theorem 1.1, we have $\mu \leq \lambda(f) \leq \omega_\eta$. This contradiction completes the proof of Corollary 1.3.

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