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On some properties of induced almost contact structures

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Abstract. Real affine hypersurfaces of the complex space $\mathbb{C}^{n+1}$ with a J-tangent transversal vector field and an induced almost contact structure (φ, ξ, η) are studied. Some properties of the induced almost contact structures are proved. In particular, we prove some properties of the induced structure when the distribution $\mathcal{D}$ is involutive. Some constraints on a shape operator when the induced almost contact structure is either normal or ξ-invariant are also given.

1. Introduction. We study real affine hypersurfaces $f : M \to \mathbb{C}^{n+1}$ of the complex space $\mathbb{C}^{n+1}$ with a J-tangent transversal vector field C and an induced almost contact structure (φ, ξ, η). The main purpose of this paper is to investigate some properties of the induced almost contact structures. In particular, we prove some properties of the induced structure when the distribution $\mathcal{D}$ is involutive. We also establish some constraints on the shape operator when the induced almost contact structure is either normal or ξ-invariant.

In Section 2, we briefly recall the basic formulas of affine differential geometry. We introduce the notion of a J-tangent transversal vector field and a J-invariant distribution $\mathcal{D}$. When the hypersurface f is additionally nondegenerate we define a 1-dimensional distribution $\mathcal{D}_h$ as the complementary orthogonal distribution to $\mathcal{D}$ in TM with respect to the second fundamental form h.

In Section 3, we recall the definition of an induced almost contact structure and some results related to this structure obtained in [SS], [S1] and [S2]. We also recall the important result of K. Yano and S. Ishihara [YI] characterizing normal almost contact structures.

Section 4 contains the main results of this paper. In this section we find equivalent conditions for $L_\xi \varphi = 0$ and for $L_\xi \eta = 0$ as well as some relations between normality of (φ, ξ, η) and these Lie derivatives. When the distribution $\mathcal{D}$ is involutive, we show the existence (at least locally) of a J-tangent

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transversal vector field such that $\tau|_{\mathcal{D}} = 0$ and $S \circ \varphi = \varphi \circ S$ on $\mathcal{D}$. In particular, we prove that locally we can always find on the manifold M a normal induced almost contact structure. We also give a local characterization of affine hypersurfaces with an involutive distribution $\mathcal{D}$ and the shape operator vanishing on $\mathcal{D}$. Finally, we show that when (φ, ξ, η) is normal and the second fundamental form is definite on $\mathcal{D}$ then the shape operator must be proportional to identity. V. Cruceanu [C] proved that an almost contact metric structure (φ, ξ, η, h) induced by a centro-affine, J-tangent transversal vector field is ξ-invariant. We show that for equiaffine hypersurfaces the converse, in a certain sense, is also true.

2. Preliminaries. We briefly recall the basic formulas of affine differential geometry. For more details, we refer to [NS]. Let $f : M \to \mathbb{R}^{n+1}$ be an orientable, connected differentiable n-dimensional hypersurface immersed in affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection D. Then, for any transversal vector field C we have

$$\mathrm{D}_X f_* Y = f_*(\nabla_X Y) + h(X, Y)C, \tag{2.1}$$

$$\mathrm{D}_X C = -f_*(SX) + \tau(X)C, \tag{2.2}$$

where X, Y are vector fields tangent to M. For any transversal vector field, ∇ is a torsion-free connection, h is a symmetric bilinear form on M, called the *second fundamental form*, S is a tensor of type $(1, 1)$, called the *shape operator*, and τ is a 1-form, called the *transversal connection form*.

If h is nondegenerate, then we say that the hypersurface or the hypersurface immersion is *nondegenerate*. We have the following

THEOREM 2.1 ([NS], Fundamental equations). *For an arbitrary transversal vector field C the induced connection ∇, the second fundamental form h, the shape operator S, and the 1-form τ satisfy the following equations:*

$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY, \tag{2.3}$$

$$(\nabla_X h)(Y, Z) + \tau(X)h(Y, Z) = (\nabla_Y h)(X, Z) + \tau(Y)h(X, Z), \tag{2.4}$$

$$(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX, \tag{2.5}$$

$$h(X, SY) - h(SX, Y) = 2d\tau(X, Y). \tag{2.6}$$

The equations (2.3), (2.4), (2.5) and (2.6) are called the *equation of Gauss*, *Codazzi for h*, *Codazzi for S* and *Ricci*, respectively.

For a hypersurface immersion $f : M \to \mathbb{R}^{n+1}$ a transversal vector field C is said to be *equiaffine* (resp. *locally equiaffine*) if $\tau = 0$ (resp. $d\tau = 0$).

Let $\dim M = 2n + 1$ and $f : M \to \mathbb{R}^{2n+2}$ be an affine hypersurface. We always assume that $\mathbb{R}^{2m} \simeq \mathbb{C}^m$ is endowed with the standard complex structure J. In particular, if $m = n+1$, we have

$$J(x_1,\dots,x_{n+1},y_1,\dots,y_{n+1}) = (-y_1,\dots,-y_{n+1},x_1,\dots,x_{n+1}).$$

Let C be a transversal vector field on M. We say that C is *J-tangent* if $JC_x \in f_*(T_xM)$ for every $x \in M$. We also define a distribution $\mathcal{D}$ on M as the biggest J invariant distribution on M, that is,

$$\mathcal{D}_x = f_*^{-1}\big(f_*(T_xM) \cap J(f_*(T_xM))\big)$$

for every $x \in M$. It is clear that $\dim \mathcal{D} = 2n$. A vector field X is called a *$\mathcal{D}$-field* if $X_x \in \mathcal{D}_x$ for every $x \in M$. We use the notation $X \in \mathcal{D}$ for vectors as well as for $\mathcal{D}$-fields.

When f is additionally nondegenerate we can define a 1-dimensional distribution $\mathcal{D}_h$ as follows:

$$\mathcal{D}_{hx} := \{X \in T_xM : h(X,Y) = 0 \ \forall Y \in \mathcal{D}_x\},$$

where h is the second fundamental form on M relative to any transversal vector field. It follows from [NS, Prop. 2.5] that the definition of $\mathcal{D}_h$ is independent of the choice of a transversal vector field. We say that the distribution $\mathcal{D}$ is *nondegenerate* if h is nondegenerate on $\mathcal{D}$. To simplify the writing, we will be omitting f_* in front of vector fields in most cases. From now on we always assume that both f and $\mathcal{D}$ are nondegenerate.

3. Almost contact structures. A $(2n+1)$-dimensional manifold M is said to have an *almost contact structure* if there exist on M a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η which satisfy

$$\varphi^2(X) = -X + \eta(X)\xi, \qquad \eta(\xi) = 1$$

for every $X \in TM$. If additionally there is a semi-Riemannian metric g on M such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for every $X, Y \in TM$ then (φ, ξ, η, g) is called an *almost contact metric structure*. We say that an almost contact structure is *normal* if

$$[\varphi,\varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi,\varphi]$ is the Nijenhuis tensor for φ.

Let $f : M \to \mathbb{R}^{2n+2}$ be a hypersurface with a J-tangent transversal vector field C. Then we can define a vector field ξ, a 1-form η and a tensor field φ of type (1,1) as follows:

$$\begin{aligned}
&\xi := JC,\\
&\eta|_{\mathcal{D}} = 0 \quad \text{and} \quad \eta(\xi) = 1,\\
&\varphi|_{\mathcal{D}} = J|_{\mathcal{D}} \quad \text{and} \quad \varphi(\xi) = 0.
\end{aligned}$$

It is easy to see that (φ, ξ, η) is an almost contact structure on M. This structure is called the *almost contact structure on M induced by C* (or simply *induced almost contact structure*).

Let $f : M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with an induced almost contact structure (φ, ξ, η). For an induced almost contact structure we have

THEOREM 3.1 ([SS]). *If (φ, ξ, η) is an induced almost contact structure on M then:*

$$\eta(\nabla_X Y) = -h(X, \varphi Y) + X(\eta(Y)) + \eta(Y)\tau(X), \tag{3.1}$$

$$\varphi(\nabla_X Y) = \nabla_X \varphi Y + \eta(Y)SX - h(X, Y)\xi, \tag{3.2}$$

$$\begin{aligned} \eta([X, Y]) = -h(X, \varphi Y) + h(Y, \varphi X) + X(\eta(Y)) - Y(\eta(X)) \\ + \eta(Y)\tau(X) - \eta(X)\tau(Y), \end{aligned} \tag{3.3}$$

$$\varphi([X, Y]) = \nabla_X \varphi Y - \nabla_Y \varphi X - \eta(X)SY + \eta(Y)SX, \tag{3.4}$$

$$\eta(\nabla_X \xi) = \tau(X), \tag{3.5}$$

$$\eta(SX) = h(X, \xi), \tag{3.6}$$

for all $X, Y \in \mathcal{X}(M)$.

From the above we immediately get

COROLLARY 3.2 ([SS]). *For all $Z, W \in \mathcal{D}$ we have*

$$\eta(\nabla_Z W) = -h(Z, \varphi W), \tag{3.7}$$

$$\eta(\nabla_\xi Z) = -h(\xi, \varphi Z), \tag{3.8}$$

$$\varphi(\nabla_Z W) = \nabla_Z \varphi W - h(Z, W)\xi, \tag{3.9}$$

$$\eta([Z, W]) = -h(Z, \varphi W) + h(W, \varphi Z), \tag{3.10}$$

$$\eta([Z, \xi]) = h(\xi, \varphi Z) + \tau(Z). \tag{3.11}$$

Moreover

$$S(\mathcal{D}) \subset \mathcal{D} \quad \text{if and only if} \quad \xi \in \mathcal{D}_h. \tag{3.12}$$

In the case when the distribution $\mathcal{D}$ is involutive we have the following simple lemma:

LEMMA 3.3 ([S2]). *If the distribution $\mathcal{D}$ is involutive then the second fundamental form is antihermitian, that is, it satisfies*

$$h(\varphi X, \varphi Y) = -h(X, Y) \qquad \text{for all } X, Y \in \mathcal{D}.$$

Normal almost contact structures can be characterized as follows:

THEOREM 3.4 ([YI, Th. 3.3]). *The induced almost contact structure (φ, ξ, η) is normal if and only if*

$$S\varphi Z - \varphi SZ + \tau(Z)\xi = 0 \qquad \text{for every } Z \in \mathcal{D}.$$

Using the same methods as in the proof of Theorem 4.7 in [S1] we get

LEMMA 3.5. *Let $f : M \to \mathbb{R}^{2n+2}$ be an affine hypersurface with an involutive distribution $\mathcal{D}$ and a J-tangent transversal vector field C such that*

$$\tau(Z) = -h(\xi, \varphi Z)$$

for all $Z \in \mathcal{D}$. Then for every point $x \in M$ there exist an open neighbourhood $U \subset M$ and linearly independent $\mathcal{D}$-fields $X_1, \ldots, X_{2n}$ defined on U such that $[X_i, X_j] = 0$ for $i, j = 1, \ldots, 2n$ and $[X_i, \xi] = 0$ for $i, \ldots, 2n$.

The following theorem gives a necessary and sufficient condition for the affine normal to be J-tangent.

THEOREM 3.6 ([S1]). *Let $f : M \to \mathbb{R}^{2n+2}$ be the Blaschke hypersurface with an affine normal field C. Then C is J-tangent if and only if the Gauss–Kronecker curvature is constant in the direction of the distribution $\mathcal{D}_h$.*

4. Main results. We start with the following lemma:

LEMMA 4.1. *Let (φ, ξ, η) be an induced almost contact structure. Then*

$$L_\xi \varphi = 0 \ \Leftrightarrow\ \begin{cases} S\varphi X - \varphi SX - h(\varphi X, \xi)\xi = 0, \\ \tau(X) = -h(\varphi X, \xi) \end{cases} \tag{4.1}$$

for all $X \in \mathcal{D}$. Moreover

$$L_\xi \eta = 0 \ \Leftrightarrow\ \tau(X) = -h(\varphi X, \xi) \text{ for every } X \in \mathcal{D}. \tag{4.2}$$

Proof. From Theorem 3.1 (formulas (3.2) and (3.4)) we have

$$(L_X\varphi)(Y) = \nabla_Y \varphi X - \nabla_{\varphi Y} X + \eta(X)SY - \eta(Y)SX$$

for all $X, Y \in \mathcal{X}(M)$. The above equality implies that $L_\xi\varphi = 0$ if and only if

$$SX = \nabla_{\varphi X}\xi \tag{4.3}$$

for all $X \in \mathcal{D}$. Applying η to both sides and using (3.5) and (3.6) we get

$$h(X, \xi) = \eta(\nabla_{\varphi X}\xi) = \tau(\varphi X)$$

for all $X \in \mathcal{D}$. Hence, we obtain a formula for τ on $\mathcal{D}$:

$$\tau(X) = -h(\varphi X, \xi) \tag{4.4}$$

for all $X \in \mathcal{D}$. Now, applying φ to both sides of (4.3) and again using (3.2) we get

$$\varphi SX = \varphi(\nabla_{\varphi X}\xi) = S\varphi X - h(\varphi X, \xi)\xi$$

for all $X \in \mathcal{D}$. Finally,

$$L_\xi \varphi = 0 \ \Leftrightarrow\ \begin{cases} S\varphi X - \varphi SX - h(\varphi X, \xi)\xi = 0 \\ \tau(X) = -h(\varphi X, \xi) \end{cases} \quad \text{for all } X \in \mathcal{D}. \tag{4.5}$$

To prove the second part of the lemma, it is enough to note that from the formula (3.3) we have

$$(L_\xi\eta)(X) = \xi(\eta(X)) - \eta([\xi, X]) = h(\varphi X, \xi) - \tau(\xi)\eta(X) + \tau(X)$$

for all $X \in \mathcal{X}(M)$. Therefore $L_\xi\eta = 0$ if and only if

$$\tau(X) = -h(\varphi X, \xi) \quad \text{for all } X \in \mathcal{D}. \ \blacksquare$$

From the above lemma and Theorem 3.4 we immediately obtain

Corollary 4.2. *If $L_\xi\varphi = 0$ then $L_\xi\eta = 0$ and (φ, ξ, η) is a normal almost contact structure. On the other hand, if (φ, ξ, η) is normal then $L_\xi\eta = 0$ if and only if $L_\xi\varphi = 0$.*

For an induced almost contact structure we have the following lemma:

Lemma 4.3. *If (φ, ξ, η) is an induced almost contact structure then*

$$h(\varphi SX, Y) - h(X, \varphi SY) + X(h(Y, \xi)) - Y(h(X, \xi)) = h([X, Y], \xi) \tag{4.6}$$

for all $X, Y \in \mathcal{X}(M)$. If $\xi \in \mathcal{D}_h$ and $X, Y \in \mathcal{D}$ then (4.6) *takes the form*

$$h(\varphi SX, Y) - h(X, \varphi SY) = h(\xi, \xi)(h(\varphi X, Y) - h(X, \varphi Y)). \tag{4.7}$$

Proof. From the Codazzi equation for S we have

$$\eta(\nabla_X SY) - \eta(S(\nabla_X Y)) - \tau(X)\eta(SY) = \eta(\nabla_Y SX) - \eta(S(\nabla_Y X)) - \tau(Y)\eta(SX).$$

Using (3.1) and (3.6) we obtain (4.6). If $\xi \in \mathcal{D}_h$ then

$$h(X, \xi) = h(\xi, \xi)\eta(X)$$

for all $X \in \mathcal{X}(M)$. Consequently, formula (4.6) takes the form

$$h(\varphi SX, Y) - h(X, \varphi SY) = h(\xi, \xi)\eta([X, Y])$$

for all $X, Y \in \mathcal{D}$. Finally, (4.7) is a consequence of Corollary 3.2 (formula (3.10)). $\blacksquare$

Now we shall prove the following

Lemma 4.4. *If the distribution $\mathcal{D}$ is involutive then for every point $x \in M$ there exist a neighborhood U and a J-tangent transversal vector field C defined in this neighborhood such that*

(1) $\tau|_{\mathcal{D}} = 0$,
(2) $S\varphi = \varphi S$ *on* $\mathcal{D}$.

This C is unique up to scaling by nonzero functions constant in the direction of $\mathcal{D}$.

Proof. Fix $x \in M$. Since $\mathcal{D}$ is involutive, the Frobenius theorem implies that there exist linearly independent vector fields $X_1, \ldots, X_{2n} \in \mathcal{D}$ defined on some neighborhood U of x and a vector field X_{2n+1} also defined on U but not belonging to $\mathcal{D}$ such that $[X_i, X_j] = 0$ for $i, j = 1, \dots, 2n+1$. It follows that for all $X \in \mathcal{D}$ we have $[X, X_{2n+1}] \in \mathcal{D}$. Now, we can find a vector field $Z \in \mathcal{D}$ such that $\xi := X_{2n+1} + Z \in \mathcal{D}_h$. Hence, for all $X \in \mathcal{D}$ we get

$$[X, \xi] = [X, X_{2n+1} + Z] = [X, X_{2n+1}] + [X, Z] \in \mathcal{D},$$

since the distribution $\mathcal{D}$ is involutive. Define $C := -J\xi$ and let (φ, ξ, η) be an almost contact structure induced by the transversal vector field C.

From (3.11) and the fact that $\xi \in \mathcal{D}_h$ we conclude that

$$\tau(Z) = 0$$

for all $Z \in \mathcal{D}$. Now, the Ricci equation (formula (2.6)) implies that

$$\begin{aligned} h(Z, SW) - h(SZ, W) &= 2d\tau(Z, W) \\ &= Z(\tau(W)) - W(\tau(Z)) - \tau([Z, W]) = 0 \end{aligned}$$

for all $Z, W \in \mathcal{D}$, because the distribution $\mathcal{D}$ is involutive and $\tau|_{\mathcal{D}} = 0$. Consequently,

$$h(Z, SW) = h(SZ, W) \tag{4.8}$$

for all $Z, W \in \mathcal{D}$. On the other hand, using (4.6) we easily get

$$h(Z, \varphi SW) = h(\varphi SZ, W) \tag{4.9}$$

for all $Z, W \in \mathcal{D}$. Since $\xi \in \mathcal{D}_h$ we see that $S(\mathcal{D}) \subset \mathcal{D}$. Now, using Lemma 3.3 and formulas (4.8) and (4.9) we obtain

$$h(\varphi SW, Z) = h(\varphi SZ, W) = h(SZ, \varphi W) = h(Z, S\varphi W)$$

for all $Z, W \in \mathcal{D}$. Finally, nondegeneracy of h on $\mathcal{D}$ implies that

$$\varphi SW = S\varphi W$$

for all $W \in \mathcal{D}$.

To prove the last part of the lemma assume that $\bar{C} = \phi C + f_* Z$ is any other J-tangent transversal vector field (note that $\phi \in C^\infty(U)$, $Z \in \mathcal{D}$) satisfying conditions (1)–(2).

From (2) it easily follows that $Z = 0$, since both $\bar{\xi} = J\bar{C} \in \mathcal{D}_h$ and $\xi \in \mathcal{D}_h$. Moreover, condition (1) implies that $X(\phi) = 0$ for all $X \in \mathcal{D}$, because $\bar{\tau} = \tau + \frac{1}{\phi} h(Z, \cdot) + d \log|\phi|$ (see [NS, Prop. 2.5, p. 35]), $\bar{\tau}|_{\mathcal{D}} = 0$, $\tau|_{\mathcal{D}} = 0$ and $Z = 0$ (here $\bar{\tau}$ is a transversal connection form induced by $\bar{C}$). ∎

Corollary 4.5. *If the distribution $\mathcal{D}$ is involutive and* $\dim M = 3$ *then for every point $x \in M$ there exist a neighbourhood U of x, a J-tangent transversal vector field C defined on U and functions $\alpha, \beta \in C^\infty(U)$ such that*

(1) $\tau|_{\mathcal{D}} = 0$,
(2) $SX = \alpha X + \beta \varphi X$ *for every* $X \in \mathcal{D}$.

Proof. Fix $x \in M$. From Lemma 4.4 there exist a neighbourhood U of x and a J-tangent transversal vector field C defined on U such that $\tau|_{\mathcal{D}} = 0$ and $S\varphi = \varphi S$ on $\mathcal{D}$. It is enough to show that S satisfies condition (2). To do this let X be an arbitrary but fixed nonvanishing $\mathcal{D}$-field on U. Then $\{X, \varphi X\}$ forms a basis of $\mathcal{D}$. There exist $a, b, c, d \in C^\infty(U)$ such that

$$SX = aX + b\varphi X \quad \text{and} \quad S\varphi X = cX + d\varphi X.$$

Since $S\varphi = \varphi S$ on $\mathcal{D}$, the above equalities imply that

$$c = -b \quad \text{and} \quad d = a.$$

Take any $Z = pX + q\varphi X \in \mathcal{D}$. Then

$$\begin{aligned} SZ &= pSX + qS\varphi X = p(aX + b\varphi X) + q(-bX + a\varphi X) \\ &= a(pX + q\varphi X) + b(-qX + p\varphi X) = aZ + b\varphi Z. \end{aligned}$$

Now, setting $\alpha := a$ and $\beta := b$ we get (2). ∎

From Lemma 4.4 and Theorem 3.4 we immediately get

Corollary 4.6. *If the distribution $\mathcal{D}$ is involutive then there exists locally on M a normal induced almost contact structure.*

As an application of Lemma 4.4, consider an affine immersion $f : M \to \mathbb{R}^{2n+2}$ with an involutive distribution $\mathcal{D}$ and the transversal vector field C from Lemma 4.4. Additionally assume that $S|_{\mathcal{D}} = 0$ and let (φ, ξ, η) be an almost contact structure on M induced by C.

From formula (3.2), and since $S|_{\mathcal{D}} = 0$, we have

$$\varphi(\nabla_X \xi) = -h(X, \xi)\xi = 0$$

for all $X \in \mathcal{D}$. We similarly conclude (using (3.5) and Lemma 4.4) that

$$\eta(\nabla_X \xi) = 0$$

for every $X \in \mathcal{D}$. Thus $\nabla_X \xi = 0$ for every $X \in \mathcal{D}$. Fix $x \in M$. Lemma 3.5 implies that in some neighbourhood of x there exist $X_1, \ldots, X_{2n} \in \mathcal{D}$ linearly independent and such that $[X_i, X_j] = 0$ and $[X_i, \xi] = 0$ for all $i, j = 1, \ldots, 2n$. Now, the Frobenius theorem implies that there exists a local coordinate system $(x_1, \ldots, x_{2n}, y)$ such that

$$\frac{\partial}{\partial x_i} = X_i \quad \text{and} \quad \frac{\partial}{\partial y} = \xi.$$

It follows that

$$\nabla_{\partial/\partial x_i} \frac{\partial}{\partial y} = 0 \quad \text{and} \quad h\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y}\right) = 0$$

for $i = 1, \ldots, 2n$. In the above coordinates f satisfies the following system of differential equations:

$$f_{x_i y} = 0$$

for $i = 1, \ldots, 2n$. It follows that f can be locally expressed in the form

$$f(x_1, \ldots, x_{2n}, y) = A(y) + B(x_1, \ldots, x_{2n}),$$

where $A : I \to \mathbb{R}^{2n+2}$, $B : U \to \mathbb{R}^{2n+2}$, I is an interval in $\mathbb{R}$ and U is some open subset in $\mathbb{R}^{2n}$. Since f is an immersion, B must be an immersion, too. Moreover, because the distribution $\mathcal{D}$ is involutive and $\partial/\partial x_1, \ldots, \partial/\partial x_{2n} \in \mathcal{D}$, the map B must be a kählerian immersion. Summarizing we have

THEOREM 4.7. *Let $f : M \to \mathbb{R}^{2n+2}$ be an affine immersion with an involutive distribution $\mathcal{D}$ and the transversal vector field from Lemma 4.4. If $S|_{\mathcal{D}} = 0$ then there exist a kählerian immersion $B : U \to \mathbb{R}^{2n+2}$ defined on some open subset in $\mathbb{R}^{2n}$ and a curve $A : I \to \mathbb{R}^{2n+2}$ such that f can be locally expressed in the form*

$$f(x_1, \ldots, x_{2n}, y) = A(y) + B(x_1, \ldots, x_{2n}).$$

EXAMPLE 4.8. Let us consider the affine immersion defined by

$$f : \mathbb{R}^3 \ni (x_1, x_2, y) \mapsto \begin{bmatrix} -2x_1^2 + 2x_2^2 + \sin y \\ -2x_2 \\ -4x_1x_2 + \cos y \\ 2x_1 \end{bmatrix} \in \mathbb{R}^4$$

with the transversal vector field

$$C : \mathbb{R}^3 \ni (x_1, x_2, y) \mapsto \begin{bmatrix} -\sin y \\ 0 \\ -\cos y \\ 0 \end{bmatrix} \in \mathbb{R}^4.$$

It is not difficult to see that C is J-tangent. Let (φ, ξ, η) be an almost contact structure induced by C. Then

$$\tau = 0, \qquad S|_{\mathcal{D}} = 0, \qquad S\xi = \xi$$

and

$$h = \begin{bmatrix} 4\sin y & 4\cos y & 0 \\ 4\cos y & -4\sin y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in the canonical base $\{\partial/\partial x_1, \partial/\partial x_2, \partial/\partial y\}$ of $\mathbb{R}^3$. Because $Jf_{x_1} = f_{x_2}$, the fields $\partial/\partial x_1, \partial/\partial x_2$ generate the distribution $\mathcal{D}$. Moreover, the form of h implies that the distribution $\mathcal{D}_h$ is generated by $\partial/\partial y$. Straightforward computations show that the Gauss–Kronecker curvature equals

$$K = -(4x_1^2 + 4x_2^2 + 1)^{-5/2},$$

so it is constant in the direction of $\mathcal{D}_h$. Now Theorem 3.6 implies that the affine normal field is J-tangent. It can be shown that in this case C is the affine normal field.

The following theorem shows that under some additional assumptions on the hypersurface, if an induced almost contact structure is normal then the shape operator must be proportional to the identity.

THEOREM 4.9. *Let (φ, ξ, η) be an almost contact structure induced by an equiaffine J-tangent transversal vector field. If (φ, ξ, η) is normal and the second fundamental form h is definite on $\mathcal{D}$ then*

$$S = h(\xi, \xi)\,\mathrm{id} \quad \textit{and} \quad h(\xi, \xi) = \mathrm{const}\,.$$

Proof. Since the structure (φ, ξ, η) is normal, we have

$$S\varphi X = \varphi S X$$

for all $X \in \mathcal{D}$. Using formula (3.6) we obtain

$$h(\varphi X, \xi) = \eta(S\varphi X) = \eta(\varphi S X) = 0$$

for all $X \in \mathcal{D}$, hence $\xi \in \mathcal{D}_h$. Now, Lemma 4.3 implies

$$h(S\varphi X, Y) - h(X, S\varphi Y) = h(\xi, \xi)(h(\varphi X, Y) - h(X, \varphi Y))$$

for all $X, Y \in \mathcal{D}$. Substituting $Y = \varphi X$ we have

$$h(S\varphi X, \varphi X) + h(X, SX) = h(\xi, \xi)(h(\varphi X, \varphi X) + h(X, X)). \tag{4.10}$$

Normality of (φ, ξ, η) implies that, if $X \in \mathcal{D}$ is an eigenvector for S corresponding to some eigenvalue λ, then φX is also an eigenvector for S and for the same eigenvalue λ. Now, the Ricci equation (2.6), equiaffinity of C and definiteness (positive or negative) of h on $\mathcal{D}$ imply that $S|_{\mathcal{D}}$ has only real eigenvalues.

Let $\{\lambda_1, \dots, \lambda_k\}$ be all eigenvalues of $S|_{\mathcal{D}}$ and let $X_1, \dots, X_k$ be the corresponding eigenvectors. Now using (4.10) we get

$$\lambda_i(h(\varphi X_i, \varphi X_i) + h(X_i, X_i)) = h(\xi, \xi)(h(\varphi X_i, \varphi X_i) + h(X_i, X_i))$$

for $i = 1, \dots, k$. Since h is definite on $\mathcal{D}$ we easily obtain $\lambda_i = h(\xi, \xi)$ for $i = 1, \dots, k$. Since $SX \in \mathcal{D}$ for all $X \in \mathcal{D}$, the Ricci equation (2.6) implies that

$$h(S\xi, X) = h(\xi, SX) = 0$$

for all $X \in \mathcal{D}$. Therefore $S\xi \in \mathcal{D}_h$. On the other hand $\eta(S\xi) = h(\xi, \xi)$ thus $S\xi = h(\xi, \xi)\xi$. Finally, since all eigenvalues of S are equal to $h(\xi, \xi)$ and S is diagonalizable on $\mathcal{D}$, we see that $S = h(\xi, \xi)\,\mathrm{id}$. Now, the Codazzi equation for S (2.5) implies that $h(\xi, \xi) = \mathrm{const}$. ∎

We say (see [C]) that an almost contact metric structure (φ, ξ, η, h) is ξ-*invariant* if $L_\xi \varphi = 0$, $L_\xi \eta = 0$, and $L_\xi h = 0$ (of course we also always have $L_\xi \xi = 0$).

V. Cruceanu [C] proved that an almost contact metric structure (φ, ξ, η, h) induced by a centro-affine, J-tangent transversal vector field is ξ-invariant. The following theorem shows that for equiaffine hypersurfaces the converse, in a certain sense, is also true.

THEOREM 4.10. *Let (φ, ξ, η) be an almost contact structure induced by an equiaffine J-tangent transversal vector field C. If*

$$L_\xi \varphi = 0, \qquad L_\xi \eta = 0, \qquad L_\xi h = 0,$$

then

$$S = h(\xi, \xi)\,\mathrm{id} \quad \textit{and} \quad h(\xi, \xi) = \mathrm{const}\,.$$

Proof. It follows from Theorem 3.1 that

$$(L_\xi h)(X, Y) = (\nabla_\xi h)(X, Y) - h(\varphi SX, Y) - h(X, \varphi SY) + \tau(X)h(\xi, Y) + \tau(Y)h(\xi, X) \tag{4.11}$$

for all $X, Y \in \mathcal{X}(M)$. Since $\tau = 0$ and $L_\xi h = 0$, the above formula implies

$$(\nabla_\xi h)(X, Y) = h(\varphi SX, Y) + h(X, \varphi SY) \tag{4.12}$$

for all $X, Y \in \mathcal{X}(M)$. Using Corollary 4.2 and the assumption that $L_\xi \varphi = 0$ we find that (φ, ξ, η) is normal, that is (see Th. 3.4),

$$S\varphi X = \varphi SX$$

for all $X \in \mathcal{D}$. In particular, $\xi \in \mathcal{D}_h$. The Codazzi equation for h (2.4) gives

$$\begin{aligned}(\nabla_\xi h)(X, Y) &= (\nabla_X h)(\xi, Y) = X(h(\xi, Y)) - h(\nabla_X \xi, Y) - h(\xi, \nabla_X Y)\\ &= -h(\nabla_X \xi, Y) - \eta(\nabla_X Y)h(\xi, \xi)\end{aligned}$$

for all $X, Y \in \mathcal{D}$. Since $\nabla_X \xi = -\varphi SX$ and $\eta(\nabla_X Y) = -h(X, \varphi Y)$ for every $X, Y \in \mathcal{D}$, the last equality takes the form

$$(\nabla_\xi h)(X, Y) = h(\varphi SX, Y) + h(X, \varphi Y)h(\xi, \xi). \tag{4.13}$$

Now, from (4.12) and (4.13) we obtain

$$h(X, \varphi SY) = h(X, \varphi Y)h(\xi, \xi)$$

for all $X, Y \in \mathcal{D}$. Nondegeneracy of h on $\mathcal{D}$ implies that

$$\varphi SY = h(\xi, \xi)\varphi Y,$$

therefore

$$SY = h(\xi, \xi)Y$$

for all $Y \in \mathcal{D}$. Formula (3.6) and the Ricci equation (2.6) imply that $S\xi = h(\xi, \xi)\xi$. Finally, for all $X \in \mathcal{X}(M)$ we have

$$SX = h(\xi, \xi)X.$$

Moreover $h(\xi, \xi)$ is constant due to the Codazzi equation for S (2.5) and equiaffinity of C. ∎

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