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On a generalization of close-to-convex functions

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Abstract. The paper of M. Ismail et al. [Complex Variables Theory Appl. 14 (1990), 77–84] motivates the study of a generalization of close-to-convex functions by means of a q-analog of the difference operator acting on analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We use the term *q-close-to-convex functions* for the q-analog of close-to-convex functions. We obtain conditions on the coefficients of power series of functions analytic in the unit disk which ensure that they generate functions in the q-close-to-convex family. As a result we find certain dilogarithm functions that are contained in this family. Secondly, we also study the Bieberbach problem for coefficients of analytic q-close-to-convex functions. This produces several power series of analytic functions convergent to basic hypergeometric functions.

1. Introduction. Denote by $\mathcal{A}$ the class of functions $f(z)$ analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0 = f'(0) - 1$. In other words, each f in $\mathcal{A}$ has a power series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{D}.$$

We denote by $\mathcal{S}$ the class of univalent (i.e. one-one) analytic functions in $\mathbb{D}$. Denote by $\mathcal{S}^*$ the subclass consisting of functions f in $\mathcal{S}$ that are *starlike* with respect to the origin, i.e. $tw \in f(\mathbb{D})$ whenever $t \in [0, 1]$ and $w \in f(\mathbb{D})$. Analytically, it is well-known that $f \in \mathcal{S}^*$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \qquad z \in \mathbb{D};$$

and a function $f \in \mathcal{A}$ is said to be *close-to-convex*, written $f \in \mathcal{K}$, if there exists $g \in \mathcal{S}^*$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) > 0, \qquad z \in \mathbb{D}.$$

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Then we say that $f \in \mathcal{K}$ *with the function* g. One can easily verify that $\mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$ (see for instance [Dur]). For several interesting geometric properties of these classes, one can refer to the standard books [Goo, Pom].

A q-analog of the class of starlike functions was first introduced in [IMS] by means of the *q-difference operator* D_q acting on functions $f \in \mathcal{A}$ by

$$(1.1)\qquad (D_q f)(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad z \in \mathbb{D} \setminus \{0\}, \quad (D_q f)(0) = f'(0),$$

where $q \in (0,1)$. Note that the q-difference operator plays an important role in the theory of hypergeometric series and in quantum physics (see for instance [And, Ern, Fin, Kir, Sla]). One can clearly see that $(D_q f)(z) \to f'(z)$ as $q \to 1^-$. This difference operator helps us to generalize the class of starlike functions $\mathcal{S}^*$ analytically. We denote by $\mathcal{S}_q^*$ the class of functions in this generalized family. For the sake of convenience, we also call functions in $\mathcal{S}_q^*$ *q-starlike functions*. This class is defined as follows:

Definition 1.1. A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_q^*$ if

$$\left| \frac{z}{f(z)}(D_q f)(z) - \frac{1}{1-q} \right| \le \frac{1}{1-q}, \quad z \in \mathbb{D}.$$

Clearly, when $q \to 1^-$, the class $\mathcal{S}_q^*$ will coincide with $\mathcal{S}^*$.

As $\mathcal{S}_q^*$ generalizes $\mathcal{S}^*$ in the above manner, a similar form of q-analog of close-to-convex functions was expected and it is defined in the following form (see [RS]).

Definition 1.2. A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{K}_q$ if there exists $g \in \mathcal{S}^*$ such that

$$\left| \frac{z}{g(z)}(D_q f)(z) - \frac{1}{1-q} \right| \le \frac{1}{1-q}, \quad z \in \mathbb{D}.$$

Then we say that $f \in \mathcal{K}_q$ *with the function* g.

In [RS], the authors have investigated some basic properties of functions that are in $\mathcal{K}_q$. Some of these results are recalled in this paper in order to exhibit their interesting consequences. As $(D_q f)(z) \to f'(z)$ as $q \to 1^-$, we observe that in the limiting sense, the closed disk $|w - (1-q)^{-1}| \le (1-q)^{-1}$ becomes the right half-plane $\mathrm{Re}(zf'(z)/g(z)) > 0$, and hence the class $\mathcal{K}_q$ clearly reduces to $\mathcal{K}$. In this paper, we refer to the functions in $\mathcal{K}_q$ as *q-close-to-convex functions*. For the sake of convenience, we use the notation $\mathcal{S}_q^*$ instead of PS_q used in [IMS], and $\mathcal{K}_q$ instead of PK_q used in [RS]. It is easy to see that $\mathcal{S}_q^* \subset \mathcal{K}_q$ for all $q \in (0,1)$. Clearly, one can easily see from the above discussion that

$$\bigcap_{0<q<1} \mathcal{K}_q \subset \mathcal{K} \subset \mathcal{S}.$$

Our main aim in this paper is to consider the following two ideas.

The first idea has its origin in the work of Friedman [Fri]. He proved that there are only nine functions in the class $\mathcal{S}$ whose coefficients are rational integers. They are

$$z, \quad \frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{(1 \pm z)^2}, \quad \frac{z}{1 \pm z + z^2}.$$

It is easy to see that these functions map the unit disk $\mathbb{D}$ onto starlike domains. Using the idea of MacGregor [Mac], we derive some sufficient conditions for functions to be in $\mathcal{K}_q$ whose coefficients are connected with certain monotonicity properties. These sufficient conditions help us to examine functions of dilogarithm type [Kir, Zag] which are in $\mathcal{K}_q$. Certain special functions, which are in the starlike and close-to-convex family, have been investigated in [HPV, MS, MM, Pon, PV, RuSi, Sil].

The second idea deals with the famous Bieberbach problem in analytic univalent function theory [DeB, Dur]. A necessary and sufficient condition for a function f to be in $\mathcal{S}_q^*$ is obtained in [IMS] by means of an integral representation of the function $zf'(z)/f(z)$ which yields the maximum moduli of the coefficients of f. Using this condition, the Bieberbach problem for q-starlike functions has been solved in the following form.

THEOREM A ([IMS, Theorem 1.18]). *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to the class $\mathcal{S}_q^*$, then $|a_n| \le c_n$, with equality holding for all n if and only if $f(z)$ is a rotation of*

$$k_q(z) := z \exp\left[\sum_{n=1}^{\infty} \frac{-2\ln q}{1-q^n} z^n\right] = z + \sum_{n=2}^{\infty} c_n z^n, \qquad z \in \mathbb{D}.$$

Note that the function k_q plays the role of the Koebe function k. By differentiating once the above expression for k_q and equating the coefficients of z^{n-1} on both sides, we get a recurrence relation for the c_n:

$$c_2 = \frac{-2\ln q}{1-q}$$

and

$$(n-1)c_n = \frac{-2\ln q}{1-q^{n-1}}(n-1) + \sum_{k=2}^{n-1} \frac{-2\ln q}{1-q^{k-1}} c_{n+1-k}(k-1), \qquad n \ge 3.$$

It can be easily verified that Theorem A turns into the famous conjecture of Bieberbach (known as the Bieberbach–de Branges theorem) for the class $\mathcal{S}^*$ if $q \to 1^-$. Comparing with the Bieberbach–de Branges theorem for close-to-convex functions, one would expect that Theorem A also holds for q-close-to-convex functions. However, this remains an open problem. Indeed, in this paper, we obtain an optimal coefficient bound for q-close-to-convex functions leading to the Bieberbach–de Branges theorem for close-to-convex functions when $q \to 1^-$. Finally, we collect a few consequences of the Bieberbach–de Branges theorem for $\mathcal{K}_q$ involving the nine starlike functions considered above.

2. Conditions for $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$ to be in $\mathcal{K}_q$. In this section, we mainly concentrate on problems in situations where the coefficients A_n of functions $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$ in $\mathcal{K}_q$ are real, non-negative and connected with certain monotone properties. Similar investigations for the class of close-to-convex functions were conducted in [Ale, Mac] (see references therein for initial contributions of Fejér and Szegö in this direction).

We obtain several sufficient conditions for $f(z) = z + \sum_{n=2}^{\infty} A_n z^n$ to be in $\mathcal{K}_q$. Rewriting this representation, we get

$$f(z) = \sum_{n=0}^{\infty} A_n z^n \quad (A_0 = 0,\ A_1 = 1). \tag{2.1}$$

If $f(z)$ is of the form (2.1), then a simple computation yields

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} \frac{A_n(1-q^n)}{1-q} z^{n-1} \tag{2.2}$$

for all $z \in \mathbb{D}$. With this, we now collect a number of sufficient conditions for functions to be in $\mathcal{K}_q$.

LEMMA 2.1 ([RS, Lemma 1.1(1)]). *Let $f(z)$ be of the form* (2.1). *Suppose that $\sum_{n=1}^{\infty} |B_{n+1} - B_n| \le 1$ with $B_n = A_n(1-q^n)/(1-q)$. Then $f \in \mathcal{K}_q$ with $g(z) = z/(1-z)$.*

As a consequence of Lemma 2.1, we have

THEOREM 2.2. *Let $\{A_n\}$ be a sequence of real numbers such that $B_n = A_n(1-q^n)/(1-q)$ for all $n \ge 1$. Suppose that*

$$1 \ge B_2 \ge \cdots \ge B_n \ge \cdots \ge 0 \quad \text{or} \quad 1 \le B_2 \le \cdots \le B_n \le \cdots \le 2.$$

Then $f(z) = z + \sum_{n=2}^{\infty} A_n z^n \in \mathcal{K}_q$ with $g(z) = z/(1-z)$.

Proof. We know that

$$\sum_{n=1}^{\infty} |B_{n+1} - B_n| = \lim_{k\to\infty} \sum_{n=1}^{k} |B_{n+1} - B_n|.$$

If $1 \ge B_2 \ge \cdots \ge B_n \ge \cdots \ge 0$, we see that

$$\lim_{k\to\infty} \sum_{n=1}^{k} |B_{n+1} - B_n| = \lim_{k\to\infty} (B_1 - B_{k+1}) \le B_1 = 1.$$

Similarly, if $1 \le B_2 \le \cdots \le B_n \le \cdots \le 2$, then we get the bound $\sum_{n=1}^{\infty} |B_{n+1} - B_n| \le 1$. Thus, by Lemma 2.1, we obtain the assertion of our theorem. ∎

Remark. Letting $q \to 1^-$ in Theorem 2.2, one can obtain the results of Alexander [Ale] and MacGregor [Mac].

Example 2.3. *The quantum dilogarithm* function is defined by

$$\mathrm{Li}_2(z;q) = \sum_{n=1}^{\infty} \frac{z^n}{n(1-q^n)}, \qquad |z| < 1,\ 0 < q < 1.$$

Note that this function is studied by Kirillov [Kir] (see also [Zag, p. 28]) and is a q-deformation of the ordinary *dilogarithm function* [Kir] defined by $\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} z^n/n^2$, $|z| < 1$, in the sense that

$$\lim_{\epsilon\to 0} \epsilon \,\mathrm{Li}_2(z; e^{-\epsilon}) = \mathrm{Li}_2(z).$$

By Theorem 2.2, one can ascertain that $(1-q)\,\mathrm{Li}_2(z;q) \in \mathcal{K}_q$.

Theorem 2.4. *Let f be defined by* (2.1) *and suppose that*

$$\sum_{n=1}^{\infty} |B_n - B_{n-1}| \le 1, \qquad B_n = \frac{A_{n+1}(1-q^{n+1})}{1-q} - \frac{A_n(1-q^n)}{1-q}.$$

Then $f \in \mathcal{K}_q$ with $g(z) = z/(1-z)^2$.

Proof. Starting with $|B_n|$, we see that

$$|B_n| = \Big|\sum_{k=1}^{n} (B_k - B_{k-1}) + 1\Big| \le \sum_{k=1}^{\infty} |B_k - B_{k-1}| + 1 \le 2.$$

Hence, for all $n \ge 2$, we have

$$\left|\frac{A_n(1-q^n)}{1-q} - \frac{A_{n-1}(1-q^{n-1})}{1-q}\right| \le 2.$$

Now, by the triangle inequality, we see that

$$\begin{aligned}\left|\frac{A_n(1-q^n)}{1-q}\right| &= \left|\frac{A_n(1-q^n)}{1-q} - \frac{A_{n-1}(1-q^{n-1})}{1-q} + \frac{A_{n-1}(1-q^{n-1})}{1-q}\right. \\ &\qquad \left. - \frac{A_{n-2}(1-q^{n-2})}{1-q} + \cdots + \frac{A_2(1-q^2)}{1-q} - 1 + 1\right| \\ &\le 2(n-1) + 1 = 2n - 1,\end{aligned}$$

and so $|A_n| \le (2n-1)/(1 + q + \cdots + q^{n-1})$. By applying the root test, one can see that the radius of convergence of $\sum_{n=0}^{\infty} A_n z^n$ is not less than unity. Therefore, $f \in \mathcal{A}$.

Since f is of the form (2.1), by using (2.2) we compute

$$\begin{aligned}
&(1-z)^2(D_qf)(z)\\
&\quad = 1+\frac{A_2(1-q^2)}{1-q}z-2z\\
&\qquad +\sum_{n=3}^{\infty}\left[\frac{A_n(1-q^n)}{1-q}-\frac{2A_{n-1}(1-q^{n-1})}{1-q}+\frac{A_{n-2}(1-q^{n-2})}{1-q}\right]z^{n-1}.
\end{aligned}$$

By the definition of B_n as given in the hypothesis, we have

$$(1-z)^2(D_qf)(z)=1+(B_1-1)z+\sum_{n=3}^{\infty}(B_{n-1}-B_{n-2})z^{n-1}.$$

Hence,

$$\frac{1}{1-q}-\left|(1-z)^2(D_qf)(z)-\frac{1}{1-q}\right|\geq 1-|B_1-1|-\sum_{n=3}^{\infty}|B_{n-1}-B_{n-2}|\geq 0$$

if $\sum_{n=2}^{\infty}|B_{n-1}-B_{n-2}|\leq 1$. This proves the assertion of our theorem. ∎

By Theorem 2.4, we immediately have the following result which generalizes a couple of results of MacGregor (see [Mac, Theorems 3 and 5]).

Theorem 2.5. *Let $\{A_n\}$ be a sequence of real numbers such that*

$$A_0=0,\qquad A_1=1\quad \textit{and}\quad B_n=\frac{A_{n+1}(1-q^{n+1})}{1-q}-\frac{A_n(1-q^n)}{1-q}.$$

Suppose that

$$1\geq B_1\geq\cdots\geq B_n\geq\cdots\geq 0\quad \textit{or}\quad 1\leq B_1\leq\cdots\leq B_n\leq\cdots\leq 2.$$

Then $f(z)=z+\sum_{n=2}^{\infty}A_nz^n\in\mathcal{K}_q$ with $g(z)=z/(1-z)^2$.

Theorem 2.6. *Let f be defined by $f(z)=z+\sum_{n=2}^{\infty}A_{2n-1}z^{2n-1}$ and suppose that*

$$\sum_{n=1}^{\infty}|B_{2n-1}-B_{2n+1}|\leq 1,\qquad B_n=\frac{A_n(1-q^n)}{1-q}.$$

Then $f\in\mathcal{K}_q$ with $g(z)=z/(1-z^2)$.

Proof. First of all we shall prove that $f(z)=z+\sum_{n=2}^{\infty}A_{2n-1}z^{2n-1}\in\mathcal{A}$. For this, we estimate

$$|B_{2n+1}|=\left|\sum_{k=1}^{n}(B_{2k-1}-B_{2k+1})-1\right|\leq 2$$

so that $|A_n|\leq 2/(1+q+\cdots+q^{n-1})$. By applying the root test, one can see that the radius of convergence of the series expansion of $f(z)$ is not less than 1. Therefore, $f\in\mathcal{A}$.

Since $f(z) = z + \sum_{n=2}^{\infty} A_{2n-1} z^{2n-1}$, by (1.1) we get

$$(1-z^2)(D_q f)(z) = 1 - \sum_{n=1}^{\infty}\left[\frac{A_{2n-1}(1-q^{2n-1})}{1-q} - \frac{A_{2n+1}(1-q^{2n+1})}{1-q}\right] z^{2n}.$$

Note that $B_n = A_n(1-q^n)/(1-q)$. So, we have

$$\frac{1}{1-q} - \left|(1-z^2)(D_q f)(z) - \frac{1}{1-q}\right| \geq 1 - \sum_{n=1}^{\infty} |B_{2n-1} - B_{2n+1}| \geq 0$$

whenever $\sum_{n=1}^{\infty} |B_{2n-1} - B_{2n+1}| \leq 1$. This proves our conclusion. ∎

By Theorem 2.6, we immediately have the following result which generalizes a result of MacGregor (see [Mac, Theorem 2]).

THEOREM 2.7. *Let $\{A_n\}$ be a sequence of real numbers and set $B_n = A_n(1-q^n)/(1-q)$ for all $n \geq 1$. Suppose that*

$$1 \geq B_3 \geq B_5 \geq \cdots \geq B_{2n-1} \geq \cdots \geq 0$$

or

$$1 \leq B_3 \leq B_5 \leq \cdots \leq B_{2n-1} \leq \cdots \leq 2.$$

Then $f(z) = z + \sum_{n=2}^{\infty} A_{2n-1} z^{2n-1} \in \mathcal{K}_q$ with $g(z) = z/(1-z^2)$.

LEMMA 2.8 ([RS, Lemma 1.1(4)]). *Let f be defined by* (2.1) *and suppose that*

$$\sum_{n=2}^{\infty} |B_n - B_{n-2}| \leq 1, \qquad B_n = \frac{A_n(1-q^n)}{(1-q)}.$$

Then $f \in \mathcal{K}_q$ with $g(z) = z/(1-z^2)$.

Lemma 2.8 leads to the following sufficient conditions for functions to be in $\mathcal{K}_q$.

THEOREM 2.9. *Let $\{A_n\}$ be a sequence of real numbers such that $A_1 = 1$ and set*

$$B_n = \frac{A_n(1-q^n)}{1-q}$$

for all $n \geq 1$. Suppose that

$$1 \geq B_1 + B_2 \geq \cdots \geq B_{n-1} + B_n \geq \cdots \geq 0$$

or

$$1 \leq B_1 + B_2 \leq \cdots \leq B_{n-1} + B_n \leq \cdots \leq 2.$$

Then $f(z) = z + \sum_{n=2}^{\infty} A_n z^n \in \mathcal{K}_q$ with $g(z) = z/(1-z^2)$.

Proof. We know that

$$\sum_{n=2}^{\infty} |B_n - B_{n-2}| = \lim_{k\to\infty} \sum_{n=2}^{k} |B_n - B_{n-2}|.$$

If $1 \geq B_1 + B_2 \geq \cdots \geq B_{n-1} + B_n \geq \cdots \geq 0$, we see that

$$\lim_{k\to\infty} \sum_{n=2}^{k} |B_n - B_{n-2}| = \lim_{k\to\infty} (1 - B_{k-1} - B_k) \leq 1 + 0 = 1.$$

Similarly, if $1 \leq B_1 + B_2 \leq \cdots \leq B_{n-1} + B_n \leq \cdots \leq 2$, then we get $\sum_{n=2}^{\infty} |B_n - B_{n-2}| \leq 1$. Thus, by Theorem 2.8, the proof is complete. ∎

As a consequence of Theorem 2.9, one can obtain the following new criteria for functions to be in the close-to-convex family.

THEOREM 2.10. *Let $\{a_n\}$ be a sequence of real numbers such that $a_1 = 1$ and set $b_n = na_n$ for all $n \geq 1$. Suppose that*

$$1 \geq b_1 + b_2 \geq \cdots \geq b_{n-1} + b_n \geq \cdots \geq 0$$

or

$$1 \leq b_1 + b_2 \leq \cdots \leq b_{n-1} + b_n \leq \cdots \leq 2.$$

Then $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is close-to-convex with $g(z) = z/(1 - z^2)$.

LEMMA 2.11 ([RS, Lemma 1.1(2)]). *Let f be defined by* (2.1) *and suppose that*

$$\sum_{n=1}^{\infty} |B_{n-1} - B_n + B_{n+1}| \leq 1, \qquad B_n = \frac{A_n(1 - q^n)}{1 - q}.$$

Then $f \in \mathcal{K}_q$ with $g(z) = z/(1 - z + z^2)$.

Lemma 2.11 yields the following sufficient condition.

THEOREM 2.12. *Let $\{A_n\}$ be a sequence of real numbers such that $A_1 = 1$ and set $B_n = A_n(1 - q^n)/(1 - q)$ for all $n \geq 1$. Suppose that either*

$$\begin{aligned} (2.3) \qquad 0 \geq B_2 - B_1 \geq B_3 \geq B_2 + B_4 \geq B_2 + B_3 + B_5 \geq \cdots \\ \geq B_2 + B_3 + B_4 + \cdots + B_{n-1} + B_{n+1} \geq -1, \end{aligned}$$

or

$$\begin{aligned} (2.4) \qquad 0 \leq B_2 - B_1 \leq B_3 \leq B_2 + B_4 \leq B_2 + B_3 + B_5 \leq \cdots \\ \leq B_2 + B_3 + B_4 + \cdots + B_{n-1} + B_{n+1} \leq 1. \end{aligned}$$

Then $f(z) = z + \sum_{n=2}^{\infty} A_n z^n \in \mathcal{K}_q$ with $g(z) = z/(1 - z + z^2)$.

Proof. We know that

$$\sum_{n=1}^{\infty} |B_{n-1} - B_n + B_{n+1}| = \lim_{k\to\infty} \sum_{n=1}^{k} |B_{n-1} - B_n + B_{n+1}|.$$

If (2.3) holds, on the one hand we see that

$$\lim_{k\to\infty} \sum_{n=1}^{k} |B_{n-1} - B_n + B_{n+1}| = \lim_{k\to\infty} -(B_2 + B_3 + B_4 + \cdots + B_{k-1} + B_{k+1}) \leq 1.$$

On the other hand, if (2.4) holds, then similarly $\sum_{n=1}^{\infty} |B_{n-1}-B_n+B_{n+1}| \le 1$. Thus, by Theorem 2.11, the proof is complete. ■

As a consequence of Theorem 2.12, one can obtain the following new criteria for functions to be in the close-to-convex family.

THEOREM 2.13. *Let $\{a_n\}$ be a sequence of real numbers such that $a_1 = 1$ and set $b_n = na_n$ for all $n \ge 1$. Suppose that*

$$0 \ge b_2 - b_1 \ge b_3 \ge b_2 + b_4 \ge b_2 + b_3 + b_5 \ge \cdots \\ \ge b_2 + b_3 + b_4 + \cdots + b_{n-1} + b_{n+1} \ge -1$$

or

$$0 \le b_2 - b_1 \le b_3 \le b_2 + b_4 \le b_2 + b_3 + b_5 \le \cdots \\ \le b_2 + b_3 + b_4 + \cdots + b_{n-1} + b_{n+1} \le 1.$$

Then $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in the close-to-convex family with $g(z) = z/(1 - z + z^2)$.

3. The Bieberbach–de Branges theorem for $\mathcal{K}_q$. A necessary and sufficient condition for membership in $\mathcal{S}_q^*$ was obtained in [IMS, Theorem 1.5]: *$f \in \mathcal{S}_q^*$ if and only if $|f(qz)/f(z)| \le 1$ for all $z \in \mathbb{D}$.*

A similar characterization for functions in $\mathcal{K}_q$ is

LEMMA 3.1. *$f \in \mathcal{K}_q$ if and only if there exists $g \in \mathcal{S}^*$ such that*

$$\frac{|g(z) + f(qz) - f(z)|}{|g(z)|} \le 1 \quad \text{for all } z \in \mathbb{D}.$$

Proof. This follows directly after substituting the formula for $(D_q f)(z)$ in Definition 1.2. ■

Lemma 3.1 will be crucial to get coefficient bounds for series representation of functions in the class $\mathcal{K}_q$; in other words, we analyze the Bieberbach–de Branges theorem for the class of q-close-to-convex functions. The Bieberbach conjecture for close-to-convex functions was proved by Reade [Rea] (see also [Goo] for more details). It states that *if $f \in \mathcal{K}$, then $|a_n| \le n$ for all $n \ge 2$.*

We now proceed to state and prove the Bieberbach–de Branges theorem for functions in the q-close-to-convex family.

THEOREM 3.2 (Bieberbach–de Branges theorem for $\mathcal{K}_q$). *If $f \in \mathcal{K}_q$, then*

$$|a_n| \le \frac{1-q}{1-q^n}\left[n + \frac{n(n-1)}{2}(1+q)\right] \quad \text{for all } n \ge 2.$$

Proof. Since $f \in \mathcal{K}_q$, by Lemma 3.1 there exists $w : \mathbb{D} \to \overline{\mathbb{D}}$ such that

$$g(z) + f(qz) - f(z) = w(z)g(z). \tag{3.1}$$

Clearly $w(0) = q$. By assuming $a_1 = 1 = b_1$, we have

$$\sum_{n=1}^{\infty}(b_n + a_n q^n - a_n)z^n = \sum_{n=1}^{\infty} q b_n z^n + \sum_{n=2}^{\infty}\Big(\sum_{k=1}^{n-1} w_{n-k} b_k\Big) z^n.$$

Equating the coefficients of z^n, for $n \geq 2$ we obtain

$$a_n(q^n - 1) = b_n(q-1) + \sum_{k=1}^{n-1} w_{n-k} b_k.$$

From the classical result [DCP], one can verify that $|w_n| \leq 1 - |w_0|^2 = 1 - q^2$ for all $n \geq 1$. Since $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S^*$, we get

$$|a_n| \leq \frac{1-q}{1-q^n}\Big[n + (1+q)\sum_{k=1}^{n-1} k\Big] \quad \text{for all } n \geq 2.$$

This proves the conclusion of our theorem. ■

Remark. When $q \to 1^-$, Theorem 3.2 yields the Bieberbach conjecture for close-to-convex functions.

It is easy to see, by the usual ratio test, that the series

$$z + \sum_{n=2}^{\infty} \frac{1-q}{1-q^n}\Big[n + \frac{n(n-1)}{2}(1+q)\Big] z^n \tag{3.2}$$

converges for $|z| < 1$. Indeed, we can ascertain by using the convergence factor for the series $\sum_{n=1}^{\infty} z^n/(1-q^n)$ (see [Sla, 3.2.2.1]) that the series given by (3.2) converges to the function

$$\frac{1+q}{2} z^2 \frac{d^2\Psi(q;z)}{dz^2} + z\frac{d\Psi(q;z)}{dz},$$

where $\Psi(q;z) := z\Phi[q,q;q^2;q,z]$ represents the corresponding Heine hypergeometric function. Note that the q-hypergeometric series was developed by Heine [Hei] as a generalization of the well-known Gauss hypergeometric series:

$$\Phi[a,b;c;q,z] = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} z^n, \qquad |q| < 1,\ 1 \neq cq^n,\ |z| < 1,$$

where the q-shifted factorial $(a;q)_n$ is defined by

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) \quad \text{and} \quad (a;q)_0 = 1.$$

This is also known as the basic hypergeometric series and its convergence function is known as the basic hypergeometric function. We refer to [Sla] for these notions. For the history of q-series related calculus and their applications, we recommend [Ern].

Due to Friedman's result mentioned in the Introduction, we now study the special cases of Theorem 3.2 with respect to the nine functions having integer coefficients. However, in this situation, it is enough to consider the identity function and four other functions which contain factors $1-z$ instead of $1 \pm z$ in the denominator. In particular, Theorem 3.2 reduces to the corollaries below. We provide proofs of the last two as they involve variations in the exponents, whereas the first three corollaries follow directly after making the corresponding substitution for the starlike functions $g(z)$.

Corollary 3.3. *If $f \in \mathcal{K}_q$ with the Koebe function $g(z) = z/(1-z)^2$, then for all $n \geq 2$ we have*

$$|a_n| \leq \frac{1-q}{1-q^n}\left[n + (1+q)\frac{n(n-1)}{2}\right].$$

If $f \in \mathcal{K}$ with $g(z) = z$, then for all $n \geq 2$ it is well-known that $|a_n| \leq 2/n$. As a generalization, we have the following:

Corollary 3.4. *If $f \in \mathcal{K}_q$ with $g(z) = z$, then for all $n \geq 2$ we have $|a_n| \leq (1-q^2)/(1-q^n)$.*

Here we note that the series $z + \sum_{n=2}^{\infty}(1-q^2)/(1-q^n)z^n$ converges to the Heine hypergeometric function $(z+qz)\Phi[q,q;q^2;q,z] - qz = z + z^2\Phi[q^2,q^2;q^3;q^2,z]$, as follows from [Sla, 3.2.2, p. 91].

If $f \in \mathcal{K}$ with $g(z) = z/(1-z)$, then for all $n \geq 2$ it is known that $|a_n| \leq (2n-1)/n$. We find the following analogous result:

Corollary 3.5. *If $f \in \mathcal{K}_q$ with $g(z) = z/(1-z)$, then for all $n \geq 2$ we have*

$$|a_n| \leq \frac{1-q}{1-q^n}\,[n + q(n-1)].$$

One can similarly verify that the series $z + \sum_{n=2}^{\infty} \frac{1-q}{1-q^n}[n + q(n-1)]$ converges to the function $z(1+q)\frac{d}{dz}\Psi(q;z) - q\Psi(q;z)$, where $\Psi(q;z) := z\Phi[q,q;q^2;q,z]$ represents the corresponding Heine hypergeometric function.

If $f \in \mathcal{K}$ with $g(z) = z/(1-z^2)$, then for all $m \geq 1$ it is known that

$$|a_n| \leq \begin{cases} 1 & \text{if } n = 2m-1, \\ 1 & \text{if } n = 2m. \end{cases}$$

As a generalization, we now state the following corollary along with an outline of its proof:

Corollary 3.6. *If $f \in \mathcal{K}_q$ with $g(z) = z/(1-z^2)$, then for all $m \geq 1$ we have*

$$|a_n| \leq \begin{cases} \dfrac{1-q}{1-q^n}\left(\dfrac{n}{2}(1+q) + \dfrac{1}{2}(1-q)\right) & \text{if } n = 2m-1, \\[2ex] \dfrac{1-q^2}{1-q^n}\,\dfrac{n}{2} & \text{if } n = 2m. \end{cases}$$

Proof. Since $g(z) = z/(1 - z^2) = \sum_{n=1}^{\infty} z^{2n-1}$, by (3.1) we get

$$\sum_{n=1}^{\infty}(q^n - 1)a_n z^n = (q - 1)\sum_{n=1}^{\infty} z^{2n-1} + \Big(\sum_{n=1}^{\infty} z^{2n-1}\Big)\Big(\sum_{n=1}^{\infty} w_n z^n\Big).$$

This is equivalent to

$$\sum_{n=1}^{\infty}(q^n - 1)a_n z^n = (q - 1)\sum_{n=1}^{\infty} z^{2n-1} + \sum_{n=2}^{\infty}\Big(\sum_{k=1}^{n-1} w_{2k}\Big)z^{2n-1} + \sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n} w_{2k-1}\Big)z^{2n}. \tag{3.3}$$

In order to prove the required optimal bound for $|a_n|$, we equate the coefficients of z^{2n-1} and z^{2n}.

In (3.3), first we equate the coefficients of z^{2n-1}, for $n \geq 2$, to get

$$(q^{2n-1} - 1)a_{2n-1} = (q - 1) + \sum_{k=1}^{n-1} w_{2k}.$$

Since $|w_k| \leq (1 - q^2)$ for all $k \geq 1$ and $q \in (0, 1)$, we have

$$|a_{2n-1}| \leq \frac{1 - q}{(1 - q^{2n-1})}(-q + (1 + q)n).$$

Secondly, by equating the coefficients of z^{2n}, for $n \geq 1$, we obtain

$$(q^{2n} - 1)a_{2n} = \sum_{k=1}^{n} w_{2k-1},$$

and similarly we get the bound

$$|a_{2n}| \leq \frac{1 - q}{1 - q^{2n}}(1 + q)n.$$

Thus, we obtain the required optimal bound for $|a_n|$. ∎

If $f \in \mathcal{K}$ with $g(z) = z/(1 - z + z^2)$, then for all $n \geq 2$ it is known that

$$|a_n| \leq \begin{cases} \dfrac{4n + 1}{3n} & \text{if } n = 3m - 1, \\ \dfrac{4}{3} & \text{if } n = 3m, \\ \dfrac{4n - 1}{3n} & \text{if } n = 3m + 1. \end{cases}$$

As a generalization, we have the following:

Corollary 3.7. *If $f \in \mathcal{K}_q$ with $g(z) = z/(1-z+z^2)$, then for all $m \geq 1$ we have*

$$|a_n| \leq \begin{cases} \dfrac{1-q}{1-q^n}\left(\dfrac{1}{3}(2-q)+\dfrac{2n}{3}(1+q)\right) & \textit{if } n=3m-1, \\ \dfrac{1-q^2}{1-q^n}\dfrac{2n}{3} & \textit{if } n=3m, \\ \dfrac{1-q}{1-q^n}\left(\dfrac{2n}{3}(1+q)+\dfrac{1}{3}(1-2q)\right) & \textit{if } n=3m+1. \end{cases}$$

Proof. By rewriting the function $g(z)=z/(1-z+z^2)$, we obtain

$$g(z)=\frac{z(1+z)}{1+z^3}=\sum_{n=1}^{\infty}(-1)^{n-1}z^{3n-2}+\sum_{n=1}^{\infty}(-1)^{n-1}z^{3n-1}.$$

Then simplifying (3.1), we get

$$\begin{aligned} &\sum_{n=1}^{\infty}(q^n-1)a_nz^n \\ &= (q-1)\Big(\sum_{n=1}^{\infty}(-1)^{n-1}z^{3n-2}+\sum_{n=1}^{\infty}(-1)^{n-1}z^{3n-1}\Big) \\ &\quad+\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}(-1)^{n-k}w_{3k-2}\Big)z^{3n-1}+\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}(-1)^{n-k}w_{3k-1}\Big)z^{3n} \\ &\quad+\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}(-1)^{n-k}w_{3k}\Big)z^{3n+1}+\sum_{n=2}^{\infty}\Big(\sum_{k=1}^{n-1}(-1)^{n-k}w_{3k}\Big)z^{3n-1} \\ &\quad+\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}(-1)^{n-k}w_{3k-2}\Big)z^{3n}+\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}(-1)^{n-k-1}w_{3k-1}\Big)z^{3n+1}. \end{aligned} \tag{3.4}$$

First equating the coefficients of z^{3n-1}, for $n\geq 2$, in (3.4), we get

$$(q^{3n-1}-1)a_{3n-1}=(-1)^{n-k}(q-1)+\sum_{k=1}^{n}(-1)^{n-k}w_{3k-2}+\sum_{k=1}^{n}(-1)^{n-k}w_{3k}.$$

Since $|w_k|\leq(1-q^2)$ for all $k\geq 1$ and $q\in(0,1)$, we have

$$|a_{3n-1}|\leq\frac{1-q}{(1-q^{3n-1})}(-q+2(1+q)n).$$

Next, for all $n\geq 1$, we equate the coefficients of z^{3n} and z^{3n+1} in (3.4), and obtain

$$|a_{3n}|\leq\frac{2(1-q)}{(1-q^{3n})}(1+q)n \quad\text{and}\quad |a_{3n+1}|\leq\frac{(1-q)}{(1-q^{3n+1})}(1+2(1+q)n).$$

Thus, the assertion of our corollary follows. ■

Remark. By making use of [IMS, Theorem 1.5], one can also obtain the Bieberbach–de Branges theorem for $\mathcal{S}_q^*$, as shown below. This also yields a Bieberbach–de Branges type theorem for $\mathcal{S}^*$, in particular. However, it differs from Theorem A.

4. Appendix. In this section, we verify that a similar technique used in the previous section yields a form of the Bieberbach–de Branges theorem for $\mathcal{S}_q^*$. This leads to a result of Bieberbach–de Branges type (different from Theorem A!) for the class $\mathcal{S}^*$, when $q \to 1^-$, as well.

THEOREM 4.1 (The Bieberbach–de Branges theorem for $\mathcal{S}_q^*$). *If $f \in \mathcal{S}_q^*$, then for all $n \geq 2$ we have*

$$|a_n| \leq \frac{1-q^2}{q-q^n} \prod_{k=2}^{n-1} \left(1 + \frac{1-q^2}{q-q^k}\right). \tag{4.1}$$

Proof. We know that $f \in \mathcal{S}_q^*$ if and only if

$$|f(qz)/f(z)| \leq 1 \quad \text{for all } z \in \mathbb{D}.$$

Then there exists $w : \mathbb{D} \to \overline{\mathbb{D}}$ such that

$$\frac{f(qz)}{f(z)} = w(z), \quad \text{i.e.} \quad f(qz) = w(z)f(z) \text{ for all } z \in \mathbb{D}.$$

Clearly, $w(0) = q$. In terms of series expansion, we get (with $a_1 = 1$ and $w_0 = q$)

$$\sum_{n=1}^{\infty} a_n q^n z^n = \Big(\sum_{n=0}^{\infty} w_n z^n\Big)\Big(\sum_{n=1}^{\infty} a_n z^n\Big) =: \sum_{n=1}^{\infty} c_n z^n,$$

where $c_n := \sum_{k=1}^{n} w_{n-k}a_k = qa_n + \sum_{k=1}^{n-1} w_{n-k}a_k$. Comparing the coefficients of z^n $(n \geq 2)$, we get

$$a_n(q^n - q) = \sum_{k=1}^{n-1} w_{n-k}a_k \quad \text{for } n \geq 2.$$

Since $|w_n| \leq 1 - |w_0|^2 = 1 - q^2$ for all $n \geq 1$, we see that

$$|a_n| \leq \frac{1-q^2}{q-q^n} \sum_{k=1}^{n-1} |a_k| \quad \text{for each } n \geq 2.$$

Thus for $n = 2$, one has $|a_2| \leq (1-q^2)/(q-q^2)$, and for $n \geq 3$, we apply a similar technique to estimate $|a_{n-1}|$ and get

$$|a_n| \leq \frac{1-q^2}{q-q^n}\left(1 + \frac{1-q^2}{q-q^{n-1}}\right) \sum_{k=1}^{n-2} |a_k|.$$

Iteratively, we conclude that

$$|a_n| \leq \frac{1-q^2}{q-q^n}\left(1 + \frac{1-q^2}{q-q^{n-1}}\right)\left(1 + \frac{1-q^2}{q-q^{n-2}}\right) \cdots \left(1 + \frac{1-q^2}{q-q^2}\right)$$

for all $n \geq 3$. This completes the proof. ∎

REMARK. One can easily verify that the right hand side of (4.1) approaches n as $q \to 1^-$, which will lead to the Bieberbach–de Branges theorem for starlike functions [Dur, Theorem 2.14].

We also find that the ratio test easily provides the convergence of the series $z + \sum_{n=2}^{\infty} A_n z^n$ in the subdisk $|z| < q/(q+1-q^2)$, where

$$A_n = \frac{1-q^2}{q-q^n} \prod_{k=1}^{n-2} \left(1 + \frac{1-q^2}{q-q^{k+1}}\right).$$

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