ANNALES
POLONICI MATHEMATICI
83.1 (2004)

Hukuhara's differentiable iteration semigroups of linear set-valued functions

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Abstract. Let K be a closed convex cone with nonempty interior in a real Banach space and let $cc(K)$ denote the family of all nonempty convex compact subsets of K. A family $\{F^t : t \geq 0\}$ of continuous linear set-valued functions $F^t : K \to cc(K)$ is a differentiable iteration semigroup with $F^0(x) = \{x\}$ for $x \in K$ if and only if the set-valued function $\Phi(t,x) = F^t(x)$ is a solution of the problem

$$D_t\Phi(t,x) = \Phi(t, G(x)) := \bigcup\{\Phi(t,y) : y \in G(x)\}, \quad \Phi(0,x) = \{x\},$$

for $x \in K$ and $t \geq 0$, where $D_t\Phi(t,x)$ denotes the Hukuhara derivative of $\Phi(t,x)$ with respect to t and $G(x) := \lim_{s\to 0+}(F^s(x) - x)/s$ for $x \in K$.

1. Let X be a vector space. Throughout this paper all vector spaces are supposed to be real. We write

$$A + B = \{a + b : a \in A,\, b \in B\}, \quad \lambda A := \{\lambda a : a \in A\}$$

for $A, B \subset X$ and $\lambda \in \mathbb{R}$. A subset $K \subset X$ is called a *cone* if $tK \subset K$ for all positive t.

Let X and Y be two vector spaces and let $K \subset X$ be a convex cone. A set-valued function $F : K \to n(Y)$, where $n(Y)$ denotes the family of all nonempty subsets of Y, is called *linear* if

$$F(x) + F(y) = F(x + y), \quad F(\lambda x) = \lambda F(x)$$

for all $x, y \in K$ and $\lambda > 0$.

Let K be a convex cone in a normed vector space and let $b(K)$, $c(K)$, and $cc(K)$ denote the sets of all bounded, compact, and convex compact members of $n(K)$, respectively. The *difference* $A - B$ of $A, B \in cc(K)$ is a set $C \in cc(K)$ such that $A = B + C$. If the difference exists, then it is unique. This is a consequence of a theorem of Rådström (see [7]).

Let $H : [0,\infty) \to cc(K)$ be a set-valued function such that the differences $H(t) - H(s)$ exist for $t, s \in [0,\infty)$ such that $t > s$. The *Hukuhara derivative*

2000 *Mathematics Subject Classification*: 39B12, 26E25.

Key words and phrases: linear set-valued functions, iterations, Hukuhara's derivative.

of H at t is defined by the formula (see [3])

$$DH(t) = \lim_{s\to t+} \frac{H(s)-H(t)}{s-t} = \lim_{s\to t-} \frac{H(t)-H(s)}{t-s},$$

whenever both limits exist with respect to the Hausdorff metric d in $cc(K)$ derived from the norm in X. Moreover,

$$DH(0) = \lim_{s\to 0+} \frac{H(s)-H(0)}{s}.$$

Now, we will prove the following

LEMMA 1. *Let X be a Banach space and $H : [0,\infty) \to cc(X)$ be a set-valued function. If H is differentiable at $t \in [0,\infty)$, then H is continuous at this point.*

Proof. For $s > t \geq 0$ we have

$$\begin{aligned} d(H(s),H(t)) &= d(H(s)-H(t),\{0\}) = (s-t)d\left(\frac{H(s)-H(t)}{s-t},\{0\}\right) \\ &\leq (s-t)\left[d\left(\frac{H(s)-H(t)}{s-t},DH(t)\right) + d(DH(t),\{0\})\right]. \end{aligned}$$

This implies that

$$\lim_{s\to t+} H(s) = H(t).$$

Similarly, for $t > 0$ and $0 < s < t$ we have

$$\begin{aligned} d(H(s),H(t)) &= d(\{0\},H(t)-H(s)) = (t-s)d\left(\{0\},\frac{H(t)-H(s)}{t-s}\right) \\ &\leq (t-s)\left[d\left(\frac{H(t)-H(s)}{t-s},DH(t)\right) + d(DH(t),\{0\})\right], \end{aligned}$$

whence

$$\lim_{s\to t-} H(s) = H(t).$$

Thus H is continuous at t.

2. Let K be a nonempty set. A family $\{F^t : t \geq 0\}$ of set-valued functions $F^t : K \to n(K)$ is said to be an *iteration semigroup* if

$$F^{s+t}(x) = F^t[F^s(x)] := \bigcup\{F^t(y) : y \in F^s(x)\}$$

for all $x \in K$ and $t,s \geq 0$.

EXAMPLE 1. Let $G : \mathbb{R}^n \to cc(\mathbb{R}^n)$ be a set-valued function. The *attainable set* $R(t,\xi)$ of the differential inclusion

$$(*) \qquad x'(s) \in G(x(s)) \quad \text{a.e. in } [0,t], \quad x(\cdot) \in AC[0,t], \quad x(0) = \xi,$$

at time t from $\xi \in \mathbb{R}^n$ is defined by the formula

$$R(t,\xi) = \{x(t) : x(\cdot) \text{ satisfies } (*) \text{ and } x(0) = \xi\}.$$

It is known that if the sets $R(t,\xi)$ are nonempty, then the set-valued functions $\xi \mapsto R(t,\xi)$ form an iteration semigroup (see e.g. [2]). Moreover, if G is locally Lipschitz on $\mathbb{R}^n$ and the sets $R([0,t],\xi)$ are compact, then G is the infinitesimal generator of this semigroup (see [13]).

Let K be a convex cone in a normed space. An iteration semigroup $\{F^t : t \geq 0\}$ of set-valued functions $F^t : K \to cc(K)$ is said to be *differentiable* if all set-valued functions $t \mapsto F^t(x)$ $(x \in K)$ have Hukuhara's derivative on $[0,\infty)$.

EXAMPLE 2. Let K be closed convex cone with nonempty interior in a Banach space. Every concave iteration semigroup $\{F^t : t \geq 0\}$ of continuous linear set-valued functions $F^t : K \to cc(K)$ with $F^0(x) = \{x\}$ for $x \in K$ is differentiable (see [10]).

EXAMPLE 3. The family $\{F^t : t \geq 0\}$, where $F^t(x) = [e^t, e^{2t}]x$ for $t \in [0,\infty)$ and $x \in \mathbb{R}$, is a differentiable iteration semigroup of continuous linear set-valued functions. This semigroup is not concave.

Let X and Y be two vector spaces and let K be a convex cone in X. A set-valued function $F : K \to n(Y)$ is called *superadditive* if

$$F(x) + F(y) \subset F(x+y)$$

for all $x, y \in K$. A set-valued function $F : K \to n(K)$ is said to be $\mathbb{Q}_+$-*homogeneous* if

$$F(\lambda x) = \lambda F(x)$$

for all $x \in K$ and all positive rational numbers λ.

We will use the following six lemmas.

LEMMA 2 (see Lemma 3 in [10]). *Let X and Y be two topological vector spaces and let K be a closed convex cone in X. Assume that $F : K \to cc(K)$ is a continuous additive set-valued function and $A, B \in cc(K)$. If the difference $A - B$ exists, then $F(A) - F(B)$ exists and $F(A) - F(B) = F(A - B)$.*

LEMMA 3 (Theorem 3 in [12], see also Lemma 4 in [9]). *Let X and Y be two normed vector spaces and let K be a convex cone in X. Suppose that $\{F_i : i \in I\}$ is a family of superadditive lower semicontinuous and $\mathbb{Q}_+$-homogeneous set-valued functions $F_i : K \to n(Y)$. If K is of the second category in K and $\bigcup_{i \in I} F_i(x) \in b(Y)$ for $x \in K$, then there exists a positive constant M such that*

$$\|F_i(x)\| := \sup\{\|y\| : y \in F_i(x)\} \leq M\|x\|$$

for every $i \in I$ and $x \in K$.

COROLLARY 1. *If X, Y and K are as in Lemma 3, then the functional*

$$F \mapsto \|F\| := \sup\{\|F(x)\|/\|x\| : x \in K,\, x \neq 0\}$$

is finite for every $\mathbb{Q}_+$*-homogeneous superadditive lower semicontinuous set-valued function* $F: K \to b(Y)$.

Lemma 4 (Lemma 5 in [9]). *Let* X *and* Y *be two normed spaces and let* d *be the Hausdorff distance derived from the norm in* Y. *Suppose that* K *is a convex cone in* X *with nonempty interior. Then there exists a positive constant* M_0 *such that for every linear continuous set-valued function* $F: K \to c(Y)$ *the inequality*

$$d(F(x), F(y)) \leq M_0 \|F\| \, \|x - y\|$$

holds for every $x, y \in K$.

Lemma 5 (Theorem 2 in [5]). *Let* (X, ϱ_X) *and* (Y, ϱ_Y) *be two metric spaces and let* d_X *and* d_Y *be the corresponding Hausdorff metrics. If* $F: X \to n(Y)$ *is a set-valued function and* M *is a positive constant such that*

$$d_Y(F(x), F(y)) \leq M \varrho_X(x, y)$$

for any $x, y \in X$, *then*

$$d_Y(F(A), F(B)) \leq M d_X(A, B)$$

for any nonempty subsets A, B *of* X.

Lemma 6 (Lemma 4 in [8]). *Let* D *be a nonempty set and* Y *be a normed space. If* $F_0, F_n: D \to c(Y)$ *are set-valued functions and the sequence* (F_n) *uniformly converges to* F_0 *on* D, *then*

$$\lim_{n \to \infty} F_n(D) = F_0(D).$$

Lemma 7. *Let* X *be a Banach space*, Y *a normed space*, *and* K *a closed convex cone in* X *with nonempty interior. Suppose that* $F_0, F_n: K \to c(Y)$ *are continuous linear set-valued functions. If*

$$F_0(y) = \lim_{n \to \infty} F_n(y) \tag{1}$$

for $y \in K$, *then the sequence* (F_n) *uniformly converges to* F_0 *on every* $D \in c(K)$.

Proof. By (1) the set $\bigcup\{F_n(y) : n = 0, 1, 2, \ldots\}$ is bounded for every $y \in K$ and by Lemma 3 there exists a positive constant M such that

$$\|F_n\| \leq M$$

for $n = 0, 1, 2, \ldots$ According to Lemma 4 there exists a positive constant M_0 such that

$$d(F_n(x), F_n(y) \leq M_0 \|F_n\| \, \|x - y\| \leq M_0 M \|x - y\|$$

for $x, y \in K$ and $n = 0, 1, 2, \ldots$ This implies that

$$|d(F_n(x), F_0(x)) - d(F_n(y), F_0(y))| \\ \leq d(F_n(x), F_n(y)) + d(F_0(x), F_0(y)) \leq 2MM_0 \|x - y\|.$$

Consequently, the family $\{d((F_n(\cdot), F_0(\cdot)) : n = 0, 1, 2, \ldots\}$ is equicontinuous in K and by (1), F_0 is the uniform limit of (F_n) on every compact subset D of K (see Theorem 3.2.4 in [4]).

3. Now, we can prove our main results.

THEOREM 1. *Let X be a Banach space and let K be a closed convex cone in X with nonempty interior. Suppose that $\{F^t : t \geq 0\}$ is a differentiable iteration semigroup of linear continuous set-valued functions $F^t : K \to cc(K)$ with $F^0(x) = \{x\}$. Then the set-valued function $(t, x) \mapsto F^t(x)$ is continuous and*

$$D_t F^t(x) = F^t(G(x))$$

for $x \in K$, $t \geq 0$, where D_t denotes the Hukuhara derivative of $F^t(x)$ with respect to t and

$$G(x) := \lim_{s \to 0+} \frac{F^s(x) - x}{s} \tag{2}$$

for $x \in K$.

Proof. It is obvious that the differences $F^s(x) - x$ exist for $s > 0$ and $x \in K$, and hence, according to Lemma 2, so do the differences

$$F^{t+s}(x) - F^t(x) = F^t[F^s(x)] - F^t(x) = F^t(F^s(x) - x)$$

and

$$F^t(x) - F^{t-s}(x) = F^{t-s}[F^s(x)] - F^{t-s}(x) = F^{t-s}(F^s(x) - x)$$

whenever $t > 0$, $s \in (0, t)$ and $x \in K$.

By Lemmas 4 and 5,

$$\begin{aligned} d\left(\frac{F^{t+s}(x) - F^t(x)}{s}, F^t(G(x))\right) &= d\left(F^t\left(\frac{F^s(x) - x}{s}\right), F^t(G(x))\right) \\ &\leq M_0 \|F^t\| d\left(\frac{F^s(x) - x}{s}, G(x)\right) \end{aligned}$$

for $x \in K$, $t > 0$, $s \in (0, t)$, so (2) implies

$$\lim_{s \to 0+} \frac{F^{t+s}(x) - F^t(x)}{s} = F^t(G(x))$$

for $t > 0$ and $x \in K$.

Similarly, for $t > 0$, $s \in (0, t)$ and $x \in K$ we have

$$\begin{aligned} & d\left(\frac{F^t(x) - F^{t-s}(x)}{s}, F^t(G(x))\right) \\ &\qquad = d\left(F^{t-s}\left(\frac{F^s(x) - x}{s}\right), F^{t-s}(F^s(G(x)))\right) \\ &\qquad \leq M_0 \|F^{t-s}\| d\left(\frac{F^s(x) - x}{s}, F^s(G(x))\right). \end{aligned} \tag{3}$$

By Lemma 1 the function $s \mapsto F^s(y)$ is continuous for every $y \in K$. Therefore the image $\bigcup_{0\le s\le t} F^s(y)$ of the interval $[0,t]$ under this set-valued function is compact (see [1, p. 110]), whence it is bounded. According to Lemma 3 there exists a positive constant M such that

$$\|F^{t-s}(y)\| \le M\|y\|$$

for $0 \le s \le t$ and $y \in K$. Consequently,

$$\|F^{t-s}\| \le M$$

for $s \in [0,t]$. This inequality and (3) imply that

$$d\biggl(\frac{F^t(x) - F^{t-s}(x)}{s}, F^t(G(x))\biggr) \le M_0 M d\left(\frac{F^s(x) - x}{s}, G(x)\right) + M_0 M d(G(x), F^s(G(x))) \tag{4}$$

for $t > 0$, $s \in (0,t)$ and $x \in K$. By Lemmas 7 and 6 we have

$$\lim_{s\to 0+} F^s(G(x)) = G(x),$$

and by (2) and (4),

$$D_t F^t(x) = \lim_{s\to 0+} \frac{F^t(x) - F^{t-s}(x)}{s} = F^t(G(x)).$$

It remains to prove that the multifunction $(t,x) \mapsto F^t(x)$ is continuous. Fix $t \ge 0$, $x \in K$ and $y \in G(x)$. By Lemmas 3–5 there are two positive constants M_0 and M such that

$$\begin{aligned} d(F^{t+s}(z), F^t(y)) &\le d(F^{t+s}(z), F^{t+s}(y)) + d(F^{t+s}(y), F^t(y)) \\ &\le M_0\|F^t\|(M\|z-y\| + d(F^s(y), \{y\})) \end{aligned}$$

for every $s \in (0,1)$ and $z \in G(x)$. Therefore

$$\limsup_{(s,z)\to(0+,y)} d(F^{t+s}(z), F^t(y)) = 0. \tag{5}$$

Similarly, fix $t > 0$, $x \in K$ and $y \in G(x)$. There exist two positive constants M_0 and M_1 for which

$$\begin{aligned} d(F^{t-s}(z), F^t(y)) &\le d(F^{t-s}(z), F^{t-s}(y)) + d(F^{t-s}(y), F^t(y)) \\ &\le M_0 M_1(\|z-y\| + d(F^s(y), \{y\})), \end{aligned} \tag{6}$$

for every $s \in (0,t)$ and $z \in G(x)$. By (5) and (6) the set-valued function $(t,y) \mapsto F^t(y)$ is continuous.

Definition 1. Let K be a convex cone in a Banach space X and let $G, \Psi : K \to cc(K)$ be two continuous linear maps. A map $\Phi : [0,\infty) \times K \to cc(K)$ is said to be a *solution* of the problem

$$D_t\Phi(t,x) = \Phi(t, G(x)) := \bigcup\{\Phi(t,y) : y \in G(x)\}, \tag{7}$$

$$\Phi(0,x) = \Psi(x), \tag{8}$$

if Φ is continuous in $[0,\infty) \times K$ and differentiable with respect to t, and satisfies (7) and (8) everywhere in $[0,\infty) \times K$ and K, respectively.

LEMMA 8 (Theorem 2 in [11]). *Let K be a closed convex cone with nonempty interior in a Banach space and let $G, \Psi : K \to cc(K)$ be two continuous linear maps. Then there exists exactly one solution of problem* (7)–(8)*. This solution is linear with respect to the second variable.*

THEOREM 2. *Let X be a Banach space, let K be a closed convex cone in X with nonempty interior, and let $G : K \to cc(K)$ be a continuous linear set-valued function. Suppose that $\Phi : [0,\infty) \times K \to cc(K)$ is a solution of problem* (7)–(8) *with $\Psi(x) = \{x\}$ for $x \in K$, such that every set-valued function $x \mapsto \Phi(t,x)$, $t \geq 0$, is linear. Then the family $\{F^t : t \geq 0\}$, where $F^t(x) := \Phi(t,x)$ for $(t,x) \in [0,\infty) \times K$, is a differentiable iteration semigroup.*

Proof. Fix $t \geq 0$ and define

$$\alpha(s,x) := \Phi(s+t,x), \qquad \beta(s,x) := \Phi(t,\Phi(s,x))$$

for $x \in K$, $s \geq 0$. We see that

$$\alpha(0,x) = \Phi(t,x), \tag{9}$$

$$\beta(0,x) = \Phi(t,\Phi(0,x)) = \Phi(t,x) \tag{10}$$

for $x \in K$. Now, we have

$$\lim_{u\to s+} \frac{\alpha(u,x) - \alpha(s,x)}{u-s} = \lim_{u\to s+} \frac{\Phi(u+t,x) - \Phi(s+t,x)}{u-s} = \Phi(s+t,G(x))$$
$$= \bigcup\{\Phi(s+t,y) : y \in G(x)\} = \bigcup\{\alpha(s,y) : y \in G(x)\} = \alpha(s,G(x))$$

for $x \in K$, $s \geq 0$, and

$$\lim_{u\to s-} \frac{\alpha(s,x) - \alpha(u,x)}{s-u} = \lim_{u\to s-} \frac{\Phi(s+t,x) - \Phi(u+t,x)}{s-u}$$
$$= \Phi(s+t,G(x)) = \alpha(s,G(x))$$

for $x \in K$, $s > 0$. Thus

$$D_s\alpha(s,x) = \alpha(s,G(x)) \tag{11}$$

for $x \in K$, $s \geq 0$. Further, by Lemma 2 we have

$$\frac{\beta(u,x) - \beta(s,x)}{u-s} = \frac{\Phi(t,\Phi(u,x)) - \Phi(t,\Phi(s,x))}{u-s}$$
$$= \Phi\left(t, \frac{\Phi(u,x) - \Phi(s,x)}{u-s}\right)$$

for $u > s \geq 0$, $x \in K$, and according to Lemma 4,

$$d\left(\frac{\beta(u,x)-\beta(s,x)}{u-s}, \Phi(t, D_s\Phi(s,x))\right) = d\left(\Phi\left(t, \frac{\Phi(u,x)-\Phi(s,x)}{u-s}\right), \Phi(t, D_s\Phi(s,x))\right) \leq M_0\|\Phi(t,\cdot)\| d\left(\frac{\Phi(u,x)-\Phi(s,x)}{u-s}, D_s\Phi(s,x)\right).$$

Thus

$$\lim_{u\to s+} \frac{\beta(u,x)-\beta(s,x)}{u-s} = \Phi(t, D_s\Phi(s,x)) = \Phi(t, \Phi(s, G(x))) = \Phi(t, \bigcup\{\Phi(s,y) : y \in G(x)\}) = \bigcup\{\Phi(t,\Phi(s,y)) : y \in G(x)\} = \bigcup\{\beta(s,y) : y \in G(x)\} = \beta(s, G(x)).$$

Similarly for $s > u \geq 0$, $x \in K$, we have

$$\frac{\beta(s,x)-\beta(u,x)}{s-u} = \frac{\Phi(t,\Phi(s,x)) - \Phi(t,\Phi(u,x))}{s-u} = \Phi\left(t, \frac{\Phi(s,x)-\Phi(u,x)}{s-u}\right).$$

and

$$d\left(\frac{\beta(s,x)-\beta(u,x)}{s-u}, \Phi(t, D_s\Phi(s,x))\right) = d\left(\Phi\left(t, \frac{\Phi(s,x)-\Phi(u,x)}{s-u}\right), \Phi(t, D_s\Phi(s,x))\right) \leq M_0\|\Phi(t,\cdot)\| d\left(\frac{\Phi(s,x)-\Phi(u,x)}{s-u}, D_s\Phi(s,x)\right).$$

Thus

$$\lim_{u\to s-} \frac{\beta(s,x)-\beta(u,x)}{s-u} = \Phi(t, D_s\Phi(s,x)) = \Phi(t, \Phi(s, G(x))) = \Phi(t, \bigcup\{\Phi(s,y) : y \in G(x)\}) = \bigcup\{\Phi(t,\Phi(s,y)) : y \in G(x)\} = \bigcup\{\beta(s,y) : y \in G(x)\} = \beta(s, G(x)).$$

Therefore

$$D_s\beta(s,x) = \beta(s, G(x)). \tag{12}$$

Equalities (9)–(12) mean that α and β are solutions of problem (7)–(8) with $\Psi(x) = \Phi(t,x)$. By Lemma 8 a solution of (7)–(8) is unique. Consequently, $\beta = \alpha$, which means that

$$F^t(F^s(x)) = \Phi(t, \Phi(s,x)) = \Phi(t+s, x) = F^{t+s}(x).$$

This completes the proof.

4. Now, we give some applications.

COROLLARY 2. *Let K be a closed convex cone with nonempty interior in a Banach space and let $\{F^t : t \geq 0\}$ be a concave iteration semigroup of continuous linear set-valued functions $F^t : K \to cc(K)$ with $F^0(x) = \{x\}$ for $x \in K$. Then the set-valued function $\Phi : [0, \infty) \times K \to cc(K)$, $\Phi(t, x) = F^t(x)$, is a solution of problem* (7)–(8) *with $\Psi(x) = \{x\}$ for $x \in K$, where G is given by* (2), *and the set-valued functions $t \mapsto \Phi(t, G(x))$, $x \in K$, are increasing.*

Proof. By the Theorem in [10] the iteration semigroup $\{F^t : t \geq 0\}$ is differentiable and the set-valued function $\Phi(t, x) := F^t(x)$ is a solution of problem (7)–(8) with $\Psi(x) = \{x\}$, where G is given by (2). Since the set-valued functions $t \mapsto \Phi(t, x)$, $x \in K$, are concave, Theorem 3.1 in [6] implies that the set-valued functions $t \mapsto D_t\Phi(t, x)$ are increasing. Thus the set-valued functions $t \mapsto F^t(G(x))$, $x \in K$, are also increasing.

COROLLARY 3. *Let K be a closed convex cone with nonempty interior in a Banach space and let $G : K \to cc(K)$ be a continuous linear set-valued function. If $\Phi : [0, \infty) \times K \to cc(K)$ is a solution of problem* (7)–(8) *with $\Psi(x) = \{x\}$ for $x \in K$, such that the set-valued functions $x \mapsto \Phi(t, x), t \geq 0$, are linear and the functions $t \mapsto \Phi(t, G(x))$, $x \in K$, are increasing, then the family $\{F^t : t \geq 0\}$, where $F^t(x) := \Phi(t, x)$, is a concave iteration semigroup of continuous linear set-valued functions.*

Proof. By Theorem 2 the family $\{F^t : t \geq 0\}$ is a differentiable iteration semigroup. According to the Proposition in [11],

$$F^t(x) = \Phi(t, x) = x + \int_0^t \Phi(s, G(x))\, ds$$

for $x \in K$, $t \geq 0$, where $\int_0^t$ denotes a Riemann-type integral. Since the set-valued functions $s \mapsto \Phi(s, G(x))$ are increasing, Corollary 4.4 in [6] implies that the set-valued functions $t \mapsto F^t(x)$, $x \in K$, are concave. That means that the iteration semigroup $\{F^t : t \geq 0\}$ is concave.

COROLLARY 4. *Let K be a closed convex cone with nonempty interior in a Banach space. Suppose that $\{F^t : t \geq 0\}$ is a differentiable iteration semigroup of continuous linear set-valued functions $F^t : K \to cc(K)$ with $F^0(x) = \{x\}$ for $x \in K$. Then this semigroup is increasing if and only if $0 \in G(x)$ for $x \in K$.*

Proof. Suppose that $\{F^t : t \geq 0\}$ is increasing. Then $x \in F^t(x)$ for $x \in K$ and

$$0 \in G(x) = \lim_{t \to 0+} \frac{F^t(x) - x}{t} \tag{13}$$

for $x \in K$, $t \geq 0$.

To prove the converse, note that by Theorem 1 the set-valued function $\Phi(t,x) := F^t(x)$ is a solution of problem (7)–(8) with $\Psi(x) = \{x\}$ and by the Proposition in [11],

$$F^t(x) = x + \int_0^t F^s(G(x))\,ds. \tag{14}$$

If (13) holds then by (14), $x \in F^t(x)$ for $x \in K, t \geq 0$, so this semigroup is increasing.

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Reçu par la Rédaction le 21.3.2002
Révisé le 24.11.2003 (1325)