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Convolution theorems for starlike and convex functions in the unit disc

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Abstract. Let $\mathcal{A}$ denote the space of all analytic functions in the unit disc Δ with the normalization $f(0) = f'(0) - 1 = 0$. For $\beta < 1$, let

$$\mathcal{P}_\beta^0 = \{f \in \mathcal{A} : \operatorname{Re} f'(z) > \beta,\ z \in \Delta\}.$$

For $\lambda > 0$, suppose that $\mathcal{F}$ denotes any one of the following classes of functions:

$$M_{1,\lambda}^{(1)} = \{f \in \mathcal{A} : \operatorname{Re}\{z(zf'(z))''\} > -\lambda,\ z \in \Delta\},$$
$$M_{1,\lambda}^{(2)} = \{f \in \mathcal{A} : \operatorname{Re}\{z(z^2 f''(z))''\} > -\lambda,\ z \in \Delta\},$$
$$M_{1,\lambda}^{(3)} = \{f \in \mathcal{A} : \operatorname{Re}\{\tfrac{1}{2}(z(z^2 f'(z))'')' - 1\} > -\lambda,\ z \in \Delta\}.$$

The main purpose of this paper is to find conditions on λ and γ so that each $f \in \mathcal{F}$ is in $\mathcal{S}_\gamma$ or $\mathcal{K}_\gamma$, $\gamma \in [0, 1/2]$. Here $\mathcal{S}_\gamma$ and $\mathcal{K}_\gamma$ respectively denote the class of all starlike functions of order γ and the class of all convex functions of order γ. As a consequence, we obtain a number of convolution theorems, namely the inclusions $M_{1,\alpha} * \mathcal{G} \subset \mathcal{S}_\gamma$ and $M_{1,\alpha} * \mathcal{G} \subset \mathcal{K}_\gamma$, where $\mathcal{G}$ is either $\mathcal{P}_\beta^0$ or $M_{1,\beta}$. Here $M_{1,\lambda}$ denotes the class of all functions f in $\mathcal{A}$ such that $\operatorname{Re}(zf''(z)) > -\lambda$ for $z \in \Delta$.

1. Introduction and useful lemmas. Let Δ be the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}$ be the space of all analytic functions in Δ, with the topology of local uniform convergence. A function $f \in \mathcal{H}$ is said to be in $\mathcal{A}$ if it satisfies the normalization conditions $f(0) = f'(0) - 1 = 0$. Let $\mathcal{K}_\gamma$ and $\mathcal{S}_\gamma$ denote respectively the well known sets of all functions f in $\mathcal{A}$ that are convex and starlike (with respect to origin) of order γ, $\gamma < 1$. We are

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interested in the following sets:

$$\begin{aligned}
\mathcal{P}_\beta &= \{f \in \mathcal{A} :\ \text{there exists an } \eta \in \mathbb{R} \text{ such that} \\
&\qquad\qquad\qquad \operatorname{Re} e^{i\eta}(f'(z) - \beta) > 0,\ z \in \Delta\}, \\
\mathcal{P}^0_\beta &= \{f \in \mathcal{A} : \operatorname{Re} f'(z) > \beta,\ z \in \Delta\}, \\
P_\beta &= \{p \in \mathcal{H} : p(0) = 1 \text{ and } \operatorname{Re} p(z) > \beta,\ z \in \Delta\}, \\
M_{1,\lambda} &= \{f \in \mathcal{A} : \operatorname{Re}(zf''(z)) > -\lambda,\ z \in \Delta\}, \\
M^{(1)}_{1,\lambda} &= \{f \in \mathcal{A} : \operatorname{Re}\{z(zf'(z))''\} > -\lambda,\ z \in \Delta\}, \\
M^{(2)}_{1,\lambda} &= \{f \in \mathcal{A} : \operatorname{Re}\{z(z^2 f''(z))''\} > -\lambda,\ z \in \Delta\}, \\
M^{(3)}_{1,\lambda} &= \left\{f \in \mathcal{A} : \operatorname{Re}\left\{\frac{1}{2}(z(z^2 f'(z))'')' - 1\right\} > -\lambda,\ z \in \Delta\right\}.
\end{aligned}$$

For functions $f(z) = \sum_{k=0}^\infty a_k z^k$ and $g(z) = \sum_{k=0}^\infty b_k z^k$ in $\mathcal{H}$ their *Hadamard product* (*convolution*) $f * g$ is defined by $(f * g)(z) = \sum_{k=0}^\infty a_k b_k z^k$. Note that $f * g$ is in $\mathcal{H}$. This definition is extended to subsets $\mathcal{M}$ and $\mathcal{N}$ by setting

$$\mathcal{M} * \mathcal{N} = \{f * g :\ f \in \mathcal{M},\ g \in \mathcal{N}\}.$$

THEOREM 1.1. *We have the following inclusions*:

(i) $M^{(1)}_{1,\lambda} \subset \mathcal{S}^*$ *for* $\lambda = 1/2(\log 4 - 1)$,
(ii) $M^{(1)}_{1,\lambda} \subset \mathcal{K}$ *for* $\lambda = 1/\log 4$,
(iii) $M^{(2)}_{1,\lambda} \subset \mathcal{S}^*$ *for* $\lambda = 1/[2 - 4\log 2 + \pi^2/6]$,
(iv) $M^{(2)}_{1,\lambda} \subset \mathcal{K}$ *for* $\lambda = 1/(\log 4)^2$,
(v) $M_{1,\lambda} * M_{1,\lambda} \subset \mathcal{S}^*$ *for* $2\lambda^2 = 1/[2 - 4\log 2 + \pi^2/6]$,
(vi) $M_{1,\lambda} * M_{1,\lambda} \subset \mathcal{K}$ *for* $2\lambda^2 = 1/(\log 4)^2$,
(vii) $M_{1,\lambda} * \mathcal{P}^0_0 \subset \mathcal{S}^*$ *for* $2\lambda = 1/2(\log 4 - 1)$,
(viii) $M_{1,\lambda} * \mathcal{P}^0_0 \subset \mathcal{K}$ *for* $2\lambda = 1/2\log 4$.

Cases (i) and (ii) of Theorem 1.1 are Theorem 2 in [8], whereas (iii) and (iv) are Theorem 1 in [8]. Similarly (v) and (vi) are Theorem 3 in [8] and (vii) and (viii) are Theorem 4 in [8].

The main aim of this paper is to obtain more general theorems which, in particular, yield all the results of Theorem 1.1 as special cases. We also obtain some additional results concerning the class $M^{(3)}_{1,\lambda}$. To prove these results we shall use a number of lemmas that have been obtained recently with the help of duality theory for convolutions. For results in duality theory, we refer to the book by Ruscheweyh [6].

LEMMA 1.2 ([2, Theorem 2.1]). *Let* $\alpha : [0,1] \to \mathbb{R}$ *be nonnegative, integrable with* $\int_0^1 \alpha(t)\,dt = 1$ *and suppose that* $\Lambda(t) = \int_t^1 \alpha(s)s^{-2}\,ds$ *satisfies the following conditions*:

(i) Λ *is not integrable on* $[0,1]$, $t\Lambda(t)$ *is integrable on* $[0,1]$, *and positive on* $(0,1)$.

(ii) *For* $0 \le \gamma \le 1/2$,

$$\frac{\Lambda(t)}{(1+t)(1-t)^{1+2\gamma}}$$

is decreasing on $(0,1)$.

For $\beta > 0$, *define*

$$F(z) = z\int_0^1 \left(1 + \frac{z\beta}{1-tz}\right)\alpha(t)\,dt * \phi(z), \tag{1.3}$$

where $\phi \in \mathcal{P}_0$. *Then* $F \in \mathcal{S}_\gamma$ *for all* $\gamma \in [0,1/2]$ *and for* β *given by*

$$\beta\int_0^1 \left(1 - \frac{1}{(1-\gamma)(1+t)} + \frac{\gamma}{1-\gamma}\,\frac{\log(1+t)}{t}\right)\frac{\alpha(t)}{t}\,dt = \frac{1}{2}. \tag{1.4}$$

A combination of Theorem 2.13 and Lemma 1.4 of [2] gives the following result.

LEMMA 1.5. *Let* $\alpha : [0,1] \to \mathbb{R}$ *be nonnegative with* $\int_0^1 \alpha(t)\,dt = 1$ *and suppose that* $\Lambda(t) = \alpha(t)/t$ *satisfies the following conditions*:

(i) Λ *is not integrable on* $[0,1]$, $t\Lambda(t)$ *is integrable on* $[0,1]$, *and positive on* $(0,1)$.

(ii) *For* $0 \le \gamma \le 1/2$,

$$\frac{\Lambda(t)}{(1+t)(1-t)^{1+2\gamma}}$$

is decreasing on $(0,1)$.

For $\beta > 0$, *let* F *be defined by* (1.3). *Then* $F \in \mathcal{K}_\gamma$ *for all* $\gamma \in [0,1/2]$ *and for* β *given by*

$$\beta\int_0^1 \left(1 - \frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2}\right)\frac{\alpha(t)}{t}\,dt = \frac{1}{2}.$$

Earlier results which led to results such as Lemmas 1.2 and 1.5 can be found in Fournier and Ruscheweyh [4] and Ponnusamy and Rønning [5].

LEMMA 1.6 ([2, Theorem 2.10]). *If* $\beta > 0$ *and* $F \in M_{1,\beta}$ *then* $F \in \mathcal{S}_\gamma$ $(0 \le \gamma \le 1/2)$ *whenever*

$$0 \le \beta \le \frac{1-2\gamma}{2\gamma + \log 4}.$$

The case $\gamma = 0$ is due to Ali, Ponnusamy and Singh [1].

Lemma 1.7 ([7]). *If $f, g \in \mathcal{H}$ and $F, G \in \mathcal{K}$ are such that $f \prec F$ and $g \prec G$, then $f * g \prec F * G$. Here $\mathcal{K}$ denotes the family of convex functions (not necessarily normalized) in Δ.*

2. Starlikeness condition for functions in $M_{1,\lambda}^{(2)}$, $M_{1,\lambda}^{(1)}$ and $M_{1,\lambda}^{(3)}$

Theorem 2.1. *Let $\gamma \in [0, 1/2]$ and let $\lambda > 0$ be given by*

$$\lambda = \frac{1-\gamma}{2 - 2\log 4 + \pi^2/6 + (4\log 4 - 6)\gamma}. \tag{2.2}$$

Then $M_{1,\lambda}^{(2)} \subset \mathcal{S}_\gamma$.

Proof. Suppose that $f \in M_{1,\lambda}^{(2)}$. Then

$$z(z^2 f''(z))'' \prec \frac{2\lambda z}{1-z}.$$

We consider a more general differential equation

$$z\{z(zf'(z))''\}' = z(z^2 f''(z))'' = \lambda(\phi'(z) - 1), \qquad \phi \in \mathcal{P}_0. \tag{2.3}$$

By comparing the coefficients of z^n on both sides of (2.3) and by a simple calculation, it is easily seen that

$$f'(z) = \left(1 + \frac{\lambda}{2} z \int_0^1 \frac{1}{1-tz}\, \alpha(t)\, dt\right) * \phi'(z)$$

so that

$$f(z) = z \int_0^1 \left(1 + \frac{\lambda z}{2(1-tz)}\right) \alpha(t)\, dt * \phi(z),$$

where $\alpha(t) = 2[-1 + t - \log t]$. In order to apply Lemma 1.2 with $\gamma \in [0, 1/2]$, we need to verify the relevant conditions of Lemma 1.2 with $\beta = \lambda/2$ and $\alpha(t)$ as above. Clearly, $\int_0^1 \alpha(t)\, dt = 1$. If we define

$$\Lambda(t) = \int_t^1 \frac{\alpha(s)}{s^2}\, ds,$$

then it is easy to see that

$$\Lambda(t) = 2[2 - 2t^{-1} - (t+1)t^{-1} \log t].$$

We observe that Λ is not integrable on $[0, 1]$ whereas $t\Lambda(t)$ is integrable on $[0, 1]$, and positive on $(0, 1)$. Finally, we show that g defined by

$$g(t) = \frac{\Lambda(t)}{(1+t)(1-t)^{1+2\gamma}}$$

is decreasing on $(0,1)$. To do this we use the idea from [3, Theorem 3.12]. Taking the logarithmic derivative of $g(t)$ and using the fact that

$$\Lambda'(t) = -\frac{\alpha(t)}{t^2},$$

we have

$$g'(t) = \left(-\frac{\alpha(t)}{t^2\Lambda(t)} + \frac{2(\gamma+(1+\gamma)t)}{1-t^2}\right)g(t).$$

From this equation, we observe that $g'(t) \le 0$ for $t \in (0,1)$ is equivalent to the inequality

$$\psi(t) = 2\Lambda(t) - \frac{(1-t^2)\alpha(t)t^{-2}}{\gamma+(1+\gamma)t} \le 0.$$

Clearly $\psi(1) = 0$, and so it suffices to show that ψ is increasing on $(0,1)$. A simple calculation shows that

$$\psi'(t) = \frac{(1-t^2)t^{-2}}{\gamma+(1+\gamma)t}\left[\alpha(t)\left(-\frac{2\gamma}{1-t} + \frac{2}{t} + \frac{1+\gamma}{\gamma+(1+\gamma)t}\right) - \alpha'(t)\right].$$

Now, ψ is increasing on $(0,1)$ if and only if

$$\alpha(t)\left[\frac{-2\gamma t + 2(1-t)}{t(1-t)} + \frac{1+\gamma}{\gamma+(1+\gamma)t}\right] \ge \alpha'(t), \tag{2.4}$$

which is equivalent to

$$\alpha(t)\left[\frac{-t+2(1-t)}{t(1-t)} + \frac{3}{1+3t} + \left(\frac{1}{1-t} + \frac{1}{(\gamma+(1+\gamma)t)(1+3t)}\right)(1-2\gamma)\right] \ge \alpha'(t).$$

In view of this observation, ψ is increasing on $(0,1)$ if the last inequality holds for $\gamma = 1/2$. Thus, we need to verify the inequality

$$\alpha(t)\left[\frac{-t+2(1-t)}{t(1-t)} + \frac{3}{1+3t}\right] \ge \alpha'(t). \tag{2.5}$$

Substituting the value of $\alpha(t)$, we have

$$[-(1-t) - \log(1-(1-t))]\left[-1 + \frac{2(1-t)}{t} + \frac{3(1-t)}{1+3t}\right] \ge -\frac{(1-t)^2}{t} = -\sum_{k=0}^{\infty}(1-t)^{k+2}.$$

After some simplification this is seen to be equivalent to the inequality

$$\sum_{k=0}^{\infty}\frac{(1-t)^k}{k+2}\left(\frac{-12t^2+6t+2}{t(1+3t)} + k + 2\right) = \sum_{k=0}^{\infty}\frac{(1-t)^k}{k+2}\left(\frac{2[3t(1-t)+(1+t)]}{t(1+3t)} + k\right) \ge 0,$$

which is clearly true for all $t \in (0,1)$. Therefore, g is decreasing on $(0,1)$ and by Lemma 1.2 it follows that $M_{1,\lambda}^{(2)} \subset \mathcal{S}_\gamma$ for all $\gamma \in [0,1/2]$ and λ given by $\lambda I = 1$, where

$$I = \int_0^1 \left\{ 1 - \frac{1}{(1-\gamma)(1+t)} + \frac{\gamma \log(1+t)}{(1-\gamma)t} \right\} \frac{\alpha(t)}{t}\, dt.$$

Thus, to complete the proof we need to compute this integral. To do this we rewrite it as

$$I = \int_0^1 \left\{ \frac{t}{1+t} - \frac{\gamma}{1-\gamma}\left(\frac{1}{1+t} - \frac{\log(1+t)}{t} \right) \right\} \frac{\alpha(t)}{t}\, dt,$$

which, by using the series expansion, can be written as

$$I = \sum_{n=0}^{\infty} (-1)^n \left[\int_0^1 \left(t^{n+1} - \frac{\gamma}{1-\gamma} \frac{n}{n+1} t^n \right) \frac{\alpha(t)}{t}\, dt \right]. \tag{2.6}$$

For each $k \in \mathbb{N} \cup \{0\}$, it can be easily seen that

$$-\int_0^1 t^k \log t \, dt = \frac{1}{(1+k)^2}.$$

If we substitute $\alpha(t) = 2[-1+t-\log t]$ in (2.6), a simple computation yields

$$\begin{aligned} I = 1 + 2\Bigg[&\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n+2} - \frac{1}{n+1} \right) + \sum_{n=1}^{\infty} (-1)^n \frac{1}{(n+1)^2} \\ &- \frac{\gamma}{1-\gamma} \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2} \right) \Bigg]. \end{aligned}$$

Using the fact that

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = \frac{\pi^2}{12} \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \log 2,$$

we find that

$$I = 2 - 2\log 4 + \frac{\pi^2}{6} - \frac{\gamma}{1-\gamma}\left(4 - 4\log 2 - \frac{\pi^2}{6} \right)$$

and therefore, $\lambda I = 1$ gives the value of λ given by (2.2). ∎

THEOREM 2.7. *Let $\gamma \in [0,1/2]$ and let $\lambda > 0$ be given by*

$$\lambda = \frac{1-\gamma}{2(\log 4 - 1) + \gamma(4 - 4\log 2 - \pi^2/6)}. \tag{2.8}$$

Then $M_{1,\lambda}^{(1)} \subset \mathcal{S}_\gamma$.

Proof. Let $f \in M_{1,\lambda}^{(1)}$. As in the proof of Theorem 2.1, we consider a general differential equation

$$z(zf'(z))'' = \lambda(\phi'(z) - 1), \quad \phi \in \mathcal{P}_0. \tag{2.9}$$

It follows that

$$f'(z) = 1 + \frac{\lambda}{2}\, z \int_0^1 \frac{1}{1-tz}\, \alpha(t)\, dt * \phi'(z)$$

and

$$f(z) = z \int_0^1 \left(1 + \frac{\lambda z}{2(1-tz)}\right) \alpha(t)\, dt * \phi(z)$$

with $\alpha(t) = 2(1-t)$. Because of the similarity with Theorem 2.1, it suffices to verify (2.5) for $\alpha(t) = 2(1-t)$. Thus, if we substitute $\alpha(t) = 2(1-t)$ in (2.5), we find that the inequality (2.5) is equivalent to

$$2(1-t) + \frac{3t(1-t)}{1+3t} \geq 0,$$

which is clearly true for each $t \in (0,1)$. Thus, we have the desired inclusion $M_{1,\lambda}^{(1)} \subset \mathcal{S}_\gamma$, where $0 \leq \gamma \leq 1/2$ and λ is given by (1.4) with $\alpha(t) = 2(1-t)$. Again, substituting $\alpha(t) = 2(1-t)$ in (1.4), we can obtain the desired λ given by (2.8). ∎

THEOREM 2.10. *Let $\gamma \in [0, 1/2]$ and let $\lambda > 0$ be given by*

$$\lambda = \frac{1-\gamma}{2(4\log 2 - 1 - \pi^2/6) - \gamma(10 + 3\zeta(3) - 16\log 2 - \pi^2/3)}, \tag{2.11}$$

where $\zeta(3) = 1.202056903\ldots$ denotes the Riemann zeta value at 3. Then $M_{1,\lambda}^{(3)} \subset \mathcal{S}_\gamma$.

Proof. Let $f \in M_{1,\lambda}^{(3)}$. Then

$$\operatorname{Re}\left\{\frac{1}{2}\,(z(z^2 f'(z))'')' - 1\right\} > -\lambda, \qquad z \in \Delta,$$

so that

$$\frac{1}{2}\,(z(z^2 f'(z))'')' - 1 \prec \frac{2\lambda z}{1-z}, \qquad z \in \Delta.$$

We consider the more general differential equation

$$\frac{1}{2}\,\{z(zf'(z))''\}' - 1 = \lambda(\phi'(z) - 1), \qquad \phi \in \mathcal{P}_0.$$

By comparison of the coefficients of the powers of z, it can be easily seen that

$$f'(z) = 1 + \frac{\lambda}{6}\, z \int_0^1 \frac{1}{1-tz}\, \alpha(t)\, dt * \phi'(z)$$

and therefore,

$$f(z) = z\int_0^1 \left(1 + \frac{\lambda z}{6(1-tz)}\right)\alpha(t)\,dt * \phi(z),$$

where $\alpha(t) = 12t(-1+t-\log t)$. We observe that f has the form (1.3) with $\beta = \lambda/6$. Further, if $\Lambda(t) = \int_t^1 (\alpha(s)/s^2)\,ds$, then a simple calculation implies that

$$\Lambda(t) = 6[2(1-t) + \log(t(2+\log t))].$$

Thus, as in the proof of Theorem 2.1, it suffices to verify (2.5) for

$$\alpha(t) = 12t(-1+t-\log t).$$

Now substituting $\alpha(t) = 12t(-1+t-\log t)$ in (2.5) yields

$$[-(1-t)-\log(1-(1-t)]\left[\frac{-t+2(1-t)}{1-t} + \frac{3t}{1+3t}\right] \geq -2(1-t)-\log(1-(1-t)),$$

that is,

$$[-(1-t)-\log(1-(1-t)]\left[-\frac{t}{1-t} + \left(\frac{3t}{1+3t}+1\right)\right] \geq -(1-t).$$

Using the series expansion for $-\log(1-(1-t))$ and deleting the common term $1-t$ on both sides and then dividing the resulting inequality by t implies that

$$\sum_{k=2}^{\infty}\frac{(1-t)^{k-1}}{k}\left[-\frac{1}{1-t} + \frac{1}{t}\left(\frac{1+6t}{1+3t}\right)\right] \geq -\frac{1}{1-(1-t)} = -\sum_{k=2}^{\infty}(1-t)^{k-2}$$

or

$$\sum_{k=2}^{\infty}\frac{(1-t)^{k-2}}{k}\left[\frac{1-t}{t}\left(\frac{1+6t}{1+3t}\right) + k - 1\right] \geq 0$$

which is clearly true for $t \in (0,1)$. By Lemma 1.2, it follows that $f \in \mathcal{S}_\gamma$ for all $\gamma \in [0,1/2]$ and λ given by $\lambda I = 3$, where (as in the proof of Theorem 2.1)

$$I = \sum_{n=0}^{\infty}(-1)^n\left[\int_0^1 \left(t^{n+1} - \frac{\gamma}{1-\gamma}\frac{n}{n+1}\,t^n\right)\right]\frac{\alpha(t)}{t}dt.$$

To compute I, we substitute $\alpha(t) = 12t(-1+t-\log t)$ to obtain

$$\begin{aligned} I = 1 + 12\Bigg[&\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n+3} - \frac{1}{n+2}\right) + \sum_{n=1}^{\infty}(-1)^n\frac{1}{(n+2)^2} \\ &- \frac{\gamma}{1-\gamma}\sum_{n=1}^{\infty}(-1)^n\Bigg\{\left(\frac{2}{n+2} - \frac{1}{n+1}\right) + \left(\frac{1}{(n+1)^2} - \frac{1}{(n+1)^3}\right) \\ &- \left(\frac{1}{n+1} - \frac{1}{(n+1)^2}\right)\Bigg\}\Bigg]. \end{aligned}$$

Using the fact that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}=\frac{\pi^2}{12},$$
$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=\log 2,$$
$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3}=\frac{3}{4}\zeta(3)=0.90154\ldots,$$

we find that

$$I=\frac{(24\log 2-6-\pi^2)+\gamma(-48\log 2-\pi^2+30+9\zeta(3))}{1-\gamma}$$

and therefore, $\lambda I=3$ yields the value of λ given by (2.11). ■

3. Convexity condition for functions in $M_{1,\lambda}^{(2)}$, $M_{1,\lambda}^{(1)}$ and $M_{1,\lambda}^{(3)}$

THEOREM 3.1. *Let $\gamma\in[0,1/2]$ and let $\lambda>0$ be given by*

$$\lambda=\frac{1-2\gamma}{(\log 4)^2+2\gamma\log 4}. \tag{3.2}$$

Then $M_{1,\lambda}^{(2)}\subset\mathcal{K}_\gamma$.

Proof. Let $f\in M_{1,\lambda}^{(2)}$. Then

$$1+\frac{1}{\lambda}z[z^2f''(z)]''\prec\frac{1+z}{1-z},\quad z\in\Delta.$$

By considering a more general differential equation

$$z[z^2f''(z)]''=z\{z(zf'(z))''\}'=\lambda(p(z)-1),\quad p\in P_0, \tag{3.3}$$

it can be easily seen that

$$z(zf'(z))''=\lambda\int_0^z\frac{p(t)-1}{t}\,dt=\lambda\int_0^1\frac{p(uz)-1}{u}\,du.$$

Since $p\in P_0$ implies that

$$\frac{1+|z|}{1-|z|}\ge\operatorname{Re}p(z)\ge\frac{1-|z|}{1+|z|},$$

we have

$$\operatorname{Re}(p(uz)-1)\ge\frac{1-|uz|}{1+|uz|}-1>-\frac{2u}{1+u}$$

and therefore,

$$\operatorname{Re}\{z(zf'(z))''\}>-\lambda\int_0^1\frac{2}{1+u}du=-2\lambda\log 2.$$

By Lemma 1.6, $zf'(z)$ is in $\mathcal{S}_\gamma$ whenever

$$2\lambda \log 2 \le \frac{1-2\gamma}{2\gamma + \log 4}$$

and the desired conclusion follows. ■

From Lemma 1.6, we can easily obtain the following result.

THEOREM 3.4. *Let $\gamma \in [0,1/2]$ and let $\lambda > 0$ be given by*

$$\lambda = \frac{1-2\gamma}{\log 4 + 2\gamma}.$$

Then $M_{1,\lambda}^{(1)} \subset \mathcal{K}_\gamma$.

THEOREM 3.5. *Let $\gamma \in [0,1/2]$ and let $\lambda > 0$ be given by*

$$\lambda = \frac{1-\gamma}{2(3-\log 16) + \gamma(2 - 2\log 16 + \pi^2/3)}. \tag{3.6}$$

Then $M_{1,\lambda}^{(3)} \subset \mathcal{K}_\gamma$.

Proof. Let $f \in M_{1,\lambda}^{(3)}$. Then as in the proof of Theorem 2.10, we notice that f has the form (1.3) with $\beta = \lambda/6$ and $\alpha(t) = 12t(-1+t-\log t)$. We set

$$\Lambda(t) = \frac{\alpha(t)}{t}$$

and apply Lemma 1.5 to complete the proof. First we observe that $\Lambda(t)$ satisfies condition (i) of Lemma 1.5. To verify (ii), we let

$$g(t) = \frac{\Lambda(t)}{(1+t)(1-t)^{1+2\gamma}}$$

and show that g is decreasing on $(0,1)$. By taking the logarithmic derivative of $g(t)$, it can be easily seen that $g'(t) \le 0$ on $(0,1)$ iff

$$\psi(t) = 2\Lambda(t) + \frac{(1-t^2)\Lambda'(t)}{\gamma + (1+\gamma)t} \le 0, \qquad t \in (0,1).$$

As $\psi(1) = 0$, it suffices to show that ψ is increasing on $(0,1)$. As usual it suffices to show this when $\gamma = 1/2$. Putting $\gamma = 1/2$, we see that

$$\psi(t) = 2\left[\Lambda(t) + \frac{1-t^2}{1+3t}\,\Lambda'(t)\right]$$

so that

$$\psi'(t) = 2\left[\Lambda'(t)\left(1 - \frac{3t^2+2t+3}{(1+3t)^2}\right) + \frac{1-t^2}{1+3t}\,\Lambda''(t)\right].$$

As

$$\Lambda'(t) = \frac{t\alpha'(t) - \alpha(t)}{t^2} = -\frac{12(1-t)}{t} \quad \text{and} \quad \Lambda''(t) = \frac{\alpha''(t)}{t} - \frac{2}{t}\Lambda'(t),$$

it follows easily that

$$\psi'(t) = \frac{2(1-t^2)}{(1+3t)t^2}\left[t\alpha''(t) - 2t\Lambda'(t) + 2t^2\Lambda'(t)\,\frac{3t^2+2t-1}{(1+3t)(1-t^2)}\right].$$

By using the expression for α'' and $\Lambda'(t)$, it can be quickly found that

$$\psi'(t) = \frac{24(1-t^2)(1-t)(1+6t)}{(1+3t)^2t^2}.$$

Therefore, ψ is increasing on $(0,1)$, which means that g is decreasing on $(0,1)$. By Lemma 1.5, $M_{1,\lambda}^{(3)} \subset \mathcal{K}_\gamma$ for all $\gamma \in [0,1/2]$ and λ given by $\lambda I = 3$, where

$$I = \int_0^1 \left(1 - \frac{1-\gamma(1+t)}{(1-\gamma)(1+t)^2}\right)\frac{\alpha(t)}{t}\,dt, \qquad \alpha(t) = 12t(-1+t-\log t).$$

Thus, to complete the proof we need to compute this integral. It is a simple exercise to see that

$$I = \sum_{n=0}^{\infty}(-1)^n(n+1)\int_0^1\left(t^{n+2} + 2t^{n+1} - \frac{\gamma}{1-\gamma}\,\frac{n}{n+1}\,t^n\right)\frac{\alpha(t)}{t}\,dt,$$

which, as in the proof of Theorem 2.1, gives

$$I = \frac{6(3-\log 16) + \gamma(6 - 6\log 16 + \pi^2)}{1-\gamma}.$$

Therefore, $\lambda I = 3$ yields the desired value of λ given by (3.6). ■

4. Convolution theorems

Theorem 4.7. *Let $\gamma \in [0,1/2]$ and $\beta < 1$. Then*:

(i) $M_{1,\alpha} * \mathcal{P}_\beta^0 \subset \mathcal{S}_\gamma$ *whenever*

$$2\alpha(1-\beta) = \frac{1-\gamma}{2(\log 4 - 1) + \gamma(4 - 4\log 2 - \pi^2/6)}.$$

(ii) $M_{1,\alpha} * \mathcal{P}_\beta^0 \subset \mathcal{K}_\gamma$ *whenever*

$$2\alpha(1-\beta) = \frac{1-2\gamma}{2\gamma + \log 4}.$$

Proof. Let $f \in M_{1,\alpha}$ and $g \in \mathcal{P}_\beta^0$. Then

$$zf''(z) \prec \frac{2\alpha z}{1-z} \quad \text{and} \quad g'(z) \prec \frac{1+(1-2\beta)z}{1-z}$$

so that, by Lemma 1.7,

$$zf''(z) * g'(z) \prec \frac{4\alpha(1-\beta)z}{1-z},$$

which is equivalent to

$$z(zh'(z))'' \prec \frac{4\alpha(1-\beta)z}{1-z},$$

where $h = f*g$. Thus, $h \in M_{1,\lambda}^{(1)}$ with $\lambda = 2\alpha(1-\beta)$. The desired conclusions follow from Theorems 2.1 and 3.1, respectively. ■

THEOREM 4.8. *Let* $\gamma \in [0, 1/2]$ *and* $\alpha, \beta > 0$. *Then*:

(i) $M_{1,\alpha} * M_{1,\beta} \subset \mathcal{S}_\gamma$ *whenever*

$$2\alpha\beta = \frac{1-\gamma}{2 - 2\log 4 + \pi^2/6 + (4\log 4 - 6)\gamma}.$$

(ii) $M_{1,\alpha} * M_{1,\beta} \subset \mathcal{K}_\gamma$ *whenever*

$$2\alpha\beta = \frac{1-2\gamma}{(\log 4)^2 + 2\gamma\log 4}.$$

Proof. Let $f \in M_{1,\alpha}$ and $g \in M_{1,\beta}$. Then

$$zf''(z) \prec \frac{2\alpha z}{1-z} \quad \text{and} \quad zg''(z) \prec \frac{2\beta z}{1-z}$$

so that, by Lemma 1.7,

$$zf''(z) * zg''(z) \prec \frac{4\alpha\beta z}{1-z},$$

which is equivalent to

$$z\{z(zh'(z))''\}' \prec \frac{4\alpha\beta z}{1-z},$$

where $h = f * g$. This implies that $h \in M_{1,\lambda}^{(2)}$ with $\lambda = 2\alpha\beta$. The desired conclusions follow from Theorems 2.1 and 3.1, respectively. ■

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