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A polynomial with $2k$ critical values at infinity

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Abstract. We construct a polynomial $f:\mathbb{C}^2\to\mathbb{C}$ of degree $4k+2$ with no critical points in $\mathbb{C}^2$ and with $2k$ critical values at infinity.

Let $f:\mathbb{C}^2\to\mathbb{C}$ be a polynomial. We call $\lambda\in\mathbb{C}$ a *critical value of f at infinity* if there exists a sequence $z_n\in\mathbb{C}^2$ such that $z_n\to\infty$, $\operatorname{grad} f(z_n)\to(0,0)$ and $f(z_n)\to\lambda$ as $n\to\infty$. Several equivalent definitions of critical values at infinity are given in [Dur]. The definition we use is due to Ha (see [Dur]).

When studying polynomial automorphisms of $\mathbb{C}^2$ we encounter polynomials with no critical points in $\mathbb{C}^2$ (see, for example [Eph], [G-P]). It is natural to ask if such a polynomial can have many critical values at infinity.

For a given integer $k>0$ we construct a polynomial $f:\mathbb{C}^2\to\mathbb{C}$ of degree $4k+2$ such that

(i) f has no critical points in $\mathbb{C}^2$,
(ii) f has exactly $2k$ critical values at infinity.

Let $A(x)=x^k-1$, $B(x)=x^{k+1}/(k+1)-x$ and

$$f(x,y)=(yA^2-x)^2+B.$$

To prove (i) recall that critical points of f are solutions of $\partial f/\partial x=\partial f/\partial y=0$, so we have

$$\begin{cases} 2(yA^2-x)(2yAA'-1)+A=0,\\ 2(yA^2-x)A^2=0.\end{cases}$$

The second equality gives $A=0$ or $yA^2-x=0$. If $A=0$ then from the first equality we obtain $x=0$, which is impossible because $A(0)\neq 0$. If we assume that $yA^2-x=0$ then $A=0$ and we again arrive at a contradiction.

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Now we will show that the critical values of f at infinity are $-\frac{k}{k+1}a$ and $a^2 - \frac{k}{k+1}a$, where a is any root of the polynomial A. Assume that $z_n = (x_n, y_n)$ is a sequence which satisfies the following four conditions:

(1) $z_n \to \infty$,

(2) $\dfrac{\partial f}{\partial x}(z_n) = 2(y_n A(x_n)^2 - x_n)(2y_n A(x_n)A'(x_n) - 1) + A(x_n) \to 0,$

(3) $\dfrac{\partial f}{\partial y}(z_n) = 2(y_n A(x_n)^2 - x_n)A(x_n)^2 \to 0,$

(4) $f(z_n) = (y_n A(x_n)^2 - x_n)^2 + B(x_n) \to \lambda \in \mathbb{C}.$

Replacing z_n by a subsequence we may additionally assume that $x_n \to \infty$ or that $x_n \to b \in \mathbb{C}$.

If x_n tends to infinity then by (3), $y_n A(x_n)^2 - x_n \to 0$. Hence $f(z_n) \to \infty$ (because $B(x_n) \to \infty$), which contradicts (4).

Now suppose that x_n converges to $b \in \mathbb{C}$ and that b is not a root of A. From (3) we have $y_n A(x_n)^2 - x_n \to 0$, which gives $y_n \to b/A(b)^2$. Hence the sequence z_n is bounded contrary to (1). We have thus proven that $x_n \to a$, where a is a root of A.

From (4) we see that the sequence $y_n A(x_n)^2 - x_n$ is convergent. If it converges to 0, then $f(z_n) \to B(a) = -\frac{k}{k+1}a$. Now suppose $y_n A(x_n)^2 - x_n \to \alpha \neq 0$. Then according to (2) we have $2y_n A(x_n)A'(x_n) - 1 \to 0$ since otherwise $(\partial f/\partial x)(z_n)$ would not be convergent to zero. Hence

$$y_n A(x_n)^2 - x_n = \frac{A(x_n)}{2A'(x_n)}\,[(2y_n A(x_n)A'(x_n) - 1) + 1] - x_n \to -a,$$

and we see from (4) that

$$f(z_n) \to a^2 + B(a) = a^2 - \frac{k}{k+1}\,a.$$

Conversely, $-\frac{k}{k+1}a$ and $a^2 - \frac{k}{k+1}a$ for $a \in A^{-1}(0)$ are critical values of f at infinity. Indeed, taking $x_n \to a$, $y_n = x_n/A(x_n)^2$ we obtain $f(z_n) \to -\frac{k}{k+1}a$ and taking $x_n \to a$, $y_n = 1/(2A(x_n)A'(x_n))$ we get $f(z_n) \to a^2 - \frac{k}{k+1}a$.

It suffices to show that the above numbers are pairwise different. We will show that if $a \neq b$ and $a, b \in A^{-1}(0)$ then

$$a^2 - \frac{k}{k+1}\,a \neq b^2 - \frac{k}{k+1}\,b,$$
$$a^2 - \frac{k}{k+1}\,a \neq -\frac{k}{k+1}\,b.$$

The first unequality can be rewritten as $(a-b)\big(a + b - \frac{k}{k+1}\big) \neq 0$, therefore it is enough to show that the equality $a + b = \frac{k}{k+1}$ does not hold. If it holds, then a and b must be conjugate, giving $a + \bar{a} = \frac{k}{k+1}$. Multiplying this by

$(k+1)a$ we obtain

$$(k+1)a^2 - ka + k + 1 = 0, \qquad a^k - 1 = 0.$$

To show that these equalities cannot both hold consider the resultant R of the polynomials $(k+1)x^2 - kx + k + 1$ and $x^k - 1$, i.e.

$$R = \begin{vmatrix} k+1 & -k & k+1 & 0 & \dots & 0 \\ 0 & k+1 & -k & k+1 & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \dots & k+1 & -k & k+1 & 0 \\ 0 & \dots & 0 & k+1 & -k & k+1 \\ 1 & 0 & \dots & & -1 & 0 \\ 0 & 1 & 0 & \dots & & -1 \end{vmatrix},$$

and the natural homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}/(k+1)\mathbb{Z}$. Then

$$\phi(R) = \begin{vmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & 0 \\ 1 & 0 & \dots & & -1 & 0 \\ 0 & 1 & 0 & \dots & & -1 \end{vmatrix} = (-1)^{k+1}.$$

Hence $R \neq 0$.

To conclude the proof we need to show that the equality

$$a^2 - \frac{k}{k+1}\,a = -\frac{k}{k+1}\,b \tag{5}$$

is false for any $a, b \in A^{-1}(0)$. Notice that if $k = 1$ then $a = b = 1$ and $a^2 - \frac{1}{2}a = \frac{1}{2} \neq -\frac{1}{2} = -\frac{1}{2}b$. Suppose that $k > 1$. Since (5) can be written as

$$\frac{k+1}{k} = a^{-1} - ba^{-2} = a^{-1} + (-ba^{-2}),$$

the numbers a^{-1} and $-ba^{-2}$ must be conjugate. Thus $-ba^{-2} = a$. Multiplying the above equality by ka we arrive at

$$ka^2 - (k+1)a + k = 0.$$

Of course $a^k - 1 = 0$. By using the same method as earlier, i.e. using a resultant modulo k, we can show that the system of equations

$$kx^2 - (k+1)x + k = 0, \qquad x^k - 1 = 0$$

has no solutions for $k > 1$. This completes the proof.

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