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Non-special projectively normal line bundles on general k-gonal curves

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Abstract. Let X be a sufficiently general smooth k-gonal curve of genus g and $R \in \mathrm{Pic}(X)$ the degree k spanned line bundle. We find an optimal integer $z > 0$ such that the line bundle $R^{\otimes z}$ is very ample and projectively normal.

1. Introduction. Let X be a smooth and connected projective curve of genus g, and L a spanned line bundle on X. Let $h_L : X \to \mathbb{P}(H^0(X, L)^*)$ denote the morphism induced by the complete linear system $|L|$. Here we assume that X is a sufficiently general k-gonal curve and explore the projective normality of non-special very ample line bundles on X.

Our main result is the following theorem giving the projective normality of several non-special line bundles L on general k-gonal curves of genus g. Indeed, often $\deg(L) \ll 3g/2$ and hence this is outside the range covered in [4]. The result (and its proof) is very different from the case of special line bundles considered in [6] (see in particular lines 4–8 of page 189 of [6]) and the references therein.

Theorem 1. *Fix integers $k \geq 3$, $g \geq 3k$ and $y \geq 1 + \lceil g/(k-1) \rceil$. Let X be a smooth and connected k-gonal curve. Let $R \in \mathrm{Pic}^k(X)$ be any line bundle computing the gonality of X. Assume that the scrollar invariants of the pair (X, R) are the ones of a general k-gonal curve of genus g, i.e. $h^0(X, R^{\otimes t}) = t + 1$ for $0 \leq t \leq [g([k-1)]$ and $h^1(X, R^{\otimes t}) = 0$ for $t > [g/(k-1)]$* (*see* [1] *in arbitrary characteristic or* [3, 1.1], [2, Prop. 1.1] *when* $\mathrm{char}(\mathbb{K}) = 0$). *Set $L := R^{\otimes y}$. Then L is very ample and projectively normal.*

Take (X, R) as in the statement of Theorem 1. If $g/(k-1) \in \mathbb{Z}$, then $h^0(X, R^{\otimes (g/(k-1))}) = 1 + g/(k-1)$ and $h_{R^{\otimes (g/(k-1))}}(X)$ is a rational normal

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curve in $\mathbb{P}^{g/(k-1)}$. Hence Theorem 1 is sharp in this case. From Theorem 1 we will easily obtain the following result.

COROLLARY 1. *Let X be a general k-gonal curve of genus $g \geq 2k+1$. Then for every integer $d \geq k\lceil g/(k-1)\rceil + k$ a general $L \in \mathrm{Pic}^d(X)$ is projectively normal.*

REMARK 1. Let X be a general k-gonal curve of genus g. If $k \approx g/4$, then Corollary 1 covers approximately the range $d \gtrsim 5g/4+5$. If $g \geq 2k+2$, then every smooth k-gonal curve of genus g is a flat limit of a flat family of $(k+1)$-gonal curves. Hence by semicontinuity if $k \geq g/4$, then on X we roughly cover the range $d \gtrsim 5g/4+5$.

Then we will explore another case and prove the following result.

PROPOSITION 1. *Fix integers $z \geq 3$ and $k \geq 3$. Let X be a smooth and connected projective curve of genus g and $R \in \mathrm{Pic}^k(X)$, and set $L := R^{\otimes z}$. Assume R is spanned, $h^0(X, R^{\otimes(z-1)}) = z$ and $h^0(X, R^{\otimes z}) = z+2$. Then $z(k-1)-1 \leq g \leq 1+k(kz-z-2)/2$. The morphism h_L is birational onto its image. L is very ample if and only if $g = 1+k(kz-z-2)/2$. We have $g = z(k-1)-1$ if and only if the pair (X,R) has the same scrollar invariants as a general k-gonal curve of genus g. Set $Y := h_L(X)$. Then $p_a(Y) = 1+k(kz-z-2)/2$ and the degree zk curve $Y \subset \mathbb{P}^{z+1}$ is the complete intersection of a cone T over a rational normal curve in $\mathbb{P}^z$ with a degree k hypersurface. We have $h^1(\mathbb{P}^{z+1}, \mathcal{I}_Y(t)) = 0$ for every $t \in \mathbb{Z}$.*

2. The proofs

Proof of Theorem 1. To fix the notation we divide the proof in two parts: in part (i) we assume $g/(k-1) \in \mathbb{Z}$, while in part (ii) we assume $g/(k-1) \notin \mathbb{Z}$.

(i) Here we assume $g/(k-1) \in \mathbb{Z}$. We will first check that $L := R^{\otimes y}$ is very ample, which is equivalent to $h^0(X, L(-Z)) = h^0(X, L)-2$ for every degree two zero-dimensional scheme $Z \subset X$. Since R is spanned, this equality is obvious if Z is not contained in a fiber of the degree k pencil $h_R : X \to \mathbb{P}^1$. Assume that Z is contained in such a fiber, say $Z+B \in |R|$ with $B \geq 0$. Assume $h^0(X, L(-Z)) \neq h^0(X, L)-2$, i.e. $h^1(X, L(-Z)) > h^1(X, L) = 0$. Since $L(-Z) \cong R^{\otimes x}(B)$ we obtain $h^1(X, R^{\otimes x}(B)) > 0$ and hence $h^1(X, R^{\otimes x}) > 0$, a contradiction.

Let $u : X \to \mathbb{P}^{ky-g}$ be the embedding associated to the complete linear system $|R^{\otimes y}|$. Fix y general divisors $D_i \in |R|$, $1 \leq i \leq y$, say $D_i = P_{i,1} + \cdots + P_{i,k}$, and let $A_i := \langle\{u(P_{i,1}), \dots, u(P_{i,k})\}\rangle$ denote the linear span of the set $\{u(P_{i,1}), \dots, u(P_{i,k})\}$. Since $h^1(X, R^{\otimes(y-1)}) = 0$, we have $\dim(A_i) = k-1$ for every i. Since each set $A_i \cap u(X)$ is finite and the divisors $D_1, \dots, D_m$ are general, and R has no base point, we may always assume $u(P_{h,j}) \notin A_i$ for all i, h, j such that $i \neq h$. Thus $\dim(\langle A_i \cup \{P_{h,j}\}\rangle) = \dim(A_i) + 1 = k$ for all i, h, j such that $i \neq h$. Similarly for any $I \subsetneq \{1, \ldots, y\}$, any $h \in \{1, \ldots, y\}$ and any j, $1 \leq j \leq k$, we may assume $u(P_{h,j}) \notin \langle \bigcup_{i \in I} A_i \rangle$.

First assume $y = g/(k-1)+1$. Since $h^0(X, L \otimes (R^{\otimes 2})^*) = h^0(X, L) - k - 1$ in this case, we have $\dim(\langle A_y \cup A_{y-1}\rangle) = k+1$ and hence $\langle A_y \cup \{u(P_{y-1,j})\}\rangle = \langle A_y \cup A_{y-1}\rangle$. Since $u(P_{y,j}) \notin A_{y-1}$ and $u(P_{y-1,j}) \notin A_y$ for all j, the points $u(P_{i,j})$, $i \in \{y-1, y\}$, $1 \leq j \leq k$, are in linearly general position in the projective space $\langle A_y \cup A_{y-1}\rangle$. By [5, Th. 2.1], the set $\{u(P_{i,j})\}$, $i \in \{y-1, y\}$, $1 \leq j \leq k$, has Properties N_0 and N_1; we will only use Property N_0. Since L is very ample and non-special, to check that it has Property N_0 (i.e. it is projectively normal) it is sufficient to find a hyperplane section of $u(X)$ with Property N_0. First we will check that the union S of the points $P_{i,j}$, $y-2 \leq i \leq y$, $1 \leq j \leq k$, has Property N_0, i.e. it imposes $3k$ independent conditions on the quadric hypersurfaces containing them. This statement is equivalent to the same statement in the linear span $\langle A_{y-2} \cup A_{y-1} \cup A_y\rangle$. Now, $\langle A_{y-1} \cup A_y\rangle$ is a hyperplane in $\langle A_{y-2} \cup A_{y-1} \cup A_y\rangle$ containing $2k$ points of S with property N_0. Set $S' := S \cap \langle A_{y-1} \cup A_y\rangle$. Notice that the points $u(P_{y-2,j})$, $1 \leq j \leq k$, are in linearly general position. Hence for every j, $1 \leq j \leq k$, there is a hyperplane H_j in $\langle A_{y-2} \cup A_{y-1} \cup A_y\rangle$ containing all $P_{y-2,h}$, $h \neq j$, but not $P_{y-2,j}$. The reducible quadric hypersurfaces $H_j \cup \langle A_{y-1} \cup A_y\rangle$, $1 \leq j \leq k$, show that $S \backslash S'$ gives k independent conditions on the linear space of all quadric hypersurfaces containing S'. Thus S imposes $3k$ independent conditions on quadrics, i.e. it has Property N_0.

Now assume $y \geq g/(k-1) + 2$. Since $h^1(X, R^{\otimes(g/(k-1))}) = 0$ and $h^0(X, R^{\otimes(g/(k-1))}) = h^0(X, R^{\otimes(g/(k-1)+1)})$, we have $\dim(\langle \bigcup_{i=g/(k-1)+1}^{y} A_i\rangle) = (y - g/(k-1))k - 1$ and $\dim(\langle \bigcup_{i=g/(k-1)}^{y} A_i\rangle) = (y - g/(k-1))k$. Follow step by step the previous proof using $\langle \bigcup_{i=g/(k-1)}^{y} A_i\rangle$ instead of A_y.

(ii) Now we assume $g/(k-1) \notin \mathbb{Z}$. First, we assume $y = \lceil g((k-1)\rceil$. In this case we have

$$h^0(X, L \otimes (R^*)^{\otimes 2}) = h^0(X, L) - k - 1 + g - (k-1)\lceil g/(k-1)\rceil$$
$$< h^0(X, L) - k - 1$$

and hence A_y is not a hyperplane in $\langle A_y + A_{y-1}\rangle$. However, $\langle A_y + A_{y-1}\rangle \cap u(X)$ contains again exactly $2k$ points in linearly independent position and hence the proof of part (i) works with only trivial modifications. The case $y > \lceil g((k-1)\rceil$ is done step by step as in part (i) without any modification.

LEMMA 1. *Let X be a smooth and connected projective curve, $A \in \mathrm{Pic}(X)$ such that $h^1(X, A) = 0$, and $P \in X$. Then $h^1(X, A(P)) = 0$. If A is very ample, then $A(P)$ is very ample. If A is very ample and projectively normal, then $A(P)$ is very ample and projectively normal.*

Proof. Obviously, $h^1(X, A(P)) \leq h^1(X, A) = 0$. A line bundle M on X is very ample if and only if for every effective degree two divisor Z on X we have $h^0(X, M(-Z)) = h^0(X, M) - 2$, i.e. $h^1(X, M(-Z)) = h^1(X, M)$. Fix an effective degree two divisor Z of X. Assume that A is very ample. Hence $h^1(X, A(-Z)) = h^1(X, A) = 0$. Thus $h^1(X, A(P)(-Z)) = 0$, proving the very ampleness of $A(P)$.

Let $\mu_A : S^2(H^0(X, A)) \to H^0(X, A^{\otimes 2})$ and $\mu_{A(P)} : S^2(H^0(X, A(P))) \to H^0(X, A^{\otimes 2}(2P))$ be the multiplication maps. Assume A is projectively normal. Thus μ_A is surjective. Since $h^1(X, A(P)) = 0$, Castelnuovo–Mumford's lemma says that to prove that $A(P)$ is projectively normal it is sufficient to check the surjectivity of $\mu_{A(P)}$. The choice of the point P allows us to view $H^0(X, A)$ as the hyperplane of $H^0(X, A(P))$ formed by all sections vanishing at P, and $H^0(X, A^{\otimes 2})$ (resp. $H^0(X, A^{\otimes 2}(P))$) as the codimension two (resp. one) linear subspace of $H^0(X, A^{\otimes 2}(2P))$ formed by the sections vanishing at P at least to order two (resp. one). The surjectivity of μ_A implies that $\mu_{A(P)}$ contains $H^0(X, A^{\otimes 2})$. Since A is spanned and $h^0(X, A(P)) = h^0(X, A) + 1$, the multiplication map $H^0(X, A) \otimes H^0(X, A(P)) \to H^0(X, A^{\otimes 2}(P))$ contains a section not vanishing at P (as an element of $H^0(X, A^{\otimes 2}(P))$), i.e. an element of $H^0(X, A^{\otimes 2}(2P))$ with exact order one at P. The spannedness of $A(P)$ implies that $\mathrm{Im}(\mu_{A(P)})$ contains an element not vanishing at P. Hence $\mathrm{Im}(\mu_A)$ has codimension at least two in $\mathrm{Im}(\mu_{A(P)})$, proving that $\mathrm{Im}(\mu_{A(P)}) = H^0(X, A^{\otimes 2}(2P))$. ∎

Proof of Corollary 1. Let $R \in \mathrm{Pic}^k(X)$ be the line bundle computing the gonality of X. Since $d \geq g+3$, a general $L \in \mathrm{Pic}^d(X)$ is non-special and very ample. By semicontinuity it is sufficient (for any fixed d) to find one non-special, very ample and projectively normal degree d line bundle on X. If $d = k\lceil g/(k-1)\rceil + k$, then take $L := R^{\otimes(d/k)}$. If $d > k\lceil g/(k-1)\rceil + k$, then apply Lemma 1 $d - k\lceil g/(k-1)\rceil - k$ times.

Proof of Proposition 1. Since $h^0(X, R^{\otimes(z-1)}) = z$ and R is spanned, for $1 \leq j \leq z-1$ we have $h^0(X, R^{\otimes j}) = j+1$, $R^{\otimes j}$ is spanned, the morphism $h_{R^{\otimes j}}(X)$ is a rational normal curve in $\mathbb{P}^j$ and the morphism $h_{R^{\otimes j}}$ has degree k onto its image. As $h^0(X, L) = z+2$ and $\deg(L) = zk$, we have $g \leq z(k-1) -1$ by Riemann's inequality. Since $h^0(X, L) > z+1$, the curve $Y := h_L(X)$ is not a rational normal curve of $\mathbb{P}^{z+1}$. Since $h^0(X, L \otimes R^*) = h^0(X, L) - 2$, the scheme $h_L(D)$ spans a line $\langle h_L(D)\rangle$ for every effective divisor $D \in |R|$. Since $h^0(X, L \otimes R^* \otimes R^*) = h^0(X, L \otimes R^*) - 1$, we have $\langle h_L(D)\rangle \cap \langle h_L(D')\rangle \neq \emptyset$ for all $D, D' \in |R|$. It is easy to check that for any infinite algebraic family $\{D_\gamma\}_{\gamma \in \Gamma}$ of mutually intersecting lines in $\mathbb{P}^{z+1}$ parametrized by an integral variety Γ either there is a plane A containing $\bigcup_{\gamma \in \Gamma} D_\gamma$ or there is $Q \in \bigcap_{\gamma \in \Gamma} D_\gamma$. In our set-up with the lines $\langle h_L(D)\rangle$, $D \in |R|$, the first case cannot occur because $Y = h_L(X)$ is non-degenerate. Hence there is a point Q common to all these lines. The projection, C, of Y from Q is a non-degenerate curve in $\mathbb{P}^z$ of degree at most z and with degree z only if $Q \notin Y$ and h_L is birational onto its image Y. Hence C is a rational normal curve in $\mathbb{P}^z$, $Q \notin Y$ and $h_L : X \to Y$ is birational. Hence $p_a(Y) \geq g$ and $p_a(Y) = g$ if and only if L is very ample.

The curve Y is contained in the cone T with base C and vertex Q. Let $\phi : S \to T$ be the blowing-up of Q. Set $h := \phi^{-1}(Q)$. It is well known that S isomorphic to the Hirzebruch surface F_z and that h is the section of the ruling $v : S \to \mathbb{P}^1$ with negative self-intersection. Let f denote any fiber of the ruling v. We have $\mathrm{Pic}(Z) \cong \mathbb{Z}h \oplus \mathbb{Z}f$, $h^2 = -z$, $h \cdot f = 1$ and $f^2 = 0$. Let $Y' \subset S$ be the strict transform of Y. Since $Q \notin Y$, we have $Y' \cap h = \emptyset$. Thus $Y' \cdot h = 0$. Since $Y' \cdot f = k$, we obtain $Y' \in |kh + kzf|$. We have $\omega_S \cong \mathcal{O}_S(-2h - (z+2)f)$. Thus $\omega_{Y'} \cong \mathcal{O}_{Y'}((k-2)h + (kz - z - 2)f)$. Hence $p_a(Y') = 1 + k(kz - z - 2)/2$. Since $Q \notin Y$, we have $Y' \cong Y$ and hence $p_a(Y) = 1 + k(kz - z - 2)/2$.

Claim. *We have* $h^1(\mathbb{P}^{z+1}, \mathcal{I}_T(x)) = h^1(T, \mathcal{O}_T(x)) = 0$ *for every* $x \in \mathbb{Z}$.

Proof of the Claim. Since C is arithmetically Cohen–Macaulay in $\mathbb{P}^z$, it follows that T is arithmetically Cohen–Macaulay in $\mathbb{P}^{z+1}$, proving the Claim.

The natural map $H^0(T, \mathcal{O}_T(k)) \to H^0(S, \mathcal{O}_S(kh + kzf))$ is surjective. Since $Y' \in |kh + kzf|$, the second vanishing of the Claim for the integer $x := k$ implies that Y is scheme-theoretically the intersection of T with a degree k hypersurface. By the first vanishing of the Claim for every integer $t > 0$ the restriction map $H^0(\mathbb{P}^{z+1}, \mathcal{O}_{\mathbb{P}^{z+1}}(t)) \to H^0(T, \mathcal{O}_T(t))$ is surjective. Since Y is scheme-theoretically the zero-locus of an element of $H^0(X, \mathcal{O}_T(k))$, the surjectivity of the restriction map $H^0(T, \mathcal{O}_T(t)) \to H^0(Y, \mathcal{O}_Y(t))$ follows from the second vanishing of the Claim for the integer $x := t - k$.

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