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Complete pluripolar curves and graphs

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Abstract. It is shown that there exist C^∞ functions on the boundary of the unit disk whose graphs are complete pluripolar. Moreover, for any natural number k, such functions are dense in the space of C^k functions on the boundary of the unit disk. We show that this result implies that the complete pluripolar closed C^∞ curves are dense in the space of closed C^k curves in $\mathbb{C}^n$. We also show that on each closed subset of the complex plane there is a continuous function whose graph is complete pluripolar.

1. Introduction. In [3] Levenberg, Martin and Poletsky studied which graphs of analytic functions in the open unit disk $\mathbb{D}$ are complete pluripolar in $\mathbb{C}^2$. Recall that a set $E \subset \mathbb{C}^n$ is *complete pluripolar* in $\mathbb{C}^n$ if there exists a plurisubharmonic function $u(z)$ defined on $\mathbb{C}^n$ such that

$$E = \{z \in \mathbb{C}^n : u(z) = -\infty\}.$$

By the graph of a function $f(z)$ defined on a subset $D \subset \mathbb{C}$ we mean the set

$$\Gamma_f(D) = \{(z, w) \in \mathbb{C}^2 : z \in D,\, w = f(z)\}.$$

It is shown in [3] that nonextendible analytic functions in the unit disk with very lacunary Taylor series have complete pluripolar graphs. (The condition is much stronger than Hadamard lacunarity.) That paper also contains an example of a function $f(z)$ which is holomorphic in the open unit disk and C^∞ on the closed unit disk, so that the graph $\Gamma_f(\overline{\mathbb{D}})$ is complete pluripolar in $\mathbb{C}^2$. We start this paper by solving the following problem that was left open in [3]:

Does there exist a C^∞ function $f : \mathbb{T} \to \mathbb{C}$ whose graph is complete pluripolar in $\mathbb{C}^2$?

Here $\mathbb{T}$ denotes the unit circle. In Section 2 we show that the answer to this question is affirmative and that such functions are not rare, that is, we prove the following theorem.

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Theorem 1. *Let k be an integer, $k \geq 0$, $g(z) \in C^k(\mathbb{T})$ and $\varepsilon > 0$. Then there exists a function $f(z) \in C^\infty(\mathbb{T})$ such that the graph of $f(z)$ is complete pluripolar in $\mathbb{C}^2$ and $\|f(z) - g(z)\|_{C^k} < \varepsilon$.*

Using a classical result of J. Wermer concerning the polynomial hull of closed real-analytic curves in $\mathbb{C}^n$ and a result on approximation of biholomorphic mappings by automorphisms of $\mathbb{C}^n$ due to Forstneric and Rosay we extend this result to arbitrary closed curves instead of graphs.

Theorem 2. *Let k be an integer, $k \geq 0$, γ a closed C^k curve in $\mathbb{C}^n$ and $\varepsilon > 0$. Then there exists an embedded closed C^∞ curve γ_1 in $\mathbb{C}^n$ such that $\|\gamma(z) - \gamma_1(z)\|_{C^k} < \varepsilon$ and γ_1 is complete pluripolar in $\mathbb{C}^n$.*

This is done in Section 3. Finally, in Section 4 we give a proof of the following theorem.

Theorem 3. *For any nonempty closed subset $F \subset \mathbb{C}$ there exists a continuous function $f(z)$ on F such that its graph $\Gamma_f(F)$ is complete pluripolar in $\mathbb{C}^2$.*

2. Proof of Theorem 1. Choose an increasing sequence $n(k)$ of natural numbers, $k = 1, 2, \ldots$, satisfying the growth condition

$$n(k+1) \geq k^2 n(k). \tag{2.1}$$

Set $a_{n(k)} = (1 - 1/k)^{n(k)}$ and define, for each natural $j \geq 2$,

$$h_j(z) = \sum_{k=j}^{\infty} a_{n(k)} (z^{n(k)} + z^{-n(k)}). \tag{2.2}$$

Lemma 1. *For each j the function $h_j(z)$ given by* (2.2) *is a smooth function on $\mathbb{T}$. Moreover, for large enough numbers j the following estimate holds*:

$$\sum_{k=j+1}^{\infty} a_{n(k)} \leq 2a_{n(j+1)}. \tag{2.3}$$

The proof of Lemma 1 is postponed until the end of this section. We now turn to the construction of the function $f(z)$. First we approximate $g(z)$ in C^k-norm by a trigonometric polynomial. Take

$$p(z) = \sum_{k=-N}^{N} b_k z^k \quad \text{so that} \quad \|g(z) - p(z)\|_{C^k} < \varepsilon/2.$$

Choose now a natural number $j_0 > N$ so that

$$\|h_j(z)\|_{C^k} < \varepsilon/2$$

and (2.3) holds for all $j > j_0$. Define

$$f(z) = p(z) + h_{j_0}(z).$$

Clearly $f(z)$ is a smooth function on $\mathbb{T}$ and $\|g(z) - f(z)\|_{C^k} < \varepsilon$. We will prove that the graph of f is complete pluripolar. We want to construct a plurisubharmonic function $u(z, w)$ which will be equal to $-\infty$ exactly on the graph of f. Define, for $k \geq j_0$,

$$f_k(z) = \sum_{j=j_0}^{k} a_{n(j)}(z^{n(j)} + z^{-n(j)}) = h_{j_0}(z) - h_{k+1}(z)$$

and put

$$u(z, w) = \sum_{k=j_0}^{\infty} \max\left\{\frac{1}{n(k+1)} \log |z^{n(k)}(w - (p(z) + f_k(z)))|, -1\right\}. \tag{2.4}$$

The following lemma holds.

LEMMA 2. *The function $u(z, w)$ given by* (2.4) *is plurisubharmonic in $\mathbb{C}^2$.*

We also postpone the proof of Lemma 2 until the end of this section and turn to the examination of the $-\infty$ locus of $u(z, w)$.

LEMMA 3. *$u(z, w) = -\infty$ if and only if (z, w) belongs to the graph of f.*

Proof. The proof is done in four steps. First we consider the case when $z = 0$ and w is a fixed complex number. Then the polynomial $z^{n(k)}(w - (p(z) + f_k(z)))$ equals $a_{n(k)}$ and by the growth condition (2.1) the series $\sum_k (1/n(k+1)) \log a_{n(k)}$ is convergent. Hence we conclude that

$$u(0, w) = \sum_{k=j_0}^{\infty} \max\left\{\frac{1}{n(k+1)} \log a_{n(k)}, -1\right\} > -\infty.$$

Secondly we consider the case when $|z| = 1$ and $w \neq f(z)$. Since $p(z) + f_k(z) \to f(z)$ as $k \to \infty$ there exists $\delta > 0$ and a natural number k_0 such that $|z^{n(k)}(w - (p(z) + f_k(z)))| \geq \delta$ when $k \geq k_0$. By the growth condition (2.1) the series $\sum_k (1/n(k+1)) \log \delta$ is convergent so we obtain

$$\begin{aligned} u(z, w) &= \sum_{k=j_0}^{\infty} \max\left\{\frac{1}{n(k+1)} \log |(w - (p(z) + f_k(z)))|, -1\right\} \\ &\geq -k_0 + \sum_{k=k_0+1}^{\infty} \frac{1}{n(k+1)} \log \delta > -\infty. \end{aligned}$$

We now consider the case when (z, w) belongs to the graph of f, that is, $|z| = 1$ and $w = f(z)$. We use the definition of the function $f(z)$ and the estimate (2.3) in Lemma 1 to conclude that

$$\begin{aligned} u(z,f(z)) &= \sum_{k=j_0}^{\infty} \max\left\{\frac{1}{n(k+1)}\log|f(z)-(p(z)+f_k(z))|,-1\right\} \\ &= \sum_{k=j_0}^{\infty} \max\left\{\frac{1}{n(k+1)}\log|h_{k+1}(z)|,-1\right\} \\ &\le \sum_{k=j_0}^{\infty} \max\left\{\frac{1}{n(k+1)}\log(4a_{n(k+1)}),-1\right\}. \end{aligned} \tag{2.5}$$

By the growth condition (2.1) we know that the series $\sum(1/n(k+1))\log 4$ is convergent and by the choice of the coefficients $a_{n(k+1)}$ we see that $(1/n(k+1))\log a_{n(k+1)}$ tends to zero as k tends to infinity. Therefore the series in (2.5) is dominated from above by const $+\sum(1/n(k+1))\log a_{n(k+1)}$. Since the sum of the latter series equals $-\infty$ we conclude that the series in (2.5) equals $-\infty$.

Finally, we consider the case when $0<|z|<1$ or $1<|z|$ and w is a fixed complex number. If $|z|>1$ then by the choice of the coefficients $a_{n(k)}$ and the growth condition (2.1) we find that $|\sum_{k=j_0}^{l} a_{n(k)}z^{-n(k)}|$ converges and $|\sum_{k=j_0}^{l} a_{n(k)}z^{n(k)}|\to\infty$ as $l\to\infty$. (For a proof see [3, pp. 519–520].) Consequently,

$$|f_k(z)| \ge \left|\left|\sum_{k=j_0}^{l} a_{n(k)}z^{n(k)}\right| - \left|\sum_{k=j_0}^{l} a_{n(k)}z^{-n(k)}\right|\right| \to \infty \qquad \text{as } l\to\infty.$$

So for a fixed (z,w) with $|z|>1$ and a fixed $\delta>0$, there exists a natural number k_0 such that if $k\ge k_0$ then $\big||w|-|f_k(z)+p(z)|\big|>\delta$. Therefore

$$\begin{aligned} u(z,w) &= \sum_{k=j_0}^{\infty} \max\left\{\frac{1}{n(k+1)}\log|z^{n(k)}(w-(p(z)+f_k(z)))|,-1\right\} \\ &\ge -k_0 + \sum_{k=k_0+1} \max\left\{\frac{n(k)}{n(k+1)}\log|z| + \frac{1}{n(k+1)}\log\delta,-1\right\}. \end{aligned} \tag{2.6}$$

By the growth condition (2.1) the series $\sum(n(k)/n(k+1))\log|z|$ and $\sum(1/n(k+1))\log\delta$ converge and hence the series in (2.6) converges, so we conclude that $u(z,w)>-\infty$. If $0<|z|<1$ then $|\sum_{k=j_0}^{l} a_{n(k)}z^{n(k)}|$ converges and $|\sum_{k=j_0}^{l} a_{n(k)}z^{-n(k)}|\to\infty$ as $l\to\infty$, so we may repeat the arguments above and conclude that also in this case $u(z,w)>-\infty$. ∎

Proof of Lemma 1. First we prove that $h(z)$ is a smooth function on $\mathbb{T}$; for this it is enough to show that, for each natural number p,

$$n(k)^p(1-1/k)^{n(k)} = o(1/k^2) \qquad \text{as } k\to\infty. \tag{2.7}$$

But this is indeed the case since the growth condition (2.1) implies that $n(k) \geq k^4$ for large k. Using this together with standard estimates we obtain, for those k,

$$\begin{aligned} k^2 n(k)^p (1-1/k)^{n(k)} &\leq n(k)^{p+1}(1-1/k)^{n(k)} = e^{(p+1)\log n(k)+n(k)\log(1-1/k)} \\ &\leq e^{\sqrt{n(k)}-n(k)/2k} \leq e^{\sqrt{n(k)}-(k/2)\sqrt{n(k)}}. \end{aligned}$$

The last expression tends to zero as $k \to \infty$, and (2.7) follows.

It therefore remains to prove the estimate (2.3) and for this it is enough to show that for all sufficiently large natural numbers k we have $a_{n(k+1)} \leq (1/2)a_{n(k)}$. But this is true since for large k we have

$$\begin{aligned} a_{n(k+1)} &= \left(1-\frac{1}{k+1}\right)^{n(k+1)} \leq \left(1-\frac{1}{k+1}\right)^{kn(k)} \\ &\leq \left(\frac{1}{2}\left(1-\frac{1}{k}\right)\right)^{n(k)} \leq \frac{1}{2}\, a_{n(k)}. \end{aligned}$$

Here the second inequality follows from the fact that

$$\left(1-\frac{1}{k+1}\right)^{k} \to \frac{1}{e} < \frac{1}{2} \qquad \text{as } k \to \infty. \ \blacksquare$$

Proof of Lemma 2. Since plurisubharmonicity is a local property it is enough to show that $u(z,w)$ is plurisubharmonic in each bidisk.

Put $C = \max_{-N\leq k\leq N} |b_j|$. Fix $r > 1$ and denote by B the bidisk $\{(z,w) \in \mathbb{C}^2 : |z|,|w| < r\}$. Fix a natural number $k \geq j_0$. By counting the number of terms in the polynomial $z^{n(k)}p(z)$ and estimating each of them we obtain the uniform estimate $|z^{n(k)}p(z)| \leq 3NCr^{2n(k)}$ when $(z,w) \in B$. Proceeding in the same way with the polynomial $z^{n(k)}(w - f_k(z))$ we get the estimate $|z^{n(k)}(w - f_k(z))| \leq 3kr^{2n(k)}$ when $(z,w) \in B$. Consequently, for $j \geq j_0$ the plurisubharmonic functions

$$\sum_{k=j_0}^{j} \max\left\{\frac{1}{n(k+1)}\log|z^{n(k)}(w-(p(z)+f_k(z)))|, -1\right\}$$

are uniformly bounded from above by the finite constant

$$\sum_{k=j_0}^{\infty}\left(\frac{\log(3(NC+k))}{n(k+1)} - \frac{2n(k)\log r}{n(k+1)}\right)$$

in B. Hence the series

$$\sum_{k=j_0}^{\infty} \max\left\{\frac{1}{n(k+1)}\log|z^{n(k)}(w-(p(z)+f_k(z)))|, -1\right\}$$

converges to a plurisubharmonic function in B. Since B was an arbitrary bidisk the function u is plurisubharmonic in $\mathbb{C}^2$. $\blacksquare$

3. Proof of Theorem 2. Let γ be a connected closed C^k curve in $\mathbb{C}^n$. Approximate it by a C^∞ curve, use Whitney's embedding theorem ([2]) and approximate the coordinate functions by trigonometric polynomials. We get a real-analytic embedding $\widetilde{\gamma} : \mathbb{T} \to \mathbb{C}^n$ which approximates γ in $C^k(\mathbb{T})$. Moreover the coordinate functions $\widetilde{\gamma}_j$ of $\widetilde{\gamma}$ are trigonometric polynomials and we may assume that $\widetilde{\gamma}_1'$ does not vanish on $\mathbb{T}$. By a theorem of Wermer ([4]), $\widetilde{\gamma}(\mathbb{T})$ is not polynomially convex if and only if

$$\int\limits_{|z|=1} \widetilde{\gamma}_1^{m_1}(z) \cdots \widetilde{\gamma}_n^{m_n}(z) \cdot \widetilde{\gamma}_1'(z)\, dz = 0 \tag{3.1}$$

for all n-tuples $(m_1, \ldots, m_n)$ of nonnegative integers. This theorem allows us to approximate $\widetilde{\gamma}$ by a polynomially convex real-analytic embedding. Indeed, suppose that all terms in (3.1) vanish. Then, in particular, the term of order -1 in the Laurent expansion of $\widetilde{\gamma}_n(z)\widetilde{\gamma}_1'(z)$ vanishes. Define $\widetilde{\gamma}_n^*(z) = \widetilde{\gamma}_n(z) + cz^N$, where N is an integer such that the Laurent expansion of $\widetilde{\gamma}_1'(z)$ contains a nonvanishing term of order $-N-1$, and $\widetilde{\gamma}_j^*(z) = \widetilde{\gamma}_j(z)$ for $j = 1, \ldots, n-1$. If $|c|$ is small enough then $\widetilde{\gamma}^*$ is close to $\widetilde{\gamma}$ in $C^k(\mathbb{T})$ and the term of order -1 in the Laurent expansion of $\widetilde{\gamma}_n^*(\widetilde{\gamma}_1^*)'$ is different from zero. Hence the integral (3.1) for the multi-index $(0, \ldots, 0, 1)$ does not vanish.

We will apply the following result of Forstneric and Rosay. Let γ_a and γ_b be real-analytic embeddings of $\mathbb{T}$ into $\mathbb{C}^n$ such that $\Gamma_a = \gamma_a(\mathbb{T})$ and $\Gamma_b = \gamma_b(\mathbb{T})$ are polynomially convex. Then there is a sequence Ψ_j of automorphisms of $\mathbb{C}^n$ which converges uniformly on a neighborhood U of Γ_a to a biholomorphic map $\Psi : U \to \Psi(U)$ satisfying $\Psi(\Gamma_a) = \Gamma_b$. Let $\gamma_a(z) = (z, \ldots, z, 1/z)$, $z \in \mathbb{T}$. Take $\gamma_b = \gamma^*$ and let Ψ_j be the sequence of holomorphic automorphisms of $\mathbb{C}^n$ obtained for those γ_a and γ_b by the theorem of Forstneric and Rosay. Then $\Psi_j \circ \gamma_a$ is close to γ^* in $C^k(\mathbb{T})$ if j is large.

Theorem 1 gives a C^∞ function f on $\mathbb{T}$ which is C^k close to $1/z$ and such that the graph $\Gamma_f(\mathbb{T}) = \{(z, w) \in \mathbb{C}^2 : z \in \mathbb{T},\, w = f(z)\}$ considered as a subset of $\mathbb{C}^2$ is complete pluripolar in $\mathbb{C}^2$. Hence the image Γ of $\mathbb{T}$ under the mapping $\gamma_f(z) = (z, \ldots, z, f(z))$ is complete pluripolar in $\mathbb{C}^n$. Indeed, since $\Gamma_f(\mathbb{T})$ is complete pluripolar in $\mathbb{C}^2$ there exists a plurisubharmonic function u defined on $\mathbb{C}^2$ which equals $-\infty$ exactly on $\Gamma_f(\mathbb{T})$. Hence the function $\max\{u(z_1, z_n), u(z_2, z_n), \ldots, u(z_{n-1}, z_n)\}$ is a plurisubharmonic function on $\mathbb{C}^n$ which is equal to $-\infty$ exactly on Γ.

Since each Ψ_j is a holomorphic automorphism of $\mathbb{C}^n$ each $\Psi_j(\Gamma)$ is complete pluripolar and $\Psi_j \circ \gamma_f$ is close to $\Psi_j \circ \gamma_a$ for a fixed j. Choosing first j suitably large and then $f(z)$ close to $1/z$ we conclude that $\Psi_j \circ \gamma_f$ approximates γ^*, and hence γ, in $C^k(\mathbb{T})$. Theorem 2 is proved. ■

4. Proof of Theorem 3. The graph of any entire function over the complex plane is complete pluripolar. Therefore we assume that F is a proper closed subset of $\mathbb{C}$. Let $\Delta = \{D(z_i, r_i)\}_{i=1}^{\infty}$ be a sequence of disks with radii < 1 and pairwise disjoint centers, and with the following properties. Each disk $D(z_i, r_i)$ is contained in the complement of the set F and the union of disks from Δ is equal to the complement of F and Δ is locally finite, i.e. each compact subset of $\mathbb{C} \setminus F$ intersects only a finite number of disks from the sequence Δ. Without loss of generality we may assume that $r_i \le r_1$ for $i = 1, 2, \ldots$. The function $f(z)$ in the statement of Theorem 3 will be of the form

$$f(z) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} \frac{1}{(z - z_i)^{n(j)}}, \tag{4.1}$$

where the natural numbers $n(j)$ and the coefficients $a_{i,j}$ are chosen in the following way. Put

$$C_k = \max_{i=1,\ldots,k} |z_i| + k \tag{4.2}$$

and choose an increasing sequence of natural numbers $n(j)$ so that the growth condition

$$n(j+1) \ge j^3 n(j) \tag{4.3}$$

is satisfied and so that the series

$$\sum_{k=1}^{\infty} \frac{kn(k)}{n(k+1)} \log C_k \tag{4.4}$$

and the series

$$\sum_{k=1}^{\infty} \frac{n(k)}{n(k+1)} \log \prod_{i=1}^{k} \operatorname{dist}(z_i, F) \tag{4.5}$$

are convergent. Here $\operatorname{dist}(z_i, F)$ denotes the Euclidean distance from the point z_i to the set F. Put

$$a_{1,j} = (r_1(1 - 1/j))^{n(j)}, \qquad a_{i,j} = (a_{1,i})^j (r_i(1 - 1/j))^{n(j)} \quad \text{for } i > 1. \tag{4.6}$$

Let $f_k(z)$ be the rational function

$$f_k(z) = \sum_{i=1}^{k} \sum_{j=1}^{k} a_{i,j} \frac{1}{(z - z_i)^{n(j)}} \tag{4.7}$$

with poles outside F and denote by $p_k(z, w)$ the polynomial

$$p_k(z, w) = \Big(\prod_{i=1}^{k} (z - z_i)^{n(k)} \Big) (w - f_k(z)).$$

Then we have the following lemmas.

Lemma 4. *The function* $f(z)$ *given by* (4.1) *is continuous on* F. *Moreover the estimate*

$$|f(z) - f_k(z)| \leq 8k\left(1 - \frac{1}{k+1}\right)^{n(k+1)} \tag{4.8}$$

holds uniformly for $z \in F$ *for all sufficiently large natural numbers* k.

Lemma 5. *If* $z \in \mathbb{C} \setminus F$ *and* $z \neq z_i$ *for all centers* z_i *of disks in* Δ *then* $|f_k(z)| \to \infty$ *as* $k \to \infty$.

Lemma 6. *Put*

$$u(z,w) = \sum_{k=1}^{\infty} \max\left\{\frac{1}{n(k+1)} \log|p_k(z,w)|, -1\right\}. \tag{4.9}$$

The series converges for each point $(z,w) \in \mathbb{C}^2$ *and* $u(z,w)$ *is plurisubharmonic in* $\mathbb{C}^2$.

The proofs of Lemmas 4–6 are postponed until the end of this section. Theorem 3 follows from the next lemma.

Lemma 7. $u(z,w) = -\infty$ *if and only if* (z,w) *belongs to the graph of* f.

Proof. First we consider the case when z belongs to the set F and $w \neq f(z)$. Since the rational function $f_k(z)$ tends to $f(z)$ as k tends to infinity there exists a $\delta > 0$ and a natural number k_0 such that if $k \geq k_0$ then $|w - f_k(z)| > \delta$. Moreover, since z belongs to F we may estimate $|z - z_i|$ from below by $\operatorname{dist}(z_i, F)$. Hence

$$\begin{aligned} u(z,w) &= \sum_{k=1}^{\infty} \max\left\{\frac{1}{n(k+1)} \log\left|\left(\prod_{i=1}^{k}(z-z_i)^{n(k)}\right)(w - f_k(z))\right|, -1\right\} \\ &\geq \text{const} + \sum_{k=k_0}^{\infty} \max\left\{\frac{n(k)}{n(k+1)} \log\prod_{i=1}^{k} \operatorname{dist}(z_i, F)\right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad \left. + \frac{1}{n(k+1)} \log\delta, -1\right\}. \end{aligned}$$

By the growth condition (4.3) it follows that the series $\sum(1/n(k+1))\log\delta$ is convergent and this together with the convergence of the series given by (4.5) implies that the last series above is convergent and hence $u(z,w) > -\infty$.

Secondly we consider the case when $z \in F$ and $w = f(z)$. The definition of $C_k = \max_{i=1,\ldots,k}|z_i| + k$ implies that $|z| + |z_i|$ is bounded from above by C_k for $k > |z|$ and $i = 1, \ldots, k$. Moreover, since $z \in F$, the estimate (4.8) holds and therefore there exists a natural number k_0 so that

$$\begin{aligned} u(z,f(z)) &= \sum_{k=1}^{\infty}\max\left\{\frac{1}{n(k+1)}\log\left|\Big(\prod_{i=1}^{k}(z-z_i)^{n(k)}\Big)(f(z)-f_k(z))\right|,-1\right\} \\ &\le \text{const} + \sum_{k=k_0}^{\infty}\max\left\{\frac{kn(k)}{n(k+1)}\log C_k\right. \\ &\qquad \left. + \frac{1}{n(k+1)}\log\left(8k\left(1-\frac{1}{k+1}\right)^{n(k+1)}\right),-1\right\} \\ &\le \text{const} + \sum_{k=k_0}^{\infty}\left\{\frac{kn(k)}{n(k+1)}\log C_k + \frac{1}{n(k+1)}\log 8k\right. \\ &\qquad \left. + \max\left\{\log\left(1-\frac{1}{k+1}\right),-1\right\}\right\}. \end{aligned}$$

Here we used the inequality $\max\{A+t,-1\} \le A + \max\{t,-1\}$ for $A>0$ and $t\in\mathbb{R}$. The first terms in braces form the convergent series (4.4). By the growth condition (4.3) it follows that the series formed by the second terms is convergent. The remaining series diverges to $-\infty$, hence $u(z,w)=-\infty$.

Finally we consider the case when z belongs to the complement of F and w is some fixed complex number. Suppose first that z equals the center of one of the disks from the covering Δ, that is, $z=z_i$ for some i. Recall that Δ has the property that each compact subset of the complement of the set F intersects only a finite number of disks from Δ. Therefore, z_i is contained in only a finite number of disks $D(z_j,r_j)$ and consequently there exists a $\delta>0$ such that $|z_i-z_j|>\delta$ for those j, $j\ne i$. Moreover, if $k\ge i$ the polynomial $p_k(z_i,w)$ is equal to $a_{i,k}\prod_{j=1,\,j\ne i}^{k}(z_i-z_j)^{n(k)}$ and therefore the absolute value of $p_k(z_i,w)$ is estimated from below by $a_{i,k}\delta^{(k-1)n(k)}$. Hence

$$\begin{aligned} u(z_i,w) &= \sum_{k=1}^{\infty}\max\left\{\frac{1}{n(k+1)}\log|p_k(z_i,w)|,-1\right\} \\ &\ge \text{const} + \sum_{k=i}^{\infty}\max\left\{\frac{1}{n(k+1)}\log a_{i,k}\delta^{(k-1)n(k)},-1\right\}. \end{aligned}$$

Now, (4.6) implies that $\log a_{i,k} = n(k)o(1)$ for $k\to\infty$. The growth condition (4.3) implies that the series above is convergent and hence $u(z_j,w)>-\infty$.

Suppose that $z\in\mathbb{C}\setminus F$ and z is not equal to the center of any disk from the covering Δ. Again, since each compact set of the complement of F intersects only a finite number of disks from Δ there exists a $\delta>0$ such that $|z-z_i|>\delta$ for all i. By Lemma 5, $|f_k(z)|\to\infty$ as $k\to\infty$. Thus, for a fixed complex number w there exists a natural number k_0 such that

$\big||w| - |f_k(z)|\big| > \delta$ for $k \geq k_0$. Hence

$$u(z,w) = \sum_{k=1}^{\infty} \max\left\{\frac{1}{n(k+1)} \log\left|\Big(\prod_{i=1}^{k}(z-z_i)^{n(k)}\Big)(w - f_k(z))\right|, -1\right\}$$
$$\geq \text{const} + \sum_{k=k_0}^{\infty} \max\left\{\frac{kn(k)}{n(k+1)} \log\delta + \frac{1}{n(k+1)}\log\delta, -1\right\}.$$

By (4.3) the series converges and hence $u(z,w) > -\infty$. ■

It remains to prove Lemmas 4–6.

Proof of Lemma 4. (4.3) implies that $8k(1-1/(k+1))^{n(k+1)}$ tends to zero for $k \to \infty$. Hence the continuity of the function $f(z)$ follows if we can prove the estimate (4.8).

If $z \in F$ then $r_i/(z-z_i) < 1$ for all i. Hence defining

$$\text{(4.10)} \quad \widetilde{a}_{1,j} = (1-1/j)^{n(j)}, \qquad \widetilde{a}_{i,j} = (\widetilde{a}_{1,i})^j(1-1/j)^{n(j)} \quad \text{for } i > 1,$$

we obtain

$$\text{(4.11)} \qquad |f(z) - f_k(z)| \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \widetilde{a}_{i,j} - \sum_{i=1}^{k}\sum_{j=1}^{k} \widetilde{a}_{i,j}.$$

The estimate (4.8) follows if we can prove that the right hand side of (4.11) does not exceed $8k(1-1/(k+1))^{n(k+1)}$. As in the proof of Lemma 1 one can find a natural number k_0 such that $\widetilde{a}_{1,k+1} \leq \frac{1}{2}\widetilde{a}_{1,k}$ for $k \geq k_0$. Then by (4.10) also $\widetilde{a}_{i,k+1} \leq \frac{1}{2}\widetilde{a}_{i,k}$ for all i whenever $k \geq k_0$. Consequently, $\sum_{l>k} \widetilde{a}_{i,l} \leq 2\widetilde{a}_{i,k+1}$. Using this twice together with the immediate estimates $\widetilde{a}_{i,j} \leq \widetilde{a}_{1,j}$ and $\widetilde{a}_{i,j} \leq \widetilde{a}_{1,i}$ we conclude that

$$\text{(4.12)} \quad \sum_{i=1}^{\infty}\sum_{j=k+1}^{\infty} \widetilde{a}_{i,j} \leq \sum_{i=1}^{\infty} 2\widetilde{a}_{i,k+1} \leq 2k\widetilde{a}_{1,k+1} + \sum_{i=k+1}^{\infty} 2\widetilde{a}_{1,i} \leq (2k+4)\widetilde{a}_{1,k+1}.$$

Moreover using the inequalities above one more time we get

$$\text{(4.13)} \qquad \sum_{j=1}^{k}\sum_{i=k+1}^{\infty} \widetilde{a}_{i,j} \leq \sum_{j=1}^{k}\sum_{i=k+1}^{\infty} \widetilde{a}_{1,i} \leq \sum_{j=1}^{k} 2\widetilde{a}_{1,k+1} \leq 2k\widetilde{a}_{1,k+1}.$$

Combining (4.12) and (4.13) we obtain the estimate (4.8) and hence Lemma 4 follows. ■

Proof of Lemma 5. We will prove first that for each $z \in \mathbb{C}\setminus F$ there exists a natural number i_0 such that for all sufficiently large j the leading term in the decomposition (4.7) of f_j at the point z is the term corresponding to the principal part of f_j at z_{i_0}; more precisely, we will prove

$$\text{(4.14)} \quad 2a_{i_0,j}\frac{1}{|z-z_{i_0}|^{n(j)}} > |f_j(z) - f_{j-1}(z)| > \frac{1}{2}\, a_{i_0,j}\frac{1}{|z-z_{i_0}|^{n(j)}}.$$

Then we will show for the i_0 above and all sufficiently large j that

$$\frac{1}{2}\, a_{i_0,j+1}\, \frac{1}{|z - z_{i_0}|^{n(j+1)}} > 6a_{i_0,j}\, \frac{1}{|z - z_{i_0}|^{n(j)}}. \tag{4.15}$$

Inequalities (4.14) and (4.15) imply $|f_{j+1}(z) - f_j(z)| > 3|f_j(z) - f_{j-1}(z)|$ for large j. Hence $|f_k(z)| = |\sum_{j=2}^{k} f_j(z) - f_{j-1}(z)|$ tends to infinity as $k \to \infty$.

In order to prove (4.14) denote by $D(z_{i_1}, r_{i_1}), \ldots, D(z_{i_n}, r_{i_n})$ the disks of the covering Δ which contain the point z. If j is larger than the i_l's, the terms in $f_j(z) - f_{j-1}(z)$ corresponding to the z_{i_l}'s have the form

$$a_{i_l,j}\, \frac{1}{(z - z_{i_l})^{n(j)}} = \begin{cases} \left(\left(1 - \frac{1}{j}\right) \frac{r_1}{z - z_1}\right)^{n(j)} & \text{if } i_l = 1, \\ (a_{1,i_l})^j \left(\left(1 - \frac{1}{j}\right) \frac{r_{i_l}}{z - z_{i_l}}\right)^{n(j)} & \text{otherwise.} \end{cases}$$

Since z is contained in the above mentioned disks, $r_{i_l}/|z - z_{i_l}| > 1$ for $l = 1, \ldots, n$ and consequently all the terms above tend to infinity as $j \to \infty$. By comparing the rate of growth of the above terms we see that one of them tends to infinity faster than the sum of the others. (In case the $r_{i_l}/|z - z_{i_l}|$ coincide for different i_l, the respective factors $(a_{1,i_l})^j$ determine the growth of the ratio of the two terms.) Moreover, a calculation similar to the one in the proof of Lemma 4 shows that the sum of the terms in $f_j(z) - f_{j-1}(z)$ which do not correspond to any of the z_{i_l}'s is uniformly bounded. Hence there exists i_0 so that (4.14) holds for all sufficiently large j.

For the proof of (4.15) we consider the ratio

$$a_{i_0,j+1}\, \frac{1}{|z - z_{i_0}|^{n(j+1)}} \Big/ a_{i_0,j}\, \frac{1}{|z - z_{i_0}|^{n(j)}} = C\left(1 + \frac{1}{j^2 - 1}\right)^{n(j+1)-n(j)}, \tag{4.16}$$

where C is a positive constant (namely $C = 1$ or $C = a_{1,i_0}$). The growth condition (4.3) implies that $n(j+1) - n(j) > j^3$ for all sufficiently large j and hence the right hand side of (4.16) is larger than $Ce^{j+o(j)}$ for $j \to \infty$. (4.15) follows. ■

Proof of Lemma 6. We will prove that for each $r > 1$ the series (4.9) converges in the bidisk $B := \{(z, w) \in \mathbb{C}^2 : |z|, |w| < r\}$ to a plurisubharmonic function. For $(z, w) \in B$ the inequality $|\prod_{i=1}^{k}(z - z_i)^{n(k)} w| \le (r + \max_{i=1,\ldots,k} |z_i|)^{kn(k)} r \le (r + C_k)^{kn(k)} r$ holds (see (4.2) for the definition of C_k). Moreover, the polynomial $(\prod_{i=1}^{k}(z - z_i)^{n(k)}) f_k(z)$ is the sum of k^2 terms and since the coefficients $a_{i,k} < 1$, the modulus of each of the k^2 terms may be estimated from above by $(r + \max_{i=1,\ldots,k} |z_i|)^{kn(k)}$ when (z, w) belongs to B. Using the same argument as above we conclude that

$|(\prod_{i=1}^{k}(z-z_i)^{n(k)})f_k(z)| \le k^2(r+C_k)^{kn(k)}$. Therefore the functions

$$\sum_{k=1}^{N} \max\left\{\frac{1}{n(k+1)}\log|p_k(z,w)|, -1\right\}$$

are plurisubharmonic in B and uniformly bounded from above by

$$\sum_{k=1}^{\infty} \frac{1}{n(k+1)}\log((r+k^2)(r+C_k)^{kn(k)}).$$

The latter series is convergent in view of conditions (4.3) and (4.4). Lemma 6 follows. ∎

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References

[1] F. Forstneric and J.-P. Rosay, *Approximation of biholomorphic mappings by automorphisms of* $\mathbb{C}^n$, Invent. Math 112 (1993), 323–349.
[2] M. W. Hirsch, *Differential Topology*, Springer, New York, 1976.
[3] N. Levenberg, G. Martin and E. A. Poletsky, *Analytic disks and pluripolar sets*, Indiana Univ. Math. J. 41 (1992), 515–532.
[4] J. Wermer, *The hull of a curve in* $\mathbb{C}^n$, Ann. of Math. 68 (1958), 550–561.

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