ANNALES
POLONICI MATHEMATICI
84.1 (2004)

A note on the Hörmander, Donnelly-Fefferman, and Berndtsson L^2-estimates for the $\bar\partial$-operator

by Zbigniew Błocki (Kraków)

Abstract. We give upper and lower bounds for constants appearing in the L^2-estimates for the $\bar\partial$-operator due to Donnelly–Fefferman and Berndtsson.

1. Introduction. Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and suppose that a form

$$\alpha=\sum_{j=1}^n \alpha_j d\bar z_j\in L^2_{\mathrm{loc},(0,1)}(\Omega)$$

is $\bar\partial$-closed (that is, $\bar\partial\alpha=0$, which means that $\partial\alpha_j/\partial\bar z_k=\partial\alpha_k/\partial\bar z_j$, $j,k=1,\dots,n$). The equation

$$\bar\partial u=\alpha \tag{1}$$

(which is equivalent to the system of equations $\partial u/\partial\bar z_j=\alpha_j$, $j=1,\dots,n$) always has a solution $u\in L^2_{\mathrm{loc},(0,1)}$ and the difference of any two solutions of (1) is a holomorphic function in Ω (see [6]). A slight modification of the proof of Hörmander's estimate [6, Lemma 4.4.1] (see e.g. [4, Théorème 4.1]) shows that for every smooth, strongly plurisubharmonic function φ in Ω we can find a solution to (1) satisfying

$$\int_\Omega |u|^2e^{-\varphi}\,d\lambda\le\int_\Omega|\alpha|^2_{i\partial\bar\partial\varphi}e^{-\varphi}\,d\lambda. \tag{H}$$

By $|\alpha|_{i\partial\bar\partial\varphi}$ we understand the pointwise norm of α with respect to the Kähler metric $i\partial\bar\partial\varphi$, that is,

$$|\alpha|^2_{i\partial\bar\partial\varphi}=\sum_{j,k=1}^n\varphi^{j\bar k}\bar\alpha_j\alpha_k,$$

2000 *Mathematics Subject Classification*: Primary 32W05.

Key words and phrases: $\bar\partial$-equation, plurisubharmonic function, L^2-estimate.

Partially supported by KBN Grant #2P03A03726.

where $(\varphi^{j\bar{k}})$ is the inverse transposed matrix of $(\partial^2\varphi/\partial z_j\partial\bar{z}_k)$. The function $|\alpha|^2_{i\partial\bar\partial\varphi}$ is the least function H satisfying

$$i\overline{\alpha}\wedge\alpha \le Hi\partial\bar\partial\varphi, \tag{2}$$

and one can obtain the estimate (H) for an arbitrary plurisubharmonic function φ in Ω, where instead of $|\alpha|^2_{i\partial\bar\partial\varphi}$ we take a function H satisfying (2) (see [3] for the approximation argument based on the proof of [6, Theorem 4.4.2]).

A very useful variation of the Hörmander estimate (H) was proved by Donnelly and Fefferman [5]. Let in addition ψ be a plurisubharmonic function in Ω satisfying

$$i\partial\psi\wedge\bar\partial\psi \le i\partial\bar\partial\psi.$$

This is equivalent to the fact that the function $-e^{-\psi}$ is plurisubharmonic, that is,

$$\psi = -\log(-v)$$

for a certain negative plurisubharmonic function v in Ω. Then one can find a solution to (1) with

$$\int_\Omega |u|^2e^{-\varphi}\,d\lambda \le C\int_\Omega |\alpha|^2_{i\partial\bar\partial\psi}e^{-\varphi}\,d\lambda, \tag{DF}$$

where C is an absolute constant.

Berndtsson [1] showed that for any δ with $0<\delta<1$ one can find a solution to (1) with

$$\int_\Omega |u|^2e^{-\varphi+\delta\psi}\,d\lambda \le \frac{4}{\delta(1-\delta)^2}\int_\Omega |\alpha|^2_{i\partial\bar\partial\psi}e^{-\varphi+\delta\psi}\,d\lambda, \tag{B}$$

where φ and ψ are as above. The Berndtsson estimate easily implies the Donnelly–Fefferman estimate—it is enough to consider the function $\varphi+\delta\psi$ instead of φ. The best choice for δ is then $\delta=1/3$, one then gets $C=27$ in the Donnelly–Fefferman estimate. In [2] Berndtsson showed that the estimate (B) follows easily from the Hörmander estimate (H). Using his arguments it was shown in [3] that the constant in the Berndtsson estimate can be improved to $1/\delta(1-\sqrt{\delta})^2$. From this with $\delta=1/4$ one gets $C=16$ in (DF).

By $C_{\rm B}(\delta)$ denote the best constant in the Berndtsson estimate. Then $C_{\rm DF}=C_{\rm B}(0)$ is the best constant in the Donnelly–Fefferman estimate. The goal of this note is to show the following result.

PROPOSITION. *We have*

$$\frac{4}{(1-\delta)(2-\delta)} \le C_{\rm B}(\delta) \le \frac{4}{(1-\delta)^2}, \qquad 0\le\delta<1.$$

COROLLARY. $2\le C_{\rm DF}\le 4.$

Note that

$$\frac{4}{(1-\delta)^2} < \frac{1}{\delta(1-\sqrt{\delta})^2} < \frac{4}{\delta(1-\delta)^2}, \qquad 0<\delta<1,$$

so the upper bound is an improvement of the constants from [1] and [3]. Concerning the lower bound, it was noted already in [1] that the best constant cannot be better than $C/(1-\delta)$, so that in particular the Berndtsson estimate does not hold for $\delta = 1$.

2. Proofs. Using the Berndtsson argument (see the proof of [2, Lemma 2.2]) we first prove the estimate

$$\int_\Omega |u|^2 e^{-\varphi+\delta\psi}\,d\lambda \le \frac{4}{(1-\delta)^2}\int_\Omega He^{-\varphi+\delta\psi}\,d\lambda, \tag{3}$$

where $i\overline{\alpha}\wedge\alpha \le Hi\partial\bar\partial\psi$, that is, the upper bound in the proposition. We will just choose the constants more carefully than in [2]. Due to the approximation argument from [3] we may assume that Ω is bounded and φ,ψ are smooth and continuous up to the boundary. Then for any real a we have the equality of sets

$$L^2(\Omega, e^{-\varphi-a\psi}) = L^2(\Omega).$$

Let u be the minimal solution to (1) in the $L^2(\Omega, e^{-\varphi-a\psi})$-norm ($a$ will be specified later). This means that u is perpendicular to the subspace $H^2(\Omega)$ of square integrable holomorphic functions in Ω in the Hilbert space $L^2(\Omega, e^{-\varphi-a\psi})$, that is,

$$\int_\Omega u\bar f e^{-\varphi-a\psi}\,d\lambda = 0, \qquad f\in H^2(\Omega).$$

Let $v := e^{b\psi}u$, where $b\in\mathbb{R}$ will be specified later. Then

$$\int_\Omega v\bar f e^{-\varphi-(a+b)\psi}\,d\lambda = 0, \qquad f\in H^2(\Omega).$$

This means that v is a minimal solution to the equation

$$\bar\partial v = \beta$$

in the $L^2(\Omega, e^{-\varphi-(a+b)\psi})$-norm, where

$$\beta = \bar\partial(e^{b\psi}u) = e^{b\psi}(\alpha + bu\bar\partial\psi).$$

If P,Q are any $(1,0)$-forms then for any $t>0$ we have

$$\begin{aligned} &i(P+Q)\wedge(\overline{P}+\overline{Q}) \\ &\quad= (1+t)iP\wedge\overline{P} + (1+t^{-1})iQ\wedge\overline{Q} - ti(P-t^{-1}Q)\wedge(\overline{P}-t^{-1}\overline{Q}) \\ &\quad\le (1+t)iP\wedge\overline{P} + (1+t^{-1})iQ\wedge\overline{Q}. \end{aligned}$$

Therefore

$$\begin{aligned} i\bar{\beta}\wedge\beta &\le e^{2b\psi}[(1+t)i\overline{\alpha}\wedge\alpha+(1+t^{-1})b^2|u|^2 i\partial\psi\wedge\bar{\partial}\psi] \\ &\le e^{2b\psi}[(1+t)H+(1+t^{-1})b^2|u|^2]i\partial\bar{\partial}\psi \\ &\le \frac{e^{2b\psi}}{a+b}[(1+t)H+(1+t^{-1})b^2|u|^2]i\partial\bar{\partial}(\varphi+(a+b)\psi) \end{aligned}$$

provided that $a+b>0$. From the Hörmander estimate (H) applied to the form β and the function $\varphi+(a+b)\psi$ we obtain

$$\int_\Omega |v|^2 e^{-\varphi-(a+b)\psi}\,d\lambda \le \frac{1}{a+b}\int_\Omega [(1+t)H+(1+t^{-1})b^2|u|^2]e^{-\varphi+(b-a)\psi}\,d\lambda.$$

Thus, taking $b=a+\delta$, we get

$$\begin{aligned} \int_\Omega |u|^2 e^{-\varphi+\delta\psi}\,d\lambda &\le \frac{1+t}{2a+\delta}\int_\Omega He^{-\varphi+\delta\psi}\,d\lambda \\ &\quad + \frac{(1+t^{-1})(a+\delta)^2}{2a+\delta}\int_\Omega |u|^2 e^{-\varphi+\delta\psi}\,d\lambda. \end{aligned}$$

We now only have to minimize the positive values of the function

$$\frac{\frac{1+t}{2a+\delta}}{1-\frac{(1+t^{-1})(a+\delta)^2}{2a+\delta}} = \frac{t(1+t)}{t(2a+\delta)-(1+t)(a+\delta)^2}$$

for $t>0$ and $a>-\delta/2$. The minimum is easily shown to be attained for $a=-\delta+t/(1+t)$ and $t=(1+\delta)/(1-\delta)$ (then $a=(1-\delta)/2$). For these values of a and t we obtain (3).

To get the lower bound in the proposition we will use the following lemma.

Lemma. *Let $\Omega=\Delta$ be the unit disc in $\mathbb{C}$. Set $\alpha=d\overline{z}$ and assume that F is a nonnegative, continuous, radially symmetric (that is, $F(z)=\gamma(|z|)$) function in Δ. Then the function $u(z)=\overline{z}$ is the minimal solution to* (1) *in the $L^2(\Delta,F)$-norm (provided that u belongs to $L^2(\Delta,F)$, that is, $\int_0^1 r^3\gamma(r)\,dr<\infty$).*

Proof. We have to show that

$$\int f\overline{u}F\,d\lambda = 0, \qquad f\in\mathcal{O}(\Delta)\cap L^2(\Delta,F).$$

Write

$$f(z)=\sum_{n=0}^{\infty} a_n z^n, \qquad z\in\Delta,$$

where the convergence is uniform on every circle in Δ. Therefore

$$\int f\overline{u}F\,d\lambda = 2\pi\int_0^1\sum_{n=0}^{\infty} a_n r^{n+2}\gamma(r)\int_0^{2\pi} e^{i(n+1)t}\,dt\,dr = 0. \ \blacksquare$$

We now consider the estimate (B) with $n=1$, $\Omega=\Delta$, $\varphi=0$ and $\psi(z)=-\log(-\log|z|)$. In this case the least value of the left-hand side of (B) is attained for $u(z)=\overline{z}$. Then

$$\int_\Delta |u|^2 e^{-\varphi+\delta\psi}\,d\lambda = 2\pi\int_0^1 r^3(-\log r)^{-\delta}\,dr$$

and

$$\int_\Delta \frac{|\alpha|^2}{\psi_{z\overline{z}}}\,e^{-\varphi+\delta\psi}\,d\lambda = 8\pi\int_0^1 r^3(-\log r)^{2-\delta}\,dr = \pi\frac{(2-\delta)(1-\delta)}{2}\int_0^1 r^3(-\log r)^{-\delta}\,dr$$

after double integration by parts.

References

[1] B. Berndtsson, *The extension theorem of Ohsawa–Takegoshi and the theorem of Donnelly–Fefferman*, Ann. Inst. Fourier (Grenoble) 46 (1996), 1083–1094.

[2] —, *Weighted estimates for the $\bar\partial$-equation*, in: Complex Analysis and Geometry (Columbus, OH, 1999), Ohio State Univ. Math. Res. Inst. Publ. 9, de Gruyter, 2001, 43–57.

[3] Z. Błocki, *The Bergman metric and the pluricomplex Green function*, Trans. Amer. Math. Soc., to appear.

[4] J.-P. Demailly, *Estimations L^2 pour l'opérateur $\bar\partial$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète*, Ann. Sci. École Norm. Sup. 15 (1982), 457–511.

[5] H. Donnelly and C. Fefferman, *L^2-cohomology and index theorem for the Bergman metric*, Ann. of Math. 118 (1983), 593–618.

[6] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, D. van Nostrand, Princeton, 1966.

Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: blocki@im.uj.edu.pl

Reçu par la Rédaction le 3.6.2004 (1523)