

Geometry of quotient spaces and proximality

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Abstract. It is proved that if X is a rotund Banach space and M is a closed and proximal subspace of X , then the quotient space X/M is also rotund. It is also shown that if Φ does not satisfy the δ_2 -condition, then h_Φ^0 is not proximal in l_Φ^0 and the quotient space l_Φ^0/h_Φ^0 is not rotund (even if l_Φ^0 is rotund). Weakly nearly uniform convexity and weakly uniform Kadec–Klee property are introduced and it is proved that a Banach space X is weakly nearly uniformly convex if and only if it is reflexive and it has the weakly uniform Kadec–Klee property. It is noted that the quotient space X/M with X and M as above is weakly nearly uniformly convex whenever X is weakly nearly uniformly convex. Criteria for weakly nearly uniform convexity of Orlicz sequence spaces equipped with the Orlicz norm are given.

1. Introduction. The notions of nearly uniform convexity (NUC for short) and uniform Kadec–Klee property (UKK) play an important role in some branches of mathematics, for example in fixed point theory (see [1] and [8]). We introduce two new geometric properties of Banach spaces: weakly nearly uniform convexity (WNUC for short) and another property, called the weakly uniform Kadec–Klee property (WUKK). We prove that a Banach space X is WNUC if and only if X is reflexive and it has the WUKK-property.

Let $(X, \|\cdot\|)$ be a Banach space and X^* be its dual space. By $B(X)$ and $S(X)$ we denote the unit ball and unit sphere of X , respectively. Recall that $x \in S(X)$ is said to be an H -point if for any sequence $\{x_n\}$ in $S(X)$ with $x_n \xrightarrow{w} x$, we have $\|x_n - x\| \rightarrow 0$. A Banach space X is said to have the *Kadec–Klee property* (property H for short) if any point of $S(X)$ is an H-point. A sequence $\{x_n\}$ in X is said to be ε -separated ($\varepsilon > 0$) if $\text{sep}(\{x_n\}) := \inf\{\|x_m - x_n\| : m \neq n\} > \varepsilon$. Further, X is said to have

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the *uniform Kadec–Klee property* (UKK for short) if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if x is a weak limit of an ε -separated sequence in $S(X)$, then $\|x\| \leq 1 - \delta(\varepsilon)$. It is well known that Banach spaces with UKK have weakly normal structure (see [1]). Replacing in the definition of UKK the phrase “for any $\varepsilon > 0$ ” by “for some $\varepsilon \in (0, 1)$ ” we define the *weakly uniform Kadec–Klee property* (WUKK for short). Also the WUKK-property of a Banach space X implies that X has weakly normal structure (the proof remains the same as for UKK).

The notion of nearly uniform convexity of a Banach space was introduced in [11]. It is an infinite-dimensional counterpart of the classical uniform convexity introduced by Clarkson in [3]. Recall that a Banach space X is said to be NUC if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any sequence $\{x_n\}$ in $B(X)$ with $\text{sep}(\{x_n\}) \geq \varepsilon$ we have $\text{conv}(\{x_n\}) \cap \delta B(X) \neq \emptyset$. It is easy to see that NUC implies UKK for any Banach space. Huff [11] has proved that a Banach space X is NUC if and only if it is reflexive and it has UKK. Define a Banach space X to be *weakly nearly uniformly convex* if there are $\varepsilon \in (0, 1)$ and $\delta \in (0, 1)$ such that $\text{conv}(\{x_n\}) \cap \delta B(X) \neq \emptyset$ for every sequence $\{x_n\}$ in $B(X)$ with $\text{sep}(\{x_n\}) \geq \varepsilon$.

Let M be a closed subspace of a Banach space X . We denote by X/M the quotient space of X modulo M . It is well known that X/M equipped with the norm $\|[x]\| = \inf\{\|y\| : y \in [x]\}$, where $[x] = \{y \in X : y - x \in M\}$, is also a Banach space. The subspace M of X is called *proximal* in X if for any $x \in X$ there is $y \in M$ such that $\|[x]\| = \|x - y\|$.

A point $x \in S(X)$ is said to be an *extreme point* of $B(X)$ if for any $y, z \in S(X)$ such that $x = (y + z)/2$ we have $z = y = x$. A Banach space X is called *rotund* (R for short) if any point of $S(X)$ is an extreme point of $B(X)$.

Let l^0 be the space of all real sequences. A Banach space X is called a *Köthe sequence space* if $X \subset l^0$ and for any $x \in l^0$ and $y \in X$ with $|x(i)| \leq |y(i)|$ for all $i \in \mathbb{N}$, we have $x \in X$ and $\|x\| \leq \|y\|$ (see [18]). Since we are interested in infinite-dimensional Köthe sequence spaces X , we may assume without loss of generality that there is an element $x \in X$ with $x(i) > 0$ for any $i \in \mathbb{N}$.

For any $x \in l^0$ we denote by $x^{(n)}$ the element

$$\overbrace{(0, \dots, 0, x(n+1), x(n+2), \dots)}^{n \text{ times}}.$$

We say that x is *absolutely continuous* if $x^{(n)} \rightarrow 0$ in X . We denote by X_a the space of all absolutely continuous elements in X . A Köthe sequence space X is said to be *absolutely continuous* if $X_a = X$. We say that X has the *semi-Fatou property* if for any sequence $\{x_n\}$ in X and any $x \in X$ satisfying $|x_n(i)| \uparrow |x(i)|$ for all $i \in \mathbb{N}$, we have $\|x_n\| \rightarrow \|x\|$.

A mapping $\Phi : \mathbb{R} \rightarrow [0, \infty)$ is said to be an *Orlicz function* if Φ vanishes only at zero, is even and convex.

Given any Orlicz function Φ we define on l^0 the convex modular I_Φ by

$$I_\Phi(x) = \sum_{i=1}^{\infty} \Phi(x(i)).$$

The linear space l_Φ defined by

$$l_\Phi = \{x \in l^0 : I_\Phi(kx) < \infty \text{ for some } k > 0\}$$

is called the *Orlicz space* generated by Φ (see [2] and [12]–[17]). We consider l_Φ equipped with the *Amemiya–Orlicz norm*

$$\|x\|^0 = \inf \left\{ \frac{1}{k} (1 + I_\Phi(kx)) : k > 0 \right\}.$$

To simplify notation we write l_Φ^0 in place of $(l_\Phi^0, \|\cdot\|^0)$. The *Luxemburg norm* in l_Φ is defined by

$$\|x\| = \inf \{ \lambda > 0 : I_\Phi(x/\lambda) \leq 1 \}.$$

For any x in $l_\Phi^0 \setminus \{0\}$ the set of all numbers $k > 0$ such that $\|x\|^0 = k^{-1}(1 + I_\Phi(kx))$ is denoted by $K(x)$. It is well known (see [2], [16] and [19]) that $K(x) = [k_x^*, k_x^{**}]$, where $k_x^* = \inf \{k > 0 : I_\Psi(p \circ k|x|) \geq 1\}$ and $k_x^{**} = \sup \{k > 0 : I_\Psi(p \circ k|x|) \leq 1\}$ if $k_x^* < \infty$ and $K(x) = \emptyset$ if $k_x^* = \infty$. Here Ψ denotes the function complementary to Φ in the sense of Young and p denotes the right derivative of Φ . It is also known that if Φ satisfies *condition* $(\infty_1) : \Phi(u)/u \rightarrow \infty$ as $u \rightarrow \infty$, then $K(x) \neq \emptyset$ for any $x \in l_\Phi \setminus \{0\}$.

We say an Orlicz function Φ satisfies the δ_2 -*condition* ($\Phi \in \delta_2$ for short) if there are $k \geq 2$ and $u_0 > 0$ such that $\Phi(2u) \leq k\Phi(u)$ whenever $|u| \leq u_0$. In what follows, h_Φ denotes the space of all $x \in l^0$ with $I_\Phi(kx) < \infty$ for all $k > 0$. It is easy to see that $h_\Phi = (l_\Phi)_a$. We write briefly h_Φ^0 in place of $(h_\Phi, \|\cdot\|^0)$. It is well known that $h_\Phi = l_\Phi$ if and only if $\Phi \in \delta_2$ (see [2] and [17]).

2. Some results on geometry of quotient spaces. Let us begin with the obvious observation that $\|[x]\| = d(x, M)$, where $d(x, M)$ denotes the distance of x from M . Next, we make the following easy remark.

REMARK 1. If X is a Banach space and M is its closed and proximal subspace, then there is $z \in S(X)$ such that $\|[z]\| = 1$.

This follows from Godini’s result [7] stating that M is proximal in X if and only if $q(B(X)) = B(X/M)$, where q is the canonical map $X \ni x \mapsto q(x) := [x] \in X/M$. However, for completeness we present a short proof.

Proof. Take $x \in X \setminus M$. By the proximality of M in X there is $y \in M$ such that $\|x - y\| = d(x, M)$. Define $z = (x - y)/\|x - y\|$. Then $\|z\| = 1$ and

$$\|[z]\| = \frac{1}{\|x - y\|} \|[x - y]\| = \frac{\|[x]\|}{\|x - y\|} = \frac{\|x - y\|}{\|x - y\|} = 1.$$

EXAMPLE 1. Denote by $C[0, 1]$ the space of all continuous real functions on the interval $[0, 1]$ equipped with the norm $\|x\|_\infty = \sup_{0 \leq x \leq 1} |x(t)|$. Define $X = \{x \in C[0, 1] : x(0) = 0\}$ and $M = \{x \in X : \int_0^1 x(t) dt = 0\}$. It is obvious that X is a Banach space and M is its closed subspace. It is known (see [18, pp. 64–65]) that there is no point on the unit sphere in X with unit distance from M . Therefore, by Remark 1, M is not proximal in X .

THEOREM 1. *If X is a rotund Banach space and M is its closed and proximal subspace, then the Banach space X/M is also rotund.*

Proof. Let $[x], [y] \in S(X/M)$, $[x] \neq [y]$, i.e. $x - y \notin M$. By the proximality of M in X , there are $x' \in [x]$ and $y' \in [y]$ such that $\|[x]\| = \|x'\| = \|y'\| = \|[y]\| = 1$. It follows from $[x] \neq [y]$ that $x' \neq y'$. Since X is rotund we get $\|\frac{x' + y'}{2}\| < 1$, whence

$$\left\| \frac{[x] + [y]}{2} \right\| = \left\| \left[\frac{x + y}{2} \right] \right\| \leq \left\| \frac{x' + y'}{2} \right\| < 1,$$

which means that X/M is rotund.

The next theorem shows that the assumption in Theorem 1 that M is proximal is essential in general.

THEOREM 2. *If Φ is an Orlicz function vanishing only at zero such that $\Phi \notin \delta_2$ and Φ satisfies condition (∞_1) , then h_Φ^0 is not proximal in l_Φ^0 and l_Φ^0/h_Φ^0 is not rotund.*

Proof. It is well known that the spaces l_Φ/h_Φ and l_Φ^0/h_Φ^0 are isometric under the identity map (see [2] and [17]). Recall (see [2]) that

$$(1) \quad \|x\| < \|x\|^0 \quad \text{for any } x \in l_\Phi \setminus \{0\}.$$

Assume that h_Φ^0 is proximal in l_Φ^0 and take any $x \in l_\Phi^0 \setminus h_\Phi^0$. We can find $y \in h_\Phi^0$ such that $\|[x]\| = \|x - y\|^0$. Hence,

$$\|x - y\|^0 = \|[x]\| \leq \|x - y\|,$$

which yields $\|x - y\|^0 = \|x - y\|$, contradicting (1).

Now, we show that l_Φ^0/h_Φ^0 is not rotund if $\Phi \notin \delta_2$ (even if Φ is strictly convex). It is well known (see [2]) that if $\Phi \notin \delta_2$, then one can find $x, y \in l_\Phi^0$ such that $\text{supp } x \subset \{2k - 1 : k \in \mathbb{N}\}$, $\text{supp } y \subset \{2k : k \in \mathbb{N}\}$, $I_\Phi(x) \leq 1/2$, $I_\Phi(y) \leq 1/2$ and $I_\Phi(\lambda x) = I_\Phi(\lambda y) = \infty$ for all $\lambda > 1$. Defining $w = x + y$ and $z = x - y$, we have $\|[w]\| = \|[z]\| = 1$, because it is well known (see

again [2]) that $\|[w]\| = \inf\{\lambda > 0 : I_\Phi(w/\lambda) < \infty\}$. We have

$$[x] = \left[\frac{w+z}{2} \right] = \frac{[w] + [z]}{2}$$

and so $\|[x]\| = 1$. This completes the proof that l_Φ^0/h_Φ^0 is not rotund.

REMARK 2. It is well known (see [4]) that if Φ is an Orlicz function satisfying condition (∞_1) , then l_Φ^0 is rotund if and only if Φ is strictly convex on the interval $[0, \Pi_\Phi(1)]$ with $\Pi_\Phi(1) = \inf\{t > 0 : \Psi(p(t)) \geq 1\}$, where Ψ denotes the function complementary to Φ in the sense of Young and p denotes the right derivative of Φ on \mathbb{R}_+ . Therefore, taking an Orlicz function Φ which is strictly convex on the real line and $\Phi \notin \delta_2$, we find that l_Φ^0 is rotund, h_Φ^0 is not proximal in l_Φ^0 (see [9]), and consequently, the quotient space l_Φ^0/h_Φ^0 is not rotund. This explains the role of the proximality of M in X in Theorem 1.

As we will see below, the situation for weakly nearly uniform convexity is different than for rotundity. However, we first need the following result.

THEOREM 3. *A Banach space X is weakly nearly uniformly convex if and only if it is reflexive and has the weakly uniform Kadec–Klee property.*

Proof. Necessity. We first prove that X is reflexive whenever it is WNUC. Assume that, on the contrary, X is WNUC but nonreflexive. Then by the James Theorem (see [6]) there exist $\{x_n\}$ in $S(X)$ and $\{x_n^*\}$ in $S(X^*)$ such that

$$x_i^*(x_j) = \begin{cases} \theta & \text{if } i \leq j, \\ 0 & \text{if } i > j, \end{cases}$$

where $\theta \in (\max\{\varepsilon, \delta\}, 1)$, with ε arbitrary in $(0, 1)$, and δ from the definition of WNUC. Hence, for $n > m$ we get

$$\|x_n - x_m\| \geq x_n^*(x_n - x_m) = x_n^*(x_n) = \theta \geq \varepsilon,$$

which gives $\text{sep}(\{x_n\}) \geq \varepsilon$. But for any $x \in \text{conv}(\{x_n\})$ we have

$$\|x\| = \left\| \sum_{i=1}^m \lambda_i x_i \right\| \geq x_m^* \left(\sum_{i=1}^m \lambda_i x_i \right) = \sum_{i=1}^m \lambda_i x_m^*(x_i) = \sum_{i=1}^m \lambda_i \theta = \theta > \delta$$

for some $m \in \mathbb{N}$ and every $\lambda_i \geq 0$ ($i = 1, \dots, m$) with $\sum_{i=1}^m \lambda_i = 1$. This contradicts the condition $\text{conv}(\{x_n\}) \cap \delta B(X) \neq \emptyset$ from the definition of WNUC.

Let us prove that X has WUKK whenever it is WNUC. Assume that $X \in \text{WNUC}$, $\{x_n\}$ is a sequence in $B(X)$ with $\text{sep}(\{x_n\}) \geq \varepsilon$, $x \in B(X)$ and $x_n \xrightarrow{w} x$ as $n \rightarrow \infty$. We have $\text{sep}(\{x_n\}_{n=m}^\infty) \geq \text{sep}(\{x_n\}) \geq \varepsilon$ for all $m \in \mathbb{N}$. Fix $m = 1$. There is $y_1 \in \text{conv}(\{x_n\}_{n=m}^\infty) \cap \delta B(X)$, i.e. there are $m_1 \in \mathbb{N}$ and $\lambda_i \geq 0$ for $i = 1, \dots, m_1$ such that $\sum_{i=1}^{m_1} \lambda_i = 1$ and $y_1 := \sum_{i=1}^{m_1} \lambda_i x_i$ satisfies $\|y_1\| \leq \delta$. Fix $m = m_1 + 1$. There is $y_2 \in \text{conv}(\{x_n\}_{n=m_1+1}^\infty) \cap \delta B(X)$, i.e.

there are $m_2 \in \mathbb{N}$, $m_2 \geq m_1 + 1$ and $\lambda_i \geq 0$ for $i = m_1 + 1, \dots, m_2$ with $\sum_{i=m_1+1}^{m_2} \lambda_i = 1$ such that $y_2 = \sum_{i=1}^{m_2} \lambda_i x_i$ satisfies $\|y_2\| \leq \delta$. In such a way we get, by induction, a sequence $\{y_n\}$ in $B(X) \cap \text{conv}(\{x_n\})$ of the form

$$y_n = \sum_{i=m_n+1}^{m_{n+1}} \lambda_i x_i, \quad m_{n+1} \geq m_n + 1 \quad (n = 1, 2, \dots)$$

with $\|y_n\| \leq \delta$ for all $n \in \mathbb{N}$. Since

$$x^*(y_n) = \sum_{i=m_n+1}^{m_{n+1}} \lambda_i x^*(x_i) \rightarrow \sum_{i=m_n+1}^{m_{n+1}} \lambda_i x^*(x) = x^*(x) \quad \text{as } n \rightarrow \infty$$

for any $x^* \in X^*$, and $\|y_n\| \leq \delta$ for all $n \in \mathbb{N}$, we get $\|x\| \leq \delta$. Therefore X has the weakly uniform Kadec–Klee property.

Sufficiency. Since X has WUKK, there exist $\varepsilon \in (0, 1)$ and $\delta \in (0, 1)$ such that for every sequence $\{x_n\}$ with $\text{sep}(\{x_n\}) \geq \varepsilon$ and $x \in B(X)$ such that $x_n \xrightarrow{w} x$, we have $\|x\| \leq \delta$.

Assume that $\{x_n\}$ is a sequence in $B(X)$ with $\text{sep}(\{x_n\}) \geq \varepsilon$. By the reflexivity of X there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $x \in B(X)$ such that $x_{n_i} \xrightarrow{w} x$. It is obvious that $\text{sep}(\{x_{n_i}\}) \geq \text{sep}(\{x_n\}) \geq \varepsilon$. Hence $\|x\| \leq \delta$. Since $x_{n_i} \xrightarrow{w} x$, we see that if $\{y_n\}$ is a sequence in $\text{conv}(\{x_{n_1}\})$, then $y_n \xrightarrow{w} x$. By the Mazur theorem we conclude that $x \in \text{conv}(\{x_n\}) \cap \delta B(X)$, which means that X is NUC. This finishes the proof of the theorem.

THEOREM 4. *If a Banach space X is weakly nearly uniformly convex and M is a closed subspace of X , then X/M has the weakly uniform Kadec–Klee property, and X/M is reflexive, so nearly uniformly convex as well.*

Proof. By Theorem 3, X is reflexive. So X/M is also reflexive, and to prove that it is WNUC, we need only show that X/M has WUKK. Since X has WUKK, there exist $\varepsilon \in (0, 1)$ and $\delta \in (0, 1)$ such that if $\{x_n\}$ is a sequence in $B(X)$ with $\text{sep}(\{x_n\}) \geq \varepsilon$ and $x_n \xrightarrow{w} x \in B(X)$, then $\|x\| \leq \delta$. Assume that $\{[x_n]\}$ is a sequence in $B(X/M)$ with $\text{sep}(\{[x_n]\}) \geq \varepsilon$ and $x \in B(X/M)$ is such that $[x_n] \xrightarrow{w} [x]$. Since X is reflexive, there is $x' \in B(X)$ such that $x_{n_i} \xrightarrow{w} x'$. Therefore $[x'] = [x]$, which shows that $x_{n_i} \xrightarrow{w} x$. By the double subsequence theorem, we get $x_n \xrightarrow{w} x$. Note that $\text{sep}(\{x_n\}) \geq \text{sep}(\{[x_n]\}) \geq \varepsilon$. By the assumption that $X \in \text{WNUC}$ we get $\|x\| \leq \delta$, whence $\|[x]\| \leq \|x\| \leq \delta$, which shows that X/M is WNUC.

THEOREM 5. *If X is a Köthe sequence space with the semi-Fatou property and the weakly uniform Kadec–Klee property, then X is absolutely continuous.*

Proof. Since X has WUKK there are $\varepsilon, \delta \in (0, 1)$ such that for every sequence $\{x_n\}$ in $B(X)$ with $\text{sep}(\{x_n\}) \geq \varepsilon$ and $x_n \xrightarrow{w} x \in B(X)$, we have

$\|x\| \leq \delta$. Suppose that X is not absolutely continuous. Since X_a is a closed subspace of X , by the Riesz Lemma, defining $\eta = \max(\varepsilon, \delta)$ one can find $x_0 \in S(X)$ such that

$$\|x_0 - y\| > \eta^{1/2} \quad \text{for all } y \in X_a.$$

Hence, by the inequality $\eta < \eta^{1/2}$, we have

$$\lim_{n \rightarrow \infty} \left\| \overbrace{(0, \dots, 0, x_0(n), x_0(n+1), \dots)}^{n-1 \text{ times}} \right\| > \eta.$$

Since X has the semi-Fatou property, there is $n_1 \in \mathbb{N}$ such that

$$\left\| \sum_{i=1}^{n_1} x_0(i)e_i \right\| \geq \eta.$$

Notice that

$$\lim_{m \rightarrow \infty} \left\| \sum_{i=n_1+1}^m x_0(i)e_i \right\| = \left\| \overbrace{(0, \dots, 0, x_0(n_1+1), x_0(n_1+2), \dots)}^{n_1 \text{ times}} \right\| > \eta.$$

So, there exists $n_2 > n_1$ such that

$$\left\| \sum_{i=n_1+1}^{n_2} x_0(i)e_i \right\| \geq \eta.$$

In such a way, we get by induction an increasing sequence $\{n_i\}$ of natural numbers such that

$$\left\| \sum_{i=n_i+1}^{n_{i+1}} x_0(i)e_i \right\| \geq \eta \quad (i = 1, 2, \dots).$$

Define $x_i = \sum_{j=n_i+1}^{n_{i+1}} x_0(j)e_j$. Then

$$(2) \quad \|x_i\| \geq \eta \quad \text{for all } i \in \mathbb{N}.$$

We will show that

$$(3) \quad x_i \xrightarrow{w} x_0 \quad \text{as } i \rightarrow \infty.$$

It is well known that for any Köthe sequence space (see [12]) we have

$$X^* = X' \oplus S,$$

where S is the space of all singular functionals over X , i.e. the functionals which vanish on the subspace X_a , and

$$X' = \left\{ y \in l^0 : \sum_{i=1}^{\infty} |x(i)y(i)| < \infty \text{ for any } x \in X \right\},$$

that is, X' is the Köthe dual of X . This means that every $x^* \in X^*$ is uniquely represented in the form

$$x^* = T_y + \phi,$$

where $\phi \in S$ and T_y is the functional generated by an element $y \in X'$ by the formula

$$T_y(x) = \sum_{i=1}^{\infty} x(i)y(i) \quad (\forall x = \{x(i)\}_{i=1}^{\infty} \in X).$$

It is obvious that $\phi(x_i) = 0$ for any $\phi \in S$ and all $i \in \mathbb{N}$. Moreover

$$T_y(x_i) = \sum_{j=n_i+1}^{n_{i+1}} x_0(j)y(j) \rightarrow 0 \quad \text{as } i \rightarrow \infty$$

because the series $\sum_{j=1}^{\infty} x_0(j)y(j)$ converges. This means that $x_i \xrightarrow{w} 0$, i.e. condition (3) holds. Defining $z_i = x_0 - x_i$ ($i = 1, 2, \dots$), we have $\|z_i\| \leq 1$ for $i = 1, 2, \dots$, $z_i \xrightarrow{w} x_0$, $\text{sep}(\{z_i\}) \geq \sup\{\|x_i\| : i \in \mathbb{N}\} \geq \eta \geq \varepsilon$ and $x_0 \in S(X)$, a contradiction finishing the proof.

THEOREM 6. *For the Orlicz sequence space l_{Φ}^0 the following assertions are equivalent:*

- (i) l_{Φ}^0 has the uniform Kadec–Klee property,
- (ii) l_{Φ}^0 has the weakly uniform Kadec–Klee property,
- (iii) $\Phi \in \delta_2$.

Proof. The implication (i) \Rightarrow (ii) is trivial. Since, by Theorem 5 and the fact that l_{Φ}^0 has the semi-Fatou property, the weakly uniform Kadec–Klee property implies order continuity (see Theorem 5), and since order continuity of l_{Φ}^0 is equivalent to $\Phi \in \delta_2$, the implication (ii) \Rightarrow (iii) is true. So, we need only prove the implication (iii) \Rightarrow (i).

We first prove that if $x \in l_{\Phi}^0 \setminus \{0\}$ and $K(x) = \emptyset$, then $\text{Card}(N(x)) < \infty$, where $N(x) = \text{supp } x$, and

$$(4) \quad \|x\|^0 = A \sum_{i \in N(x)} |x(i)|,$$

where $A = \lim_{u \rightarrow \infty} \Phi(u)/u$. Note that if $K(x) = \emptyset$ we must have $A < \infty$. It is clear that if $\text{Card}(N(x)) = \infty$, then there exists $k > 0$ such that $I_{\Psi}(p \circ k|x|) \geq 1$, i.e. $k_x^* < \infty$. Hence $K(x) \neq \emptyset$, a contradiction, which proves that $K(x) = \emptyset$ implies that $\text{Card}(N(x)) < \infty$. The fact that $K(x) = \emptyset$ yields (4) has been proved in [4].

Take an arbitrary $\varepsilon > 0$, any sequence $\{x_n\}$ in $B(l_{\Phi}^0)$ with $\text{sep}(\{x_n\}) > \varepsilon$ and any $x \in l_{\Phi}^0$ with $x_n \xrightarrow{w} x$. It is clear that $\text{sep}(\{\sum_{i=m}^{\infty} x_n(i)e_i\}) > \varepsilon$ for any $m \in \mathbb{N}$. Without loss of generality, we may assume that $\|\sum_{i=m}^{\infty} x_n(i)e_i\|^0 >$

$\varepsilon/2$ for all $m \in \mathbb{N}$. Since $\Phi \in \delta_2$, there exists $\varepsilon_1 \in (0, \varepsilon)$ depending only on ε such that $I_\Phi(\sum_{i=m}^\infty x_n(i)e_i) > \varepsilon_1$ for all $m \in \mathbb{N}$.

Take m so large that $\|\sum_{i=m}^\infty x_n(i)e_i\|^0 > \|x\|^0 - \varepsilon_1/4$. Since $x_n(i) \rightarrow x(i)$ for $i = 1, 2, \dots$, we conclude that there exists $m_0 \in \mathbb{N}$ such that $\|\sum_{i=1}^m x_n(i)e_i\|^0 > \|x\|^0 - \varepsilon_1/4$ for all $n > m_0$.

We divide the remaining part of the proof into two cases.

CASE I: $K(x_n) \neq \emptyset$ for $n = 1, 2, \dots$. Then for $k_n \in K(x_n)$ we have

$$\begin{aligned} 1 &\geq \|x_n\|^0 = \frac{1}{k_n} \left(1 + \sum_{i=1}^m \Phi(k_n x_n(i)) + \sum_{i=m+1}^\infty \Phi(k_n x_n(i)) \right) \\ &= \frac{1}{k_n} \left(1 + \sum_{i=1}^m \Phi(k_n x_n(i)) \right) + \frac{1}{k_n} \sum_{i=m+1}^\infty \Phi(k_n x_n(i)) \\ &\geq \left\| \sum_{i=1}^m x_n(i)e_i \right\|^0 + \sum_{i=m+1}^\infty \Phi(x_n(i)) > \|x\|^0 - \frac{\varepsilon_1}{4} + \varepsilon_1, \end{aligned}$$

whence $\|x\|^0 < 1 - \frac{3}{4}\varepsilon_1$ for $n > m$.

CASE II: $K(x_n) = \emptyset$ for $n = 1, 2, \dots$. Then

$$\begin{aligned} 1 &\geq \|x_n\|^0 = \left\| \sum_{i=1}^m x_n(i)e_i \right\|^0 + \left\| \sum_{i=m+1}^\infty x_n(i)e_i \right\|^0 \\ &> \|x\|^0 - \frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{2} = \|x\|^0 + \frac{\varepsilon_1}{4}, \end{aligned}$$

whence $\|x\|^0 < 1 - \varepsilon_1/4$ for all $n > m$.

Notice that it is enough to consider only cases I and II because defining $N_1 = \{n \in \mathbb{N} : K(x_n) \neq \emptyset\}$ and $N_2 = \mathbb{N} \setminus N_1$, we have $\text{Card}(N_1) = \infty$ or $\text{Card}(N_2) = \infty$. If $\text{Card}(N_i) = \infty$, we can assume without loss of generality that $N_i = \mathbb{N}$. The proof is finished.

Let us recall here that property UKK for Musielak–Orlicz sequence spaces equipped with the Luxemburg norm has been characterized in [5].

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