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On asymptotic solutions of analytic equations

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Abstract. Sufficient conditions for the existence of an analytic solution of analytic equations in the complex and real cases are given.

1. Introduction. Let $K = \mathbb{R}$ or $\mathbb{C}$. We will denote by $K\{x\}$ the ring of convergent power series in the variables $x = (x_1, \ldots, x_n)$ with coefficients in K, and by $K[[x]]$ the formal power series ring. We will denote by $\mathfrak{m}$ the maximal ideal of $K\{x\}$, and by $\widehat{\mathfrak{m}}$ the maximal ideal of $K[[x]]$. Consider an arbitrary system of analytic equations:

$$f(x, y) = 0, \tag{1.1}$$

where $f \in K\{x, y\}^s$, $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_p)$. We ask for solutions of (1.1) in which y_ν are convergent series in x. M. Artin showed in [A1] that any formal solution of (1.1) can be approximated to any desired degree of accuracy (in the (x)-adic topology) by a convergent solution. Namely we have

Artin's Approximation Theorem. *Let $f = (f_1, \ldots, f_s) \in K\{x, y\}^s$ be such that $f(0, 0) = 0$. Consider a solution $\widehat{y}(x) = (\widehat{y}_1(x), \ldots, \widehat{y}_p(x)) \in K[[x]]^p$ of the system $f(x, y) = 0$. Then for every integer $L \geq 1$ there exists a solution $y(x) = (y_1(x), \ldots, y_p(x)) \in K\{x\}^p$ of $f(x, y) = 0$ such that $y(x) = \widehat{y}(x) \pmod{\widehat{\mathfrak{m}}^L}$ (the congruence just means that the coefficients of monomials of degree less than L are the same in $y_\nu(x)$ and $\widehat{y}_\nu(x)$).*

The Artin Theorem is indispensable in the study of complex analytic structures, particularly in deformation theory where it is used to provide a transition from formal information to "actual" (i.e. convergent) information. There are some improvements of this theorem essentially stating that to determine whether a convergent solution exists one needs only a finite

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amount of information (compare with [W] and [A2] in the polynomial case) or sharper results with parameters (see [P]).

In this paper, for a fixed holomorphic function $F(z,t)$ defined in a neighborhood of $(a,b) \in \mathbb{C}^{n+1}_{z,t}$ we show the existence of a number $N \in \mathbb{N}$ such that the existence of a continuous quasi-solution g of order at least N (i.e. $F(z,g(z)) = O|z-a|^N$) implies the existence of a holomorphic solution (i.e. $F(z,h(z)) \equiv 0$ with some holomorphic function h defined in a neighborhood of a, and $h(a) = b$).

By the Weierstrass Preparation Theorem we can assume that $F(z,t) = t^k + a_1(z)t^{k-1} + \ldots + a_k(z)$ and $D(\operatorname{red} F) = HD_0$, where $\inf |H| > 0$, $D_0(u,s) = s^r + b^1(u)s^{r-1} + \ldots + b^r(u)$. Let $D(\operatorname{red} D_0)(u) = \sum_{|\mu| \geq q} c_\mu u^\mu$ ($c_q \neq 0$). In fact, we show that it suffices to take $N > krq$ in our theorem.

For the convenience of the reader we recall some basic definitions.

Consider a monic polynomial P in $t \in \mathbb{C}$ whose coefficients $a_1, \ldots, a_n$ are holomorphic functions in an open subset Ω of $\mathbb{C}^n$, i.e.

$$P(z,t) = t^n + a_1(z)t^{n-1} + \ldots + a_n(z).$$

The function

$$D(P)(z) = \prod_{i<j} (t_i(z) - t_j(z))^2 = (-1)^{\binom{k}{2}} \prod_{j=1}^{k} \frac{\partial P}{\partial t}(z, t_j(z)),$$

where $t_1(z), \ldots, t_k(z)$ is the complete sequence of roots of the polynomial $t \mapsto P(z,t)$, is called the *discriminant* of the polynomial P.

We say that a holomorphic function $f(z,t)$ in a neighborhood of zero in $\mathbb{C}^n \times \mathbb{C}$ is *t-regular* if $f(0,t) \not\equiv 0$ in a neighborhood of zero in $\mathbb{C}$.

Let $P(z,t)$ be a distinguished polynomial for which $DP(z) \equiv 0$ in some neighborhood of $0 \in \mathbb{C}^n$. Then there exists a distinguished polynomial $\operatorname{red} P(z,t)$ for which $D(\operatorname{red} P)(z) \not\equiv 0$ and $\{P = 0\} = \{\operatorname{red} P = 0\}$ in some neighborhood of 0.

Observe that our Theorem 1 implies the Artin Theorem in the case $p = s = 1$. Indeed, take N from our theorem for a convergent power series f. If $f(z, y(z)) = 0$ with some formal power series $y(z) = \sum a_\nu z^\nu$, then $f(z, \sum_{\nu \leq m} a_\nu z^\nu) \in \mathfrak{m}^N$ for sufficiently large m. Hence, knowing that $\sum_{\nu \leq m} a_\nu z^\nu$ is a continuous function, we get from Theorem 1 a convergent solution. As an application of Theorem 1 we also get a sufficient condition for the existence of a Nash solution in the real case (compare with [G]).

2. Main result

LEMMA 1. *Let K_0, K be open discs in $\mathbb{C}$, $\overline{K_0} \subset K$ and $a \in K_0$. Let $P(z,t)$ be a monic polynomial in t with coefficients holomorphic in K. Assume that $D(P)(z) \neq 0$ in $K \setminus \{a\}$. Then each function h holomorphic in $K \setminus \overline{K_0}$ for which $P(z,h(z)) \equiv 0$ extends holomorphically onto K.*

Proof. Each of the open sets $B_\nu = K \cap \Pi_\nu$, where Π_1, Π_2 and Π_3, Π_4 are open halfplanes cut off from $\mathbb{C}$ by two different lines passing through a, is homeomorphic to an open disc, hence each of the sets $W_{B_\nu} = W \cap (B_\nu \times \mathbb{C})$, where $W = \{P(z,t) = 0\}$, is a finite sum of graphs of holomorphic functions on B_ν. Therefore the restrictions of h to the open connected sets $B_{0\nu} = (K \setminus \overline{K}_0) \cap \Pi_\nu$ has holomorphic extensions h_ν onto B_ν, which are compatible. Thus $\bigcup h_\nu$ is an extension of h, which extends holomorphically to a by the Riemann Theorem. ∎

LEMMA 2. *Let $P(z,t)$ be a monic polynomial of degree k with holomorphic coefficients for which $D(\operatorname{red} P)(z) \neq 0$ in an open set $G \subset \mathbb{C}^n_z$. Define*

$$\tau(z) := \tfrac{1}{2} \min_{i \neq j} |t_i(z) - t_j(z)|, \qquad \eta(z,t) := \min_i |t - t_i(z)|,$$

where $t_1(z), \ldots, t_l(z)$ is the sequence of all different roots of the polynomial $t \mapsto P(z,t)$. If there exists a continuous function g for which $\eta(z, g(z)) < \tau(z)$ in G, then there exists a holomorphic function h such that $P(z, h(z)) \equiv 0$ and $|h(z) - g(z)| \leq |P(z, g(z))|^{1/k}$ in G.

Proof. It follows from the assumption that for each $a \in G$ there exists a uniquely determined root $h(a)$ of the polynomial $t \mapsto P(a,t)$ such that $|h(a) - g(a)| < \tau(a)$. Hence $|h(a) - g(a)| = \eta(a, g(a)) \leq |P(a, g(a))|^{1/k}$. Since g is a continuous function, the implicit function theorem shows that the root $h(z)$ must coincide in some neighborhood of a with a holomorphic root $t(z)$ of the polynomial P. ∎

THEOREM 1. *Let $F(z,t)$ be a holomorphic function in a neighborhood of $(a,b) \in \mathbb{C}^{n+1}_{z,t}$, t-regular in (a,b), and let $L > 0$. Then there exists $N \in \mathbb{N}$ such that if $F(z, g(z)) = O(|z-a|^N)$ as $z \to a$ with some function g continuous in a neighborhood of a such that $g(a) = b$, then there exists a function h holomorphic in a neighborhood of a such that $h(a) = b$ and $F(z, h(z)) \equiv 0$. Moreover, if g is holomorphic, then $h(z) - g(z) = O(|z-a|^L)$ as $z \to a$.*

Proof. We can assume that $a = b = 0$, and by the Weierstrass Preparation Theorem that $F(z,t) = t^k + a_1(z)t^{k-1} + \ldots + a_k(z)$ is a distinguished polynomial of degree $k > 0$ with bounded holomorphic coefficients a_j in some open neighborhood $\Omega \times \Delta$ of $0 \in \mathbb{C}^n$, where $\Omega \subset \mathbb{C}^{n-1}_u$ and $\Delta \subset \mathbb{C}^1_s$, $z = (u,s)$. We can also assume that the polynomial $\operatorname{red} F$ has bounded holomorphic coefficients in $\Omega \times \Delta$ and that $\{F = 0\} = \{\operatorname{red} F = 0\}$. Moreover, by the Weierstrass Preparation Theorem we can assume that $D(\operatorname{red} F) = HD_0$ in $\Omega \times \Delta$, where $D_0(u,s) = s^r + b_1(u)s^{r-1} + \ldots + b_r(u)$ is a distinguished polynomial with coefficients b_j holomorphic and bounded in Ω and $\inf |H| > 0$ in $\Omega \times \Delta$. Finally, we can assume that $\operatorname{red} D_0$ has coefficients holomorphic and bounded in Ω. Take τ and η from Lemma 2 for the polynomial $\operatorname{red} F$ in the set $D(\operatorname{red} F) \neq 0$. Of course in this set we have $|D(\operatorname{red} F)(z)| \leq M\tau(z)$ with some constant $M > 1$ and $|F(z,t)| \geq \eta(z,t)^k$. Hence, due to Lemma 2 for each open set $G \subset \{D(\operatorname{red} F) \neq 0\}$ the inequality

$$\text{(i)} \qquad |F(z,g(z))| < M^{-k}|D(\operatorname{red} F)(z)|^k$$

with some continuous function g on G gives us the existence of a holomorphic root h of the polynomial F in G such that

$$\text{(ii)} \qquad |h(z) - g(z)| \leq |F(z,g(z))|^{1/k} \quad \text{in } G.$$

Put $|u| = \sigma$. The roots of the polynomial $s \mapsto D_0(u,s)$ are $O(\sigma^{1/r})$. Hence, if we take M sufficiently large, there exists $\delta > 0$ such that for $\sigma > 0$ sufficiently small, on the set

$$G_\sigma := B_\sigma \times (K'_\sigma \setminus \overline{K}_\sigma),$$

where $B_\sigma = \{|u| < 2\delta\}$, $K_\sigma = \{|s| < M\sigma^{1/r}\}$, $K'_\sigma = \{|s| < 2M\sigma^{1/r}\}$, we have the inequality $|D_0(u,s)| \geq \delta\sigma$ and $G_\sigma \subset \Omega \times \Delta$. Let $D(\operatorname{red} D_0)(u) = \sum_{|\mu|\geq q} c_\mu u^\mu$. Hence there exists $\varepsilon \in (0,1)$ such that for sufficiently small $\sigma > 0$ we have

$$\max_{|u|=\sigma} |D(\operatorname{red} D_0)(u)| \geq 3\varepsilon\sigma^q.$$

It follows that for some u_0 with $|u_0| = \sigma$, we have $|D(\operatorname{red} D_0)(u)| \geq 3\varepsilon\sigma^q$. But when $\sigma > 0$ is sufficiently small, the roots $s_i(u_0)$ of the polynomial $s \mapsto D_0(u_0,s)$ satisfy $|s_i(u_0)| \leq 1/2$ and therefore $|s_i(u_0) - s_j(u_0)| \geq 3\varepsilon\sigma^q$ for $s_i(u_0) \neq s_j(u_0)$. Observe that for

$$K_{\sigma_i} = \{|s - s_i(u_0)| < \varepsilon\sigma^k\}$$

we have $\overline{K}_{\sigma_i} \cap \overline{K}_{\sigma_j} = \emptyset$ (for different roots) and in the complement of $\bigcup_i \overline{K}_{\sigma_i}$ we have the inequality $|D_0(u,s)| \geq \varepsilon^r\sigma^{rq}$. Finally, diminishing $\delta > 0$, when σ is sufficiently small, we have

$$|D(\operatorname{red} F)(z)| \geq \delta\sigma^{rq}$$

in

$$G = G_\sigma \cup \Big(U_\sigma \times \Big(K_\sigma \setminus \bigcup_i \overline{K}_{\sigma_i}\Big)\Big) \subset \Omega \times \Delta$$

for some open neighborhood $U_\sigma \subset B_\sigma$ of u_0. If we take $N > krq$, the condition $F(z,g(z)) = O(|z|^N)$ implies the inequality (i) for σ sufficiently small. Then there exists a holomorphic root h of the polynomial F in G. According to Lemma 1, if $u \in U_\sigma$, the function $h(u,\cdot)$ extends holomorphically onto $K_{\sigma'}$. Hence h extends holomorphically onto $U_\sigma \times K_{\sigma'}$ (use the Cauchy formula for $h(u,\cdot)$ on $\{|s| = \frac{3}{2}M\sigma^{1/2}\}$). Thus, from the Hartogs Lemma, h extends holomorphically onto $B_\sigma \times K_{\sigma'}$. In case g is holomorphic take moreover $N > kLr$. According to (ii) we have $|h(z) - g(z)| \leq \widetilde{M}|z|^{Lr}$

with some $\widetilde{M}$ in the set G_σ when σ is small enough. Hence in G_σ we have the inequality $|h(z) - g(z)| \le M'\sigma^L$ with some constant M', and due to the maximum principle this inequality also holds in $B_\sigma \times K_{\sigma'}$. So $h - g = O(|z|^L)$. ∎

REMARK. From the above proof it is easy to see that if $F(z,t)$ is a polynomial monic in t, then the $N \in \mathbb{N}$ can be taken the same for all points $z \in \mathbb{C}^n$.

THEOREM 2. *Let $Q(x,t)$ be a complex polynomial on $\mathbb{R}^{n+1}_{x,t}$, monic in t. Then there exists $m > 0$ such that if f, g are complex, continuous roots: $Q(x,f(x)) = Q(x,g(x)) = 0$ in a neighborhood of $a \in \mathbb{R}^n$, then*

$$f(x) - g(x) = O(|x-a|^m) \;\Rightarrow\; f = g$$

in a neighborhood of a.

Proof. We can assume that $a = 0$ and $D(Q)(z) \neq 0$ (replacing Q by $\operatorname{red} Q$). Obviously, when $|x| \le 1$ we have $|t''(x) - t'(x)| \ge M|D(Q)(x)|$ with some constant M for different roots $t''(x), t'(x)$ of the polynomial $t \mapsto Q(x,t)$. Take an open ball $B = \{|x| < R\}$, $R < 1$, in which f and g are defined. The number of connected components of the set $B \cap \{D(Q) \neq 0\}$ is finite and each connected component is semi-algebraic. Let S be one of them. Assume that $0 \in \overline{S}$. The function

$$d : r \mapsto \sup_{S(r)} |D(r)|,$$

where $S(r) = S \cap \{|x| = r\}$, is positive semi-algebraic. Hence $d(r) \ge r^{m-1}$ in $(0,\delta)$ with $m > 0$ and $\delta > 0$ common for all such components S. The assumption $f \not\equiv g$ in each neighborhood of zero implies $f \neq g$ on one of these components S, because the set $\{f = g\} \cap S$ is open-closed in S. But $|f(x) - g(x)| \ge M|D(Q)(x)|$ on S, which implies $\sup_{S(r)} |f(x) - g(x)| \ge Mr^{m-1}$ for sufficiently small r. The last inequality contradicts our assumption that $f(x) - g(x) = O(|x|^m)$. ∎

THEOREM 3. *Let Q be a real polynomial on $\mathbb{R}^{n+1}_{x,t}$, monic in t, and let $L > 0$. Then there exists $N \in \mathbb{N}$ such that if $Q(x,\psi(x)) = O(|x-a|^N)$ as $x \to a$, with some real-analytic function ψ defined in a neighborhood of $a \in \mathbb{R}^n$, then there exists an analytic function φ defined in a neighborhood of $a \in \mathbb{R}^n$ such that $Q(x,\varphi(x)) \equiv 0$ and $\varphi - \psi = O(|x-a|^L)$ as $x \to a$.*

Proof. Replacing Q and ψ by their complexifications, and taking m from Theorem 2 we get (due to Theorem 1 with $L > m$) a holomorphic function φ defined in a neighborhood of a such that $Q(z,\varphi(z)) \equiv 0$ and $\varphi - \psi = O(|z-a|^m)$. But then also $Q(z,\overline{\varphi}(z)) \equiv 0$. Moreover $\overline{\varphi} - \psi = O(|z-a|^m)$, which implies $\overline{\varphi} - \varphi = O(|z-a|^m)$. According to Theorem 2 we get $\overline{\varphi} = \varphi$, i.e. the function φ is real. ∎

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