ANNALES
POLONICI MATHEMATICI
82.1 (2003)

Interval criteria for oscillation of second order self-adjoint matrix differential systems

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Abstract. By employing the matrix Riccati technique and the integral averaging technique, new interval oscillation criteria are established for second-order matrix differential systems of the form $[P(t)Y']' + Q(t)Y = 0$.

1. Introduction. Consider the second order matrix differential systems of the form

$$[P(t)Y']' + Q(t)Y = 0, \qquad t \geq t_0, \tag{1.1}$$

where $t_0 \geq 0$ and $Y(t)$, $P(t)$, and $Q(t)$ are $n \times n$ real continuous matrix functions with $P(t), Q(t)$ symmetric and $P(t)$ positive-definite for $t \in [t_0, \infty)$ ($P(t) > 0$, $t \geq t_0$).

A solution $Y(t)$ of (1.1) is said to be *nontrivial* if $\det Y(t) \neq 0$ for at least one $t \in [t_0, \infty)$, and a nontrivial solution $Y(t)$ of (1.1) is said to be *prepared* if

$$Y^*(t)P(t)Y'(t) - Y^{*\prime}(t)P(t)Y(t) \equiv 0, \qquad t \in [t_0, \infty),$$

where for any matrix A, the transpose of A is denoted by A^*. A nontrivial prepared solution $Y(t)$ of (1.1) is said to be *oscillatory* if $\det Y(t)$ has arbitrarily large zeros; otherwise, it is said to be *nonoscillatory*. System (1.1) is said to be *oscillatory* if every nontrivial prepared solution of the system is oscillatory.

The oscillation problem for system (1.1) and its various particular cases have been studied extensively in recent years (see e.g. [1–5, 7] and the references therein). Note that in Kong [2], Wang [5] and Yang [7], the authors employed the techniques of Philos [3] for the equation $y''(t) + q(t)y(t) = 0$,

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2000 *Mathematics Subject Classification*: 34A30, 34C10.

Key words and phrases: matrix differential systems, oscillation, interval criteria, matrix Riccati technique, integral averaging technique.

Project supported by China Postdoctoral Science Foundation (no. 2002031294) and the National Education Committee Doctoral Foundation of China (no. 20020558092).

and of Kong [1] for the equation $(p(t)y'(t))' + q(t)y(t) = 0$, and presented several interval criteria for oscillation of system (1.1). As an application, Wang [5] derived the following result for system (1.1).

Theorem A. *Let $P(t) \equiv P$ be a constant matrix for $t \ge t_0$. Then system* (1.1) *is oscillatory provided that for each $\tau \ge t_0$ and for some $r > 1$, one of the following conditions holds*:

(i)
$$\limsup_{t\to\infty} \frac{1}{t^{r-1}} \lambda_1\Big[\int_\tau^t (t-s)^r Q(s)\,ds\Big] > \frac{r^2}{4(r-1)} \lambda_1[P],$$
$$\limsup_{t\to\infty} \frac{1}{t^{r-1}} \lambda_n\Big[\int_\tau^t (s-\tau)^r Q(s)\,ds\Big] > \frac{r^2}{4(r-1)} \lambda_1[P],$$

and

(ii)
$$\limsup_{t\to\infty} \frac{1}{t^{r-1}} \int_\tau^t (t-s)^r \operatorname{tr} Q(s)\,ds > \frac{r^2}{4(r-1)} \operatorname{tr} P,$$
$$\limsup_{t\to\infty} \frac{1}{t^{r-1}} \int_\tau^t (s-\tau)^r \operatorname{tr} Q(s)\,ds > \frac{r^2}{4(r-1)} \operatorname{tr} P,$$

where for any real symmetric $n \times n$ matrix A, $\lambda_1[A]$ (resp. $\lambda_n[A]$) indicates the largest (resp. smallest) eigenvalue of A and $\operatorname{tr} A$ denotes the trace of A.

In 1999, Wong [6] established an interval criterion for oscillation of the forced linear differential equation

$$(p(t)y'(t))' + q(t)y(t) = f(t), \tag{1.2}$$

namely:

Theorem B. *Suppose that for any $T \ge 0$, there exist $T \le s_1 < t_1 \le s_2 < t_2$ such that*

$$f(t) \begin{cases} \le 0, & t \in [s_1, t_1], \\ \ge 0, & t \in [s_2, t_2]. \end{cases}$$

Define $D(s_i, t_i) = \{u \in C^1[s_i, t_i] : u(t) \not\equiv 0,\ u(s_i) = u(t_i) = 0\}$, $i = 1, 2$. If there exists $u \in D(s_i, t_i)$ such that

$$Q_i(u) = \int_{s_i}^{t_i} (qu^2 - pu'^2)\,dt \ge 0$$

for $i = 1, 2$, then equation (1.2) *is oscillatory.*

Motivated by the idea of Wong [6], in the present paper we will, by employing the matrix Riccati technique and the integral averaging technique, establish new interval criteria for oscillation of the matrix differential system (1.1), which are different from those of Kong [2], Wang [5] and Yang [7].

Two interesting examples are also included to point out the versatility of our results.

2. Main results

THEOREM 2.1. *Suppose that all conditions on* (1.1) *stated in Section* 1 *are satisfied. Let* $D(a,b) := \{u \in C^1[a,b] : u(t) \not\equiv 0,\ u(a) = u(b) = 0\}$. *If for any* $T \geq t_0$, *there exist* a, b *with* $b > a \geq T$ *and* $u \in D(a,b)$ *such that*

$$\int_a^b [u^2(t)Q(t) - u'^2(t)P(t)]\,dt \geq 0, \tag{2.1}$$

then system (1.1) *is oscillatory.*

Proof. Suppose to the contrary that there exists a prepared solution $Y(t)$ of (1.1) which is nonoscillatory. Then there exists a $t_1 \geq t_0$ such that $\det Y(t) \neq 0$ for $t \geq t_1$. For $t \geq t_1$ define

$$V(t) = -P(t)Y'(t)Y^{-1}(t). \tag{2.2}$$

Then $V(t)$ is well defined, real symmetric, and solves the Riccati equation

$$V'(t) = V(t)P^{-1}(t)V(t) + Q(t) \tag{2.3}$$

on $[t_1,\infty)$. From the hypothesis, it follows that for any $T \geq t_1$, there exist a, b with $b > a \geq T$ and $u \in D(a,b)$ such that (2.1) holds. Multiplying (2.3) by u^2, integrating over $[a,b]$ and using integration by parts, we have

$$\begin{aligned}\int_a^b u^2(t)Q(t)\,dt &= \int_a^b u^2(t)[V'(t) - V(t)P^{-1}(t)V(t)]\,dt\\ &= \int_a^b [-2u(t)u'(t)V(t) - u^2(t)V(t)P^{-1}(t)V(t)]\,dt\\ &= -\int_a^b R_1^{-1}(t)\{u(t)[R_1(t)V(t)R_1(t)] + u'(t)E_n\}^2 R_1^{-1}(t)\,dt\\ &\quad + \int_a^b u'^2(t)P(t)\,dt,\end{aligned}$$

where $R_1(t) = [P^{-1}(t)]^{1/2}$ and E_n is the $n \times n$ identity matrix. This and (2.1) yield

$$u'(t)E_n + u(t)[R_1(t)V(t)R_1(t)] = R_1^{-1}(t)Y(t)(u(t)Y^{-1}(t))'R_1(t) = 0$$

on $[a,b]$. Also $\det Y(t) \neq 0$ for $t \geq t_1$, so it follows that $u(t)Y^{-1}(t) = C$ on $[a,b]$ for some constant matrix C. Because $u(t) \in D(a,b)$ and $u \not\equiv 0$, this is incompatible with the fact that $\det Y(t) \neq 0$ for $t \geq t_1$. This contradiction completes the proof.

By employing a more general matrix Riccati substitution than (2.2) (see (2.5) below), we can get the following criterion.

THEOREM 2.2. *Suppose that all conditions on* (1.1) *stated in Section* 1 *are satisfied. If for any* $T \ge t_0$, *there exist* a, b *with* $b > a \ge T$, $u \in D(a,b)$ *and* $\alpha \in C^1([t_0,\infty),(0,\infty))$ *such that* $\alpha' P$ *is differentiable and*

$$\lambda_1\Big[\int_a^b \{u^2(t)\Phi(t) - \alpha(t)u'^2(t)P(t)\}\,dt\Big] > 0, \tag{2.4}$$

then system (1.1) *is oscillatory*, *where* $D(a,b)$ *is defined in Theorem* 2.1 *and*

$$\Phi(t) = \alpha(t)Q(t) - \frac{\alpha'^2(t)}{4\alpha(t)}\,P(t) + \frac{1}{2}\,[\alpha'(t)P(t)]'.$$

Proof. Suppose to the contrary that there exists a prepared solution $Y(t)$ of (1.1) which is nonoscillatory. Then there exists a $t_1 \ge t_0$ such that $\det Y(t) \ne 0$ for $t \ge t_1$. From the hypothesis, it follows that for any $T \ge t_1$, there exist a, b with $b > a \ge T$, $u \in D(a,b)$ and $\alpha \in C^1([t_0,\infty),(0,\infty))$ such that $\alpha' P$ is differentiable and (2.4) holds.

Now, for $t \ge t_1$ we define

$$V(t) = -\alpha(t)P(t)\Big[Y'(t)Y^{-1}(t) - \frac{\alpha'(t)}{2\alpha(t)}\,E_n\Big]. \tag{2.5}$$

Then $V(t)$ is well defined, real symmetric, and solves the Riccati equation

$$V'(t) = \frac{1}{\alpha(t)}\,V(t)P^{-1}(t)V(t) + \Phi(t) \tag{2.6}$$

on $[t_1,\infty)$. Multiplying (2.6) by u^2, integrating over $[a,b]$ and using integration by parts, we have

$$\begin{aligned}
\int_a^b u^2(t)\Phi(t)\,dt &= \int_a^b u^2(t)\Big[V'(t) - \frac{1}{\alpha(t)}\,V(t)P^{-1}(t)V(t)\Big]\,dt \\
&= \int_a^b \Big[-2u(t)u'(t)V(t) - \frac{u^2(t)}{\alpha(t)}\,V(t)P^{-1}(t)V(t)\Big]\,dt \\
&= -\int_a^b R_2^{-1}(t)\{u(t)[R_2(t)V(t)R_2(t)] + u'(t)E_n\}^2 R_2^{-1}(t)\,dt \\
&\quad + \int_a^b \alpha(t)u'^2(t)P(t)\,dt \\
&\le \int_a^b \alpha(t)u'^2(t)P(t)\,dt,
\end{aligned}$$

where $R_2(t) = \left[\frac{1}{\alpha(t)} P^{-1}(t)\right]^{1/2}$. Thus

$$\int_a^b \{u^2(t)\Phi(t) - \alpha(t)u'^2(t)P(t)\}\, dt \le 0,$$

which contradicts assumption (2.4). This contradiction completes the proof.

3. Examples. In this section, we give two examples of applications of our oscillation criteria. We will see that the systems in the examples are oscillatory basing on the results of Section 2, though the oscillations cannot be demonstrated by means of Theorems A, B and most other known criteria. The first example illustrates Theorem 2.1.

EXAMPLE 3.1. Consider the second-order matrix differential system

$$[\sqrt{t+1}\, Y]' + \operatorname{diag}(q_1(t), \dots, q_n(t))Y = 0, \tag{3.1}$$

where $t \ge 0$, $q_i \in C([0,\infty),(0,\infty))$ and $q_i(t) \ge 1/(4\sqrt{t+1})$, $i = 1, \dots, n$.

For any $T \ge 0$, choose $k \in \mathbb{N}$ sufficiently large so that $(k\pi)^2 - 1 \ge T$ and set $a = (k\pi)^2 - 1$, $b = (k+1)^2\pi^2 - 1$. Taking $u(t) = \sin\sqrt{t+1}$, it is easy to verify that $u \in D((k\pi)^2 - 1, (k+1)^2\pi^2 - 1)$ and

$$\begin{aligned}
&\int_a^b \{u^2(t)q_i(t) - \sqrt{t+1}\, u'^2(t)\}\, dt \\
&\qquad \ge \int_{(k\pi)^2-1}^{(k+1)^2\pi^2-1} \left\{ \frac{1}{4\sqrt{t+1}} \sin^2\sqrt{t+1} - \frac{1}{4\sqrt{t+1}} \cos^2\sqrt{t+1} \right\} dt \\
&\qquad = \int_{k\pi}^{(k+1)\pi} \left\{ \frac{1}{2}\sin^2 t - \frac{1}{2}\cos^2 t \right\} dt = 0.
\end{aligned}$$

It follows that (2.1) holds and from Theorem 2.1 that system (3.1) is oscillatory.

The second example illustrates Theorem 2.2.

EXAMPLE 3.2. For any $\alpha \in C^2([0,\infty),(0,\infty))$, consider the second-order matrix differential system

$$\left[\frac{\sqrt{t+1}}{\alpha(t)} Y'\right]' + \operatorname{diag}(q_1(t), \dots, q_n(t))Y = 0, \tag{3.2}$$

where $t \ge 0$, $q_i \in C([0,\infty),\mathbb{R})$ $(i = 1, \dots, n)$ and $\max_{1\le i\le n}\{\phi_i(t)\} \ge 1/(2\sqrt{t+1})$,

$$\phi_i(t) = \alpha(t)q_i(t) - \frac{\alpha'^2(t)}{4\alpha^2(t)}\sqrt{t+1} + \frac{1}{2}\left[\alpha'(t)\,\frac{\sqrt{t+1}}{\alpha(t)}\right]', \qquad i = 1, \dots, n.$$

For any $T \geq 0$, choose $k \in \mathbb{N}$ sufficiently large so that $(k\pi)^2 - 1 \geq T$ and also set $a = (k\pi)^2 - 1$, $b = (k+1)^2\pi^2 - 1$. Taking $u(t) = \sin\sqrt{t+1}$ for $t \geq 0$ in Theorem 2.2, we have

$$u^2(t)\Phi(t) - \alpha(t)u'^2(t)P(t) = u^2(t)\operatorname{diag}(\phi_1(t), \dots, \phi_n(t)) - u'^2(t)\sqrt{t+1}\,E_n.$$

Since

$$\max_{1\leq i\leq n}\{u^2(t)\phi_i(t) - \sqrt{t+1}\,u'^2(t)\} \geq \frac{1}{2\sqrt{t+1}}\,u^2(t) - \sqrt{t+1}\,u'^2(t),$$

it follows that

$$\begin{aligned} &\lambda_1\Big[\int_a^b \{u^2(t)\Phi(t) - \alpha(t)u'^2(t)P(t)\}\,dt\Big] \\ &\qquad \geq \int_{(k\pi)^2-1}^{(k+1)^2\pi^2-1} \left\{\frac{1}{2\sqrt{t+1}}\sin^2\sqrt{t+1} - \frac{1}{4\sqrt{t+1}}\cos^2\sqrt{t+1}\right\}dt \\ &\qquad = \int_{k\pi}^{(k+1)\pi} \left\{\sin^2 t - \frac{1}{2}\cos^2 t\right\}dt = \frac{1}{4}\pi > 0. \end{aligned}$$

Thus, we conclude that (2.4) holds and from Theorem 2.2 that system (3.2) is oscillatory.

Remark 3.3. In Example 3.2, if we choose $\alpha(t) \equiv 1$ for $t \geq 0$ then $\phi_i(t) = q_i(t)$ $(i = 1, \dots, n)$; if we choose $\alpha(t) = (t+1)^2$ for $t \geq 0$, then $\phi_i(t) = (t+1)^2 q_i(t) - \frac{3}{2}(t+1)^{-3/2}$ $(i = 1, \dots, n)$ and the condition "$\max_{1\leq i\leq n}\{\phi_i(t)\} \geq 1/(2\sqrt{t+1})$" reduces to "$\max_{1\leq i\leq n}\{q_i(t)\} \geq \frac{1}{2}(t+1)^{-5/2} + \frac{3}{2}(t+1)^{-7/2}$".

Acknowledgements. The author is greatly indebted to Professors Qingkai Kong and J. S. W. Wong for their kind help. He is also grateful to the referee for his valuable comments and helpful suggestions.

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Reçu par la Rédaction le 10.3.2003
Révisé le 22.5.2003 (1431)