ANNALES
POLONICI MATHEMATICI
108.3 (2013)

Unique decomposition for a polynomial of low rank

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Abstract. Let F be a homogeneous polynomial of degree d in $m+1$ variables defined over an algebraically closed field of characteristic 0 and suppose that F belongs to the sth secant variety of the d-uple Veronese embedding of $\mathbb{P}^m$ into $\mathbb{P}^{\binom{m+d}{d}-1}$ but that its minimal decomposition as a sum of dth powers of linear forms requires more than s summands. We show that if $s \le d$ then F can be uniquely written as $F = M_1^d + \cdots + M_t^d + Q$, where $M_1, \dots, M_t$ are linear forms with $t \le (d-1)/2$, and Q is a binary form such that $Q = \sum_{i=1}^q l_i^{d-d_i} m_i$ with l_i's linear forms and m_i's forms of degree d_i such that $\sum (d_i+1) = s-t$.

Introduction. In this paper we will always work over an algebraically closed field K of characteristic 0. Let $X_{m,d} \subset \mathbb{P}^N$, with $m \ge 1$, $d \ge 2$ and $N := \binom{m+d}{m} - 1$, be the classical Veronese variety obtained as the image of the d-uple Veronese embedding $\nu_d : \mathbb{P}^m \to \mathbb{P}^N$. The *$s$th secant variety* $\sigma_s(X_{m,d})$ of the Veronese variety $X_{m,d}$ is the Zariski closure in $\mathbb{P}^N$ of the union of all linear spans $\langle P_1, \dots, P_s \rangle$ with $P_1, \dots, P_s \in X_{m,d}$. For any point $P \in \mathbb{P}^N$, we indicate with $\mathrm{sbr}(P)$ the minimum integer s such that $P \in \sigma_s(X_{m,d})$. This integer is called the *symmetric border rank* of P.

Since $\mathbb{P}^m \simeq \mathbb{P}(K[x_0, \dots, x_m]_1) \simeq \mathbb{P}(V^*)$ with V an $(m+1)$-dimensional vector space over K, the generic element belonging to $\sigma_s(X_{m,d})$ is the projective class of a form (symmetric tensor) of type

$$F = L_1^d + \cdots + L_r^d \quad (T = v_1^{\otimes d} + \cdots + v_r^{\otimes d}). \tag{1}$$

The minimum $r \in \mathbb{N}$ such that F can be written as in (1) is the *symmetric rank* of F and we denote it $\mathrm{sr}(F)$ ($\mathrm{sr}(T)$, if we replace F with T).

The decomposition of a homogeneous polynomial with a minimum number of terms and a minimum number of variables is a problem that is receiving a great deal of attention not only in classical algebraic geometry

2010 *Mathematics Subject Classification*: 15A21, 15A69, 14N15.
Key words and phrases: Waring problem, polynomial decomposition, symmetric rank, symmetric tensors, Veronese varieties, secant varieties.

DOI: 10.4064/ap108-3-2

([2], [9], [7], [8], [10]), but also from the point of view of applications like computational complexity ([11]) and signal processing ([12]).

At the workshop on Tensor Decompositions and Applications (September 13–17, 2010, Monopoli, Bari, Italy), A. Bernardi presented a work in collaboration with E. Ballico where a possible structure of small rank homogeneous polynomials with border rank smaller than rank was characterized (see [3]). It is well known that, if a homogeneous polynomial F is such that $\mathrm{sbr}(F) < \mathrm{sr}(F)$, then there are infinitely many decompositions of F as in (1). Our purpose in [3] was to find, among all the possible decompositions of F, a "best" one in terms of the number of variables. Namely: Does there exist a canonical choice of two variables such that most of the terms involved in the decomposition (1) of F depend only on those two variables? The precise statement of that result is the following:

- ([3, Corollary 1]) Let $F \in K[x_0, \dots, x_m]_d$ be such that $\mathrm{sbr}(F) + \mathrm{sr}(F) \le 2d+1$ and $\mathrm{sbr}(F) < \mathrm{sr}(F)$. Then there are an integer $t \ge 0$, linear forms $L_1, L_2, M_1, \dots, M_t \in K[x_0, \dots, x_m]_1$, and a form $Q \in K[L_1, L_2]_d$ such that $F = Q + M_1^d + \cdots + M_t^d$, $t \le \mathrm{sbr}(F) + \mathrm{sr}(F) - d - 2$, and $\mathrm{sr}(F) = \mathrm{sr}(Q) + t$. Moreover t, $M_1, \dots, M_t$ and the linear span of L_1, L_2 are uniquely determined by F.

In terms of tensors this can be translated as follows:

- ([3, Corollary 2]) Let $T \in S^dV^*$ be such that $\mathrm{sbr}(T) + \mathrm{sr}(T) \le 2d+1$ and $\mathrm{sbr}(T) < \mathrm{sr}(T)$. Then there are an integer $t \ge 0$, vectors $v_1, v_2, w_1, \dots, w_t \in S^1V^*$, and a symmetric tensor $v \in S^d(\langle v_1, v_2 \rangle)$ such that $T = v + w_1^{\otimes d} + \cdots + w_t^{\otimes d}$, $t \le \mathrm{sbr}(T) + \mathrm{sr}(T) - d - 2$, and $\mathrm{sr}(T) = \mathrm{sr}(v) + t$. Moreover t, $w_1, \dots, w_t$ and $\langle v_1, v_2 \rangle$ are uniquely determined by T.

The natural questions raised by applied people at the Monopoli workshop mentioned above were about the possible uniqueness of the binary form Q in [3, Corollary 1] (i.e. the vector v in [3, Corollary 2]) and a bound on the number t of linear forms (i.e. rank 1 symmetric tensors). We are finally able to give an answer as complete as possible to these questions. The main result of the present paper is the following:

Theorem 1. *Let $P \in \mathbb{P}^N$ with $N = \binom{m+d}{d} - 1$. Suppose that*

$$\mathrm{sbr}(P) < \mathrm{sr}(P) \quad \textit{and} \quad \mathrm{sbr}(P) + \mathrm{sr}(P) \le 2d+1.$$

Let $\mathcal{S} \subset X_{m,d}$ be a 0-dimensional reduced subscheme that realizes the symmetric rank of P, and let $\mathcal{Z} \subset X_{m,d}$ be a 0-dimensional non-reduced subscheme such that $P \in \langle \mathcal{Z} \rangle$ and $\deg \mathcal{Z} \le \mathrm{sbr}(P)$. There is a unique rational normal curve $C_d \subset X_{m,d}$ such that $\deg(C_d \cap (\mathcal{S} \cup \mathcal{Z})) \ge d+2$. Then for all *points* $P \in \mathbb{P}^N$ *as above we have*

$$\mathcal{S} = \mathcal{S}_1 \sqcup \mathcal{S}_2, \qquad \mathcal{Z} = \mathcal{Z}_1 \sqcup \mathcal{S}_2,$$

where $\mathcal{S}_1 = \mathcal{S} \cap C_d$, $\mathcal{Z}_1 = \mathcal{Z} \cap C_d$ *and* $\mathcal{S}_2 = (\mathcal{S} \cap \mathcal{Z}) \setminus \mathcal{S}_1$. *Moreover* C_d, $\mathcal{S}_2$ *and* $\mathcal{Z}$ *are unique,* $\deg(\mathcal{Z}) = \mathrm{sbr}(P)$, $\deg(\mathcal{Z}_1) + \deg(\mathcal{S}_1) = d+2$, $\mathcal{Z}_1 \cap \mathcal{S}_1 = \emptyset$ *and* $\mathcal{Z}$ *is the unique zero-dimensional subscheme* N *of* $X_{m,d}$ *such that* $\deg(N) \le \mathrm{sbr}(P)$ *and* $P \in \langle N \rangle$.

In the language of polynomials, Theorem 1 can be rephrased as follows.

Corollary 1. *Let* $F \in K[x_0, \dots, x_m]_d$ *be such that* $\mathrm{sbr}(F) + \mathrm{sr}(F) \le 2d+1$ *and* $\mathrm{sbr}(F) < \mathrm{sr}(F)$. *Then there are an integer* $0 \le t \le (d-1)/2$, *linear forms* $L_1, L_2, M_1, \dots, M_t \in K[x_0, \dots, x_m]_1$, *and a form* $Q \in K[L_1, L_2]_d$ *such that* $F = Q + M_1^d + \cdots + M_t^d$, $t \le \mathrm{sbr}(F) + \mathrm{sr}(F) - d - 2$, *and* $\mathrm{sr}(F) = \mathrm{sr}(Q) + t$. *Moreover the line* $\langle L_1, L_2 \rangle$, *the forms* $M_1, \dots, M_t$ *and* Q *such that* $Q = \sum_{i=1}^{q} l_i^{d-d_i} m_i$ *with* l_i*'s linear forms and* m_i*'s forms of degree* d_i *such that* $\sum (d_i + 1) = s - t$, *are uniquely determined by* F.

An analogous corollary can be stated for symmetric tensors.

Corollary 2. *Let* $T \in S^d V^*$ *be such that* $\mathrm{sbr}(T) + \mathrm{sr}(T) \le 2d + 1$ *and* $\mathrm{sbr}(T) < \mathrm{sr}(T)$. *Then there are an integer* $0 \le t \le (d-1)/2$, *vectors* $v_1, v_2, w_1, \dots, w_t \in S^1 V^*$, *and a symmetric tensor* $v \in S^d(\langle v_1, v_2 \rangle)$ *such that* $T = v + w_1^{\otimes d} + \cdots + w_t^{\otimes d}$, $t \le \mathrm{sbr}(T) + \mathrm{sr}(T) - d - 2$, *and* $\mathrm{sr}(T) = \mathrm{sr}(v) + t$. *Moreover the line* $\langle v_1, v_2 \rangle$, *the vectors* $v_1, \dots, v_t$ *and the tensor* v *such that* $v = \sum_{i=1}^{q} u_i^{\otimes(d-d_i)} \otimes z_i$ *with* $u_i \in \langle v_1, v_2 \rangle$ *and* $z_i \in S^{d_i}(\langle v_1, v_2 \rangle)$ *such that* $\sum (d_i + 1) = s - t$, *are uniquely determined by* T.

Moreover, by introducing the notion of linearly general position of a scheme (Definition 1), we can give a finer geometric description of the condition for the uniqueness of the scheme $\mathcal{Z}$ of Theorem 1. This is the main purpose of Theorem 2 and Corollary 4 below. In terms of homogeneous polynomials and symmetric tensors, they can be phrased as follows:

Corollary 3. *Fix integers* $m \ge 2$ *and* $d \ge 4$. *Fix a degree* d *homogeneous polynomial* F *in* $m+1$ *variables* (*resp.* $T \in S^d V$) *such that* $\mathrm{sbr}(F) \le d$ (*resp.* $\mathrm{sbr}(T) \le d$). *Let* $Z \subset \mathbb{P}^m$ *be any smoothable zero-dimensional scheme such that* $\nu_d(Z)$ *evinces* $\mathrm{sbr}(F)$ (*resp.* $\mathrm{sbr}(T)$). *Assume that* Z *is in linearly general position. Then* Z *is the unique scheme which evinces* $\mathrm{sbr}(F)$ (*resp.* $\mathrm{sbr}(T)$).

1. Proofs. The existence of a scheme $\mathcal{Z}$ as in Theorem 1 is known from [4] and [5] (see Remark 1 of [3]).

Lemma 1. *Fix integers* $m \ge 2$ *and* $d \ge 2$, *a line* $\ell \subset \mathbb{P}^m$ *and any finite set* $E \subset \mathbb{P}^m \setminus \ell$ *such that* $\sharp(E) \le d$. *Then* $\dim(\langle \nu_d(E) \rangle) = \sharp(E) - 1$ *and* $\langle \nu_d(\ell) \rangle \cap \langle \nu_d(E) \rangle = \emptyset$.

Proof. Since $h^0(\ell \cup E, \mathcal{O}_{\ell\cup E}(d)) = d+1+\sharp(E)$, to get both statements it is sufficient to prove $h^1(\mathcal{I}_{\ell\cup E}(d)) = 0$. Let $H \subset \mathbb{P}^m$ be a general hyperplane containing ℓ. Since E is finite and H is general, we have $H \cap E = \emptyset$. Hence the residual exact sequence of the scheme $\ell \cup E$ with respect to the hyperplane H is the following exact sequence on $\mathbb{P}^m$:

$$0 \to \mathcal{I}_E(d-1) \to \mathcal{I}_{\ell\cup E}(d) \to \mathcal{I}_{\ell,H}(d) \to 0. \tag{2}$$

Since $h^1(\mathcal{I}_E(d-1)) = h^1(H, \mathcal{I}_{\ell,H}(d)) = 0$, we get the lemma. ∎

Proof of Theorem 1. All the statements are contained in [3, Theorem 1], except the uniqueness of $\mathcal{Z}$, the fact that $\deg(\mathcal{Z}_1) + \deg(\mathcal{S}_1) = d+2$ and $\mathcal{Z}_1 \cap \mathcal{S}_1 = \emptyset$. Let $\ell \subset \mathbb{P}^m$ be the line such that $\nu_d(\ell) = C_d$. Take $Z, S, Z_1, S_1, S_2 \subset \mathbb{P}^m$ such that $\nu_d(Z) = \mathcal{Z}$, $\nu_d(S) = \mathcal{S}$, $\nu_d(Z_1) = \mathcal{Z}_1$, and $\nu_d(S_i) = \mathcal{S}_i$ for $i = 1, 2$. Assume the existence of another subscheme $\mathcal{Z}' \subset X_{m,d}$ such that $P \in \langle \mathcal{Z}' \rangle$ and $\deg(\mathcal{Z}') \le \mathrm{sbr}(P)$. Set $\mathcal{Z}'_1 := \mathcal{Z}' \cap C_d$. The fact that $\mathcal{Z}' = \mathcal{Z}'_1 \sqcup S_2$ is actually given by the proof of [3, Theorem 1(b)–(d)]. At the end of step (a) (last five lines) of that proof, there is a description of the next steps (b)–(d) needed to prove that $\mathcal{Z} = (\mathcal{Z} \cap C_d) \sqcup \mathcal{S}_2$ for a certain scheme $\mathcal{Z}$. The role played by $\mathcal{Z}$ in [3, Theorem 1] is the same that $\mathcal{Z}'$ plays here, hence the same steps (b)–(d) give $\mathcal{Z}' = \mathcal{Z}'_1 \sqcup \mathcal{S}_2$ as we want here (one just needs to write $\mathcal{Z}'$ instead of $\mathcal{Z}$).

Since C_d is a smooth curve, $\mathcal{Z}_1 \cup \mathcal{Z}'_1 \subset C_d$, $\mathcal{S}_2 \cap C_d = \emptyset$, and $\mathcal{Z} \cup \mathcal{Z}' = (\mathcal{Z}_1 \cup \mathcal{Z}'_1) \sqcup \mathcal{S}_2$, the schemes $\mathcal{Z}$ and $\mathcal{Z}'$ are curvilinear. Hence all subschemes of $\mathcal{Z}$ and $\mathcal{Z}'$ are smoothable. Hence any subscheme of either $\mathcal{Z}$ or $\mathcal{Z}'$ may be used to compute the border rank of some point of $\mathbb{P}^N$. Since $\deg(\ell \cap (Z \cup S)) \ge d+2$, $\nu_d((Z \cup S) \cap \ell)$ spans $\langle C_d \rangle$. Lemma 1 implies $\langle C_d \rangle \cap \langle \mathcal{S}_2 \rangle = \emptyset$. Since $P \in \langle \mathcal{S}_1 \cup \mathcal{S}_2 \rangle$ and $\sharp(S) = \mathrm{sr}(P)$, we have $P \notin \langle \mathcal{A} \rangle$ for any $\mathcal{A} \subsetneq \mathcal{S}$. Therefore $\langle \{P\} \cup \mathcal{S}_2 \rangle \cap \langle \mathcal{S}_1 \rangle$ is a unique point. Call it P_1. Similarly, $\langle \mathcal{Z}_1 \rangle \cap \langle \mathcal{S}_2 \rangle$ is a unique point, say P_2. Similarly, $\langle \mathcal{Z}'_1 \rangle \cap \langle \mathcal{S}_2 \rangle$ is a unique point, P_3. Since $\langle C_d \rangle \cap \langle \mathcal{S}_2 \rangle = \emptyset$, the set $\langle C_d \rangle \cap \langle \{P\} \cup \mathcal{S}_2 \rangle$ is at most one point. Since $P_i \in \langle C_d \rangle \cap \langle \{P\} \cup \mathcal{S}_2 \rangle$, $i = 1, 2, 3$, we have $P_1 = P_2 = P_3$ and $\{P_1\} = \langle C_d \rangle \cap \langle \{P\} \cup \mathcal{S}_2 \rangle$. Since $P_1 = P_3$, we have $P_1 \in \langle \mathcal{Z}'_1 \rangle \cap \langle \mathcal{S}_1 \rangle$. Take any $E \subseteq \mathcal{Z}_1$ such that $P_1 \in \langle E \rangle$. Since $P \in \langle \{P_1\} \cup \mathcal{S}_2 \rangle \subseteq \langle E \cup \mathcal{S}_2 \rangle$ and $P \notin \langle \mathcal{U} \rangle$ for any $\mathcal{U} \subsetneq \mathcal{Z}$, we get $E \cup \mathcal{S}_2 = \mathcal{Z}$. Hence $E = \mathcal{Z}_1$. Therefore $\mathcal{Z}_1$ computes $\mathrm{sbr}(P_1)$ with respect to C_d. Similarly, $\mathcal{Z}'_1$ computes $\mathrm{sbr}(P_2)$ with respect to the same rational normal curve C_d. For any $Q \in \langle C_d \rangle$ with $\mathrm{sbr}(Q) < (d+2)/2$ (equivalently $\mathrm{sbr}(Q) \ne (d+2)/2$), there is a unique zero-dimensional subscheme of $\langle C_d \rangle$ which evinces $\mathrm{sbr}(Q)$ ([9, Proposition 1.36]; in [9, Definition 1.37], this scheme is called the canonical form of the polynomial associated to P). Since $P_1 = P_2$, we have $\mathcal{Z}'_1 = \mathcal{Z}_1$. ∎

Definition 1. A scheme $Z \subset \mathbb{P}^m$ is said to be in *linearly general position* if for every linear subspace $R \subsetneq \mathbb{P}^m$ we have $\deg(R \cap Z) \le \dim(R) + 1$.

Notice that the conclussion of the next theorem is false if either $d = 2$ or $m = 1$. Moreover if $d = 3$ and $m > 1$, then it essentially says that a point in the tangential variety of a Veronese variety belongs to a unique tangent line. This is a consequence of the well known Sylvester theorem on decompositions of binary forms ([4], [10]).

Theorem 2. *Fix integers $m \geq 2$ and $d \geq 4$. Fix $P \in \mathbb{P}^N$. Let $Z \subset \mathbb{P}^m$ be any smoothable zero-dimensional scheme such that $P \in \langle \nu_d(Z) \rangle$ and $P \notin \langle \nu_d(\overline{Z}) \rangle$ for any $\overline{Z} \subsetneq Z$. Assume that* $\deg(Z) \leq d$ *and Z is in linearly general position. Then Z is the unique scheme $Z' \subset \mathbb{P}^m$ such that* $\deg(Z') \leq d$ *and $P \in \langle \nu_d(Z') \rangle$. Moreover $\nu_d(Z)$ evinces* $\mathrm{sbr}(P)$.

Proof. Since $\deg(Z) \leq d$ and Z is smoothable, [4, Proposition 11 (last sentence)] gives $\mathrm{sbr}(P) \leq d$. Hence there is a scheme which evinces $\mathrm{sbr}(P)$ ([3, Remark 3]). The existence of such a scheme follows from [3, Remark 1], and the inequality $\mathrm{sbr}(P) \leq d$. Fix any scheme $Z' \subset \mathbb{P}^m$ such that $Z' \neq Z$, $\deg(Z') \leq d$, $P \in \langle \nu_d(Z') \rangle$, and $P \notin \langle \nu_d(Z'') \rangle$ for any $Z'' \subsetneq Z'$. Since $\deg(Z \cup Z') \leq 2d + 1$ and $h^1(\mathbb{P}^m, \mathcal{I}_{Z \cup Z'}(d)) > 0$ ([3, Lemma 1]), there is a line $D \subset \mathbb{P}^m$ such that $\deg(D \cap (Z \cup Z')) \geq d + 2$ ([4, Lemma 34]). Since Z is in linearly general position and $m \geq 2$, we have $\deg(Z \cap D) \leq 2$. Hence $\deg(Z' \cap D) \geq d$, so $\deg(Z') = d$. Since $\deg(Z') = d$, we get $Z' \subset D$. Hence $P \in \langle \nu_d(D) \rangle$, so $\mathrm{sbr}(P) = d$. The secant varieties of any non-degenerate curve have the expected dimension ([1, Remark 1.6]). Hence $\mathrm{sbr}(P) \leq \lfloor (d + 2)/2 \rfloor$. Since $\deg(Z') = d$, we assumed $\deg(Z') \leq \mathrm{sbr}(P)$, contradicting the assumption $d \geq 4$. ∎

Corollary 4. *Fix integers $m \geq 2$ and $d \geq 4$. Fix $P \in \mathbb{P}^N$ such that* $\mathrm{sbr}(P) \leq d$*. Let $Z \subset \mathbb{P}^m$ be any smoothable zero-dimensional scheme such that $\nu_d(Z)$ evinces* $\mathrm{sbr}(P)$*. Assume that Z is in linearly general position. Then Z is the unique scheme which evinces* $\mathrm{sbr}(P)$.

Acknowledgments. The authors were partially supported by CIRM of FBK Trento (Italy), Project Galaad of INRIA Sophia Antipolis Méditerranée (France), Marie Curie: Promoting science (FP7-PEOPLE-2009-IEF), MIUR and GNSAGA of INdAM (Italy).

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Received 9.5.2012
and in final form 7.8.2012 (2788)