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# Hermitian $(a,b)$-modules and Saito's "higher residue pairings"

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**Abstract.** Following the work of Daniel Barlet [Pitman Res. Notes Math. Ser. 366 (1997), 19–59] and Ridha Belgrade [J. Algebra 245 (2001), 193–224], the aim of this article is to study the existence of $(a,b)$-hermitian forms on regular $(a,b)$-modules. We show that every regular $(a,b)$-module $E$ with a non-degenerate bilinear form can be written in a unique way as a direct sum of $(a,b)$-modules $E_i$ that admit either an $(a,b)$-hermitian or an $(a,b)$-anti-hermitian form or both; all three cases are possible, and we give explicit examples.

As an application we extend the result of Ridha Belgrade on the existence, for all $(a,b)$-modules $E$ associated with the Brieskorn module of a holomorphic function with an isolated singularity, of an $(a,b)$-bilinear non-degenerate form on $E$. We show that with a small transformation Belgrade's form can be considered $(a,b)$-hermitian and that the result satisfies the axioms of Kyoji Saito's "higher residue pairings".

**1. Introduction.** In this article we will study the self-duality properties of $(a,b)$-modules and more precisely the conditions under which they admit a non-degenerate hermitian form.

The $(a,b)$-modules were introduced by D. Barlet [Bar93] as a formal completion of the Brieskorn module ([Bri70])

$$D := \frac{\Omega_0^{n+1}}{df \wedge d\Omega_0^{n-1}}$$

associated to a holomorphic function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ with an isolated singularity at the origin, where we denote by $\Omega_0^p$ the germs of holomorphic $p$-forms at 0.

We briefly recall the basic results about $(a,b)$-modules and refer the reader to the articles [Bar93] and [Bar97] for further details.

Definition 1.1. Let $\mathbb{C}[[b]]$ be the ring of formal series in the variable $b$. An $(a,b)$-*module* is a free $\mathbb{C}[[b]]$-module $E$ of finite rank equipped with a

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$\mathbb{C}$-linear map $a : E \to E$ that satisfies the commutation relation

$$ab - ba = b^2, \tag{1.1}$$

where $b : E \to E$ is multiplication by $b \in \mathbb{C}[[b]]$.

For a complex number $\lambda \in \mathbb{C}$ and an $(a,b)$-module $E$, we define a *monomial of type* $(\lambda, 0)$ to be an element $x \in E$ that satisfies the relation $ax = \lambda bx$. The simplest $(a,b)$-modules are those generated over $\mathbb{C}[[b]]$ by a monomial $e_\lambda$ of type $(\lambda, 0)$. These modules are called *elementary* and denoted $E_\lambda$.

Given an $(a,b)$-module $E$, a $\mathbb{C}[[b]]$-submodule $F$ of $E$ closed under multiplication by $a$ is called an $(a,b)$-*submodule.* Since the quotient of an $(a,b)$-module $E$ by an $(a,b)$-submodule $F$ is not necessarily $b$-torsion free, an $(a,b)$-submodule $F$ of $E$ will be called *normal* if $E/F$ is free and hence has an induced $(a,b)$-module structure.

The $(a,b)$-modules associated to a Brieskorn module are all *regular*, i.e. they are $(a,b)$-submodules of an $(a,b)$-module $E$ satisfying $aE \subset bE$ (a *simple-pole* $(a,b)$-module). The composition series of a regular $(a,b)$-module have the following property:

Proposition 1.2. *Let $E$ be a regular $(a,b)$-module. Then all its composition series are of the form*

$$0 = E_0 \subsetneq \cdots \subsetneq E_{n-1} \subsetneq E_n = E,$$

*with $E_i/E_{i-1}$ elementary $(a,b)$-modules $E_\lambda$.*

As proven in [Bar93], the quotients of two composition series of an $(a,b)$-module $E$ are not unique, even if we ignore the permutations of the quotients.

**2.** $(a,b)$**-modules and their duality.** The dual and bi-dual structures on $(a,b)$-modules were first introduced in [Bar97] and [Bel01] and then expanded in our thesis (cf. [Kar09]). We will therefore begin by giving a formal definition of the duality structures we work with.

In the spirit of category theory we will define an $(a,b)$-*morphism* as a map $\phi : E \to F$ between two $(a,b)$-modules $E$ and $F$, which is a morphism of the underlying $\mathbb{C}[[b]]$-modules and respects the $a$-structure: $\phi(ax) = a\phi(x)$ for any $x \in E$. We will call $\phi$ an *isomorphism* (resp. *endomorphism*) of $(a,b)$-modules if it is bijective (resp. $E = F$).

**2.1.** $(a,b)$**-linear maps and dual** $(a,b)$**-modules.** Let $E$ and $F$ be $(a,b)$-modules. As defined by D. Barlet [Bar97], the $\mathbb{C}[[b]]$-module $\mathrm{Hom}_{\mathbb{C}[[b]]}(E,F)$ of $\mathbb{C}[[b]]$-linear maps from $E$ to $F$ has a natural structure of $(a,b)$-module provided by an operator $\Lambda$ that satisfies

$$(\Lambda\phi)(x) = a_F(\phi(x)) - \phi(a_E x), \tag{2.1}$$

where $\phi \in \mathrm{Hom}_{\mathbb{C}[[b]]}(E,F)$, $x \in E$, and $a_E$ and $a_F$ are the $a$-structures of $E$ and $F$ respectively. For this $(a,b)$-module we use the notation $\mathrm{Hom}_{(a,b)}(E,F)$.

For simplicity we will denote $a_E$, $a_F$ and $\Lambda$ all by the letter $a$; to avoid confusion we write

$$a \cdot \phi(x)$$

for $(\Lambda\phi)(x)$, and

$$a\phi(x)$$

for $a_E(\phi(x))$. Thus, we will write equation (2.1) as $a \cdot \phi(x) = a\phi(x) - \phi(ax)$.

By choosing $E_0$ as the codomain of the morphisms, we can give the following definition:

Definition 2.1 (Barlet). Let $E$ be an $(a,b)$-module and $E_0$ the elementary $(a,b)$-module of parameter 0. Then we call the module

$$\mathrm{Hom}_{(a,b)}(E,E_0)$$

the *dual* $(a,b)$*-module* of $E$ and denote it by $E^*$.

Remark 2.2. When considering only the $b$-structure of $E$, $E^*$ as a $\mathbb{C}[[b]]$-module is exactly the dual of a $\mathbb{C}[[b]]$-module, since $E_0 = \mathbb{C}[[b]]e_0$.

The duality functor $^*$ is exact (cf. [Bar97]).

**2.2. Conjugate $(a,b)$-module.** In [Bel01] R. Belgrade uses another definition of dual $(a,b)$-module which is not equivalent to the one of D. Barlet. In order to be able to express one concept in terms of the other, we will introduce an operation that exchanges the signs of both $a$ and $b$, whose behaviour is similar to that of conjugation of the complex field.

As in the case of the complex field $\mathbb{C}$, the ring of formal series $\mathbb{C}[[b]]$ also admits a natural involution

$$\check{} : \mathbb{C}[[b]] \to \mathbb{C}[[b]], \qquad S(b) \mapsto \check{S}(b) = S(-b),$$

where $S(b) \in \mathbb{C}[[b]]$. This allows us to define the conjugate of an $(a,b)$-module in the same way as one defines the conjugate of a complex vector space.

Definition 2.3. Let $E$ be an $(a,b)$-module. We define the $(a,b)$*-conjugate* of $E$, denoted by $\check{E}$, to be the set $E$ endowed with the $a$- and $b$-structure given by

$$a \cdot_{\check{E}} v = -a \cdot_E v, \qquad b \cdot_{\check{E}} v = -b \cdot_E v,$$

where $\cdot_{\check{E}}$ and $\cdot_E$ denote the $(a,b)$-structures of $\check{E}$ and $E$ respectively.

Since we change the signs of both $a$ and $b$, the formula $ab - ba = b^2$ is still satisfied.

Remark 2.4. An $(a,b)$-module is not necessarily isomorphic to its conjugate. Take, for example, the $(a,b)$-module of rank 2 generated by two elements $x$ and $y$ that satisfy

$$ax = \lambda bx, \quad ay = \lambda by + (1 + \alpha b)x,$$

where $\lambda$ and $\alpha \in \mathbb{C}$ and $\alpha \neq 0$. Its conjugate satisfies

$$ax = \lambda bx, \quad ay = \lambda by + (1 - \alpha b)x,$$

and the classification of rank 2 regular $(a, b)$-modules, given in [Bar93], implies that the two modules are not isomorphic.

One can see immediately that for every $(a, b)$-module $E$ the double conjugate $(\check{E})\check{}$ is the $(a, b)$-module $E$ itself.

On the other hand, let $E$ and $F$ be $(a, b)$-modules and $\phi$ a morphism between $E$ and $F$. Since $\phi(-ax) = -a\phi(x)$ and $\phi(-bx) = -b\phi(x)$, for all $x \in E$, the map $\phi$ is also a morphism between the conjugates $\check{E}$ and $\check{F}$. The *conjugation functor* associates to every $(a, b)$-module its conjugate and to every morphism, the morphism itself. This functor is exact.

For an $(a, b)$-module $E$ we will be especially interested in a particular kind of conjugate, the conjugate of the dual, which we call the *adjoint* $(a, b)$-module and denote by $\check{E}^*$.

**2.3. Bilinear forms and tensor products.** To define $\mathrm{Hom}_{(a,b)}(E, F)$ we used an equivalent object for the underlying $b$-structure. We can proceed in a similar way to obtain the concept of $(a, b)$-bilinear maps:

DEFINITION 2.5. Let $E$, $F$ and $G$ be $(a, b)$-modules. A $\mathbb{C}[[b]]$-linear map $\Phi : E \times F \to G$ is $(a, b)$-*bilinear* if

$$a\Phi(x, y) = \Phi(ax, y) + \Phi(x, ay).$$

REMARK 2.6. If $\Phi : E \times F \to G$ is $(a, b)$-bilinear and $v \in E$, then

$$\Phi_v := \Phi(v, \cdot) : w \mapsto \Phi(v, w), \quad w \in F,$$

is not necessarily an $(a, b)$-morphism. However the map $\pi : v \mapsto \Phi_v$ is an $(a, b)$-morphism between $E$ and $\mathrm{Hom}_{(a,b)}(F, G)$:

$$\pi(av)(x) = \Phi_{av}(x) = a\Phi_v(x) - \Phi_v(ax) = a \cdot \Phi_v(x) = a\pi(v).$$

Inherently linked to the concept of $(a, b)$-bilinear maps is that of tensor products, which allows a more practical manipulation of these objects.

DEFINITION 2.7. Let $E$ and $F$ be $(a, b)$-modules. The $(a, b)$-*tensor product* of $E$ and $F$, denoted by $E \otimes_{(a,b)} F$, is the $\mathbb{C}[[b]]$-module

$$E \otimes_{\mathbb{C}[[b]]} F$$

endowed with the $a$-structure defined by

$$a(v \otimes w) = (av) \otimes w + v \otimes (aw) \quad \text{for } v \in E \text{ and } w \in F.$$

This $a$-structure is well defined:

$$\begin{aligned} a(bv \otimes w) &= a(bv) \otimes w + bv \otimes a(w) = ba(v) \otimes w + b^2 v \otimes w + v \otimes ba(w) \\ &= a(v) \otimes bw + v \otimes a(bw) = a(v \otimes bw), \end{aligned}$$

for each $v \in E$, $w \in F$ and it satisfies $ab - ba = b^2$:

$$\begin{aligned} &a(bv \otimes w) - ba(v \otimes w) \\ &\qquad = ba(v) \otimes w + b^2 v \otimes w + bv \otimes a(w) - ba(v) \otimes w - bv \otimes a(w) \\ &\qquad = b^2(v \otimes w). \end{aligned}$$

We can easily verify that the above tensor product has the usual universal property: there exists a bilinear map

$$\Phi : E \times F \to E \otimes_{(a,b)} F$$

such that for every bilinear map $\Psi$ on $E \times F$ with values in an $(a,b)$-module $G$, there exists a unique $(a,b)$-morphism $\tilde{\Psi}$ from $E \otimes_{(a,b)} F$ into $G$ that makes the following diagram commutative:

$$\begin{tikzcd} E \times F \arrow[r, "\Psi"] \arrow[d, "\Phi"] & G \\ E \otimes_{(a,b)} F \arrow[ur, dashed, "\tilde{\Psi}"] & \end{tikzcd}$$

Indeed, we can take

$$\begin{aligned} \Phi : E \times F \to E \otimes_{(a,b)} F, &\quad (v,w) \mapsto v \otimes_{(a,b)} w, \\ \tilde{\Psi} : E \otimes_{(a,b)} F \to G, &\quad v \otimes_{(a,b)} w \mapsto \Psi(v,w). \end{aligned}$$

The unicity of $\tilde{\Psi}$ follows directly from the universal property of the tensor product of $\mathbb{C}[[b]]$-modules. We need only verify that the map is $a$-linear. We check it on the generators $v \otimes_{(a,b)} w$ for $v \in E$ and $w \in F$:

$$\begin{aligned} \tilde{\Psi}(a(v \otimes_{(a,b)} w)) &= \tilde{\Psi}((av) \otimes_{(a,b)} w + v \otimes_{(a,b)} (aw)) \\ &= \Psi(av, w) + \Psi(v, aw) = a\Psi(v,w) = a\tilde{\Psi}(v \otimes_{(a,b)} w). \end{aligned}$$

Exploiting the properties of the tensor product of $\mathbb{C}[[b]]$-modules, we can derive in a similar manner the properties of the tensor product in the theory of $(a,b)$-modules.

LEMMA 2.8. *Let $E$, $F$ and $G$ be $(a,b)$-modules. Then the tensor product has the following properties:*

(i) $E \otimes_{(a,b)} F \simeq F \otimes_{(a,b)} E$,
(ii) $(E \otimes_{(a,b)} F) \otimes_{(a,b)} G \simeq E \otimes_{(a,b)} (F \otimes_{(a,b)} G)$,
(iii) $(E \otimes_{(a,b)} F)^* \simeq E^* \otimes_{(a,b)} F^*$,
(iv) $(E \otimes_{(a,b)} F)\check{} \simeq \check{E} \otimes_{(a,b)} \check{F}$,

(v) *The $(a,b)$-morphism*

$$\Phi : E \to E \otimes_{(a,b)} E_0, \qquad v \mapsto v \otimes_{(a,b)} e_0,$$

*where $e_0$ is a generator of the elementary $(a,b)$-module $E_0$, is an isomorphism.*

(vi) *We have the following isomorphism of $(a,b)$-modules:*

$$E^* \otimes_{(a,b)} F \to \mathrm{Hom}_{(a,b)}(E, F \otimes_{(a,b)} E_0),$$
$$\phi \otimes_{(a,b)} y \mapsto (\Phi : x \mapsto y \otimes_{(a,b)} \phi(x)),$$

*where $\phi \in E^*$, $x \in E$ and $y \in F$.*

REMARK 2.9. In [Bel01], R. Belgrade defines the concept of $\delta$-dual of an $(a,b)$-module $E$:

DEFINITION 2.10 (Belgrade). Let $E$ be an $(a,b)$-module and $\delta \in \mathbb{C}$. Then the $\delta$-*dual* of $E$ is the set $\mathrm{Hom}_{\mathbb{C}[[b]]}(E, E_\delta)$ with the $(a,b)$-structure defined by

$$[a \cdot \phi](x) = \phi(ax) - a\phi(x), \qquad [b \cdot \phi](x) = -b\phi(x) = \phi(-bx).$$

From properties (v) and (vi) of the previous lemma we obtain the isomorphism $E^* \otimes_{(a,b)} F \simeq \mathrm{Hom}_{(a,b)}(E, F)$, which in turn yields an alternative description of the $\delta$-dual. In fact from Definition 2.10 it is easy to show that the $\delta$-dual of an $(a,b)$-module is the module

$$\mathrm{Hom}_{(a,b)}(\check{E}, E_\delta),$$

which in turn can be rewritten as $\check{E}^* \otimes_{(a,b)} E_\delta$.

We will call an $(a,b)$-bilinear map $E \times F \to E_0$ an $(a,b)$-*bilinear form*. In the rest of this section we will deal with the existence of non-degenerate hermitian forms on $(a,b)$-modules. We need the following definitions.

DEFINITION 2.11. Let $E$ and $F$ be $(a,b)$-modules and $\Phi$ a bilinear form on $E \times F$. We say that $\Phi$ is *non-degenerate* if the $(a,b)$-morphism $v \mapsto \Phi(v, \cdot)$ is an isomorphism of $E$ onto $F^*$.

DEFINITION 2.12. Let $E$ be an $(a,b)$-module. A *sesquilinear form* on $E$ is a bilinear form on $E \times \check{E}$.

REMARK 2.13. Since a non-degenerate sequilinear form on an $(a,b)$-module $E$ induces an isomorphism of $E$ and its adjoint $\check{E}^*$, it follows that not every $(a,b)$-module admits such a form (e.g. $E_\lambda$ with $\lambda \neq 0$ does not).

Consider now a sesquilinear form $\Phi$ on $E$. By applying to it the conjugate functor we obtain a bilinear map $\check{\Phi}$ on $\check{E} \times E$ with values in $\check{E}_0$. If we fix an isomorphism of $\check{E}_0$ with $E_0$, we can consider $\check{\Phi}$ as a sequilinear form on $\check{E}$. Under this assumption, we define $(a,b)$-hermitian and anti-$(a,b)$-hermitian forms:

Definition 2.14. Let $E$ be an $(a,b)$-module. An $(a,b)$-sesquilinear form $H$ on $E$ is called *$(a,b)$-hermitian*, respectively *anti-$(a,b)$-hermitian*, if it satisfies

$$H(v,w) = \check{H}(w,v),$$

respectively

$$H(v,w) = -\check{H}(w,v),$$

where $v \in E$, $w \in \check{E}$ and $\check{H}$ is the sesquilinear form on $\check{E}$ defined above.

We have already shown that in order to admit a non-degenerate sesquilinear form, an $(a,b)$-module must be self-adjoint. We now refine the concept of self-adjoint:

Definition 2.15. Let $E$ be a self-adjoint $(a,b)$-module. We say that $E$ is *hermitian* (resp. *anti-hermitian*) if it admits a non-degenerate hermitian (resp. anti-hermitian) form.

Let $E$ be an $(a,b)$-module endowed with a hermitian form and let $\Phi : E \to \check{E}^*$ be the linear form associated to the hermitian form via Remark 2.6.

We can translate the hermitian property into the identity between $\Phi$ and its adjoint $\check{\Phi}^* : E \to \check{E}^*$. In fact while $\Phi(v)$ for $v \in E$ is the linear map

$$\phi : w \mapsto \Phi(v,w), \qquad w \in \check{E},$$

the adjoint map $\check{\Phi}^*$ sends an element $v \in E = E^{**}$ to the map

$$\phi : w \mapsto v(\check{\Phi}(w,\cdot)) = \check{\Phi}(w,v).$$

We will use this formulation extensively in the next section.

Note moreover that having an isomorphism from an $(a,b)$-module $E$ onto its $\delta$-dual $\check{E}^* \otimes_{(a,b)} E_\delta$ is equivalent to specifying an isomorphism between $E \otimes_{(a,b)} E_{-\delta/2}$ and

$$\check{E}^* \otimes_{(a,b)} E_\delta \otimes_{(a,b)} E_{-\delta/2} \simeq \check{E}^* \otimes_{(a,b)} E_{\delta/2}.$$

Since

$$\big(\widetilde{E \otimes_{(a,b)} E_{-\delta/2}}\big)^* \simeq \check{E}^* \otimes_{(a,b)} \check{E}^*_{-\delta/2} \simeq \check{E}^* \otimes_{(a,b)} E_{\delta/2},$$

we can identify an isomorphism of $E$ with its $\delta$-dual with a hermitian form on $E \otimes_{(a,b)} E_{-\delta/2}$.

**3. Existence of hermitian forms.** In this section we will analyze the existence of non-degenerate hermitian forms on regular $(a,b)$-modules. We will proceed in two steps: in the first two subsections we will restrict ourselves to a subclass of $(a,b)$-modules called indecomposable and show that every regular $(a,b)$-module can be uniquely decomposed into the direct sum of indecomposable ones (Theorem 3.7).

In the last subsection we will show that a self-adjoint $(a,b)$-module which is indecomposable admits a hermitian or an anti-hermitian form. The result is optimal since there are examples that admit only a hermitian or only an anti-hermitian form (Theorem 3.13).

### 3.1. Indecomposable $(a, b)$-modules

DEFINITION 3.1. Let $E$ be an $(a, b)$-module. We say that $E$ is *indecomposable* if it cannot be written as a direct sum $F \oplus G$ of non-zero $(a, b)$-modules.

Whenever we decompose an $(a, b)$-module $E$ into a (non-trivial) direct sum of $(a, b)$-modules, the rank of each component is strictly less than the rank of $E$, hence by induction for every $(a, b)$-module $E$ we can find a decomposition

$$E = \bigoplus_{i=1}^{r} F_i,$$

where $r \in \mathbb{N}$ and each $F_i$ is an indecomposable $(a, b)$-submodule.

We are interested in whether the isomorphism classes of the $F_i$ are unique and do not depend upon the decomposition. To clarify this, we will need the following result:

PROPOSITION 3.2. *Let $E$ be a regular and indecomposable $(a, b)$-module. Then every endomorphism of $E$ is either bijective or nilpotent.*

The proof will need several steps beginning with a definition:

DEFINITION 3.3. Let $E$ be a regular $(a, b)$-module and $\lambda \in \mathbb{C}$. We define

$$V_\lambda = \sum_{F_i \subset E,\, F_i \simeq E_\lambda} F_i,$$

the sum of all $(a, b)$-submodules of $E$ isomorphic to $E_\lambda$.

Clearly $V_\lambda$ is an $(a, b)$-submodule. We will use $V_\lambda$ as an induction step in the proof of Proposition 3.2, by choosing a $\lambda$ such that $V_\lambda$ is normal:

PROPOSITION 3.4. *Let $E$ be a regular $(a, b)$-module, let $\lambda \in \mathbb{C}$ and let*

$$\lambda_{\min} = \inf_j \{\lambda + j \mid \exists x \in E,\ ax = (\lambda + j)bx\}$$

*be the minimal $\lambda + j$ such that $E$ contains a monomial of type $(\lambda + j, 0)$. Then $V_{\lambda_{\min}}$ is a normal $(a, b)$-submodule of $E$ isomorphic as an $(a, b)$-module to the direct sum of a finite number of copies of $E_{\lambda_{\min}}$.*

*Proof.* We will use two facts.

First, for every $(a, b)$-submodule $W \simeq \bigoplus E_{\lambda_{\min}}$ of $E$, $W$ is normal in $E$. Indeed, let $\{e_i\}_{i=1}^{p}$ be a basis of $W$ with $p$ the rank of $W$. Assume for contradiction that there exists $x \in W$ which is in $bE$, but not in $bW$.

By possibly translating $x$ by an element of $bW$, we can assume $x = \sum_{i=1}^{p} \alpha_i e_i$, $\alpha_i \in \mathbb{C}$. We can easily verify that $ax = \lambda_{\min} bx$ but now if $x = by$

we must have $ay = (\lambda_{\min} - 1)by$, and since $y \in E$ this contradicts the minimality of $\lambda_{\min}$.

On the other hand, $V_{\lambda_{\min}}$ is a direct sum of copies of $E_{\lambda_{\min}}$. In fact let $W$ be the largest (inclusionwise) direct sum of copies of $E_{\lambda_{\min}}$ included in $V_{\lambda_{\min}}$. We remark that since $W$ is normal, for any $(a,b)$-submodule $F$ isomorphic to $E_{\lambda_{\min}}$ only one of two cases is possible: either

$$W \cap F = \{0\} \quad \text{or} \quad F \subset W.$$

If $W \cap F \neq \{0\}$, let $e$ be the generator of $F$ and $S(b)b^n e \in W$ with $S(0) \neq 0$; then $S(b)e \in W$ by normality and $e = S^{-1}(b)S(b)e \in W$ so $F \subset W$.

If $W$ contains every $(a,b)$-submodule isomorphic to $E_{\lambda_{\min}}$, then it is equal to $V_{\lambda_{\min}}$: otherwise there is an $F$ such that $W \cap F = \{0\}$, hence $W \oplus F$ is still in $V_{\lambda_{\min}}$, which contradicts the maximality of $W$. ■

We will now use the $(a,b)$-submodule $V_{\lambda_{\min}}$ to prove

Proposition 3.5. *Let $E$ be a regular $(a,b)$-module and $\phi : E \to E$ an $(a,b)$-morphism. Then $\phi$ is bijective if and only if $\phi$ is injective.*

*Proof.* To show that bijectivity follows from injectivity, we will proceed by induction on the rank of the module.

If $E$ is of rank 1 the statement is satisfied: in fact $E$ must be isomorphic to one of the $E_\lambda$ and the only $b$-linear morphisms from $E_\lambda$ to itself that are also $a$-linear are those that send the generator $e$ to $\alpha e$, $\alpha \in \mathbb{C}$. They are all bijective for $\alpha \neq 0$.

Let now $E$ be of rank $n > 1$. We can find a $\lambda_{\min}$ (cf. [Bar93]) that has the minimality property of the previous proposition. Hence the module $V_{\lambda_{\min}}$ is normal and isomorphic to a direct sum of copies of $E_{\lambda_{\min}}$.

Let $\{e_i\}$ be a basis of $V_{\lambda_{\min}}$ composed of monomials of type $(\lambda_{\min}, 0)$ and let $x$ be any monomial of type $(\lambda_{\min}, 0)$. We want to show that $x$ is a linear combination of elements of the basis, with coefficients in $\mathbb{C} \subset \mathbb{C}[[b]]$.

From the definition of $V_{\lambda_{\min}}$ it follows that $x \in V_{\lambda_{\min}}$. Suppose now that $x = \sum_i S_i(b)e_i$ and apply $a$ to both sides. We obtain

$$ax = \sum_i (\lambda_{\min} S_i(b)be_i + S_i'(b)b^2 e_i) = \lambda_{\min} bx + \sum_i S_i'(b)b^2 e_i.$$

Since $x$ is a monomial of type $(\lambda_{\min}, 0)$, we must have $S_i'(b) = 0$ for all $i$, and therefore

$$x = \sum_i S_i(0)e_i,$$

as desired.

Let $\phi : E \to E$ be an injective endomorphism and $\{e_i\}$ a basis of $V_{\lambda_{\min}}$. Every $\phi(e_i)$ is a monomial of type $(\lambda_{\min}, 0)$ and therefore an element of $V_{\lambda_{\min}}$.

Hence

$$\phi|_{V_{\lambda_{\min}}} : V_{\lambda_{\min}} \to V_{\lambda_{\min}}.$$

Moreover since the coefficients of the $\phi(e_i)$ in our basis are complex constants, $\phi|_{V_{\lambda_{\min}}}$ behaves as a linear map between finite-dimensional spaces; in particular, if it is injective, it is also surjective.

In order to apply our induction hypothesis consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & V_{\lambda_{\min}} \arrow[r, hook] \arrow[d, "\phi"] & E \arrow[r, two heads] \arrow[d, "\phi"] & E/V_{\lambda_{\min}} \arrow[r] \arrow[d, "\tilde{\phi}"] & 0 \\
0 \arrow[r] & V_{\lambda_{\min}} \arrow[r, hook] & E \arrow[r, two heads] & E/V_{\lambda_{\min}} \arrow[r] & 0
\end{tikzcd}$$

where $\tilde{\phi}$ is the $(a,b)$-linear morphism induced on the quotient. As we showed above, the first vertical arrow is bijective.

The third arrow $\tilde{\phi}$ is injective: indeed, suppose that two classes with representatives $x, y \in E$ map to the same class modulo $V_{\lambda_{\min}}$. Then $\phi(x-y)$ is in $V_{\lambda_{\min}}$. From the bijectivity of $\phi|_{V_{\lambda_{\min}}}$ we can find $v \in V_{\lambda_{\min}}$ such that $\phi(x-y) = \phi(v)$, so $x - y = v$ by the injectivity of $\phi$, that is, $x$ and $y$ are in the same class modulo $V_{\lambda_{\min}}$.

Since the rank of $E/V_{\lambda_{\min}}$ is strictly inferior to the rank of $E$, we can apply the induction hypothesis to show that $\tilde{\phi}$ is also bijective.

By homological algebra, the second arrow is also bijective. ∎

We can now consider endomorphisms that are not necessarily injective. Once again the structure of $(a,b)$-modules does not differ essentially from that of finite-dimensional vector spaces over $\mathbb{C}$:

Lemma 3.6. *Let $E$ be a regular $(a,b)$-module and $\phi$ an endomorphism of $E$. Then $E$ splits into the direct sum of two $\phi$-stable $(a,b)$-submodules $F$ and $N$, with $\phi$ bijective on $F$ and nilpotent on $N$.*

*Proof.* Consider the sequence of normal $(a,b)$-submodules

$$K_n = \operatorname{Ker}\phi^n, \quad n \in \mathbb{N}.$$

Since two normal $(a,b)$-submodules $F \subset G$ are equal if and only if they have the same rank, the sequence $K_n$ stabilizes: $K_m = K_{m+1} = \cdots$ for some $m$.

On the other hand, consider the sequence $I_n = \operatorname{Im}\phi^n$. The restriction

$$\phi|_{I_m} : I_m \to I_{m+1} \subset I_m$$

is injective: if $y = \phi^m(x) \in \operatorname{Ker}\phi$, then $x \in K_{m+1} = K_m$, hence $y = 0$. From the previous proposition we deduce that this restriction is in fact bijective, which means that $I_{m+1} = \phi(I_m) = I_m$.

We can now take $F = I_m$ and $N = K_m$. They are clearly stable under $\phi$. We will show that $E = F \oplus N$.

We have $\operatorname{Ker} \phi \cap F = \{0\}$, since $\phi|_{I_m}$ is injective. *A fortiori*, since $K \subset \operatorname{Ker} \phi$ we have $F \cap N = \{0\}$.

Pick $x \in E$. Since $I_m = I_{2m}$ we can find $y \in E$ such that $\phi^m(x) = \phi^{2m}(y)$. Set $k = x - \phi^m(y)$. Thus

$$x = \phi^m(y) + k$$

with $\phi^m(y) \in I_m$ and $k \in K_m$, which implies that $E = N \oplus F$.

The restriction of $\phi$ to $N$ is nilpotent, since $\phi|_N^m = 0$, while we already showed that the restriction to $I_m = F$ is bijective. ∎

*Proof of Proposition 3.2.* Let $\phi$ be an endomorphism of $E$. Then by Lemma 3.6, $E$ splits into a sum $N \oplus F$ of two $(a, b)$-modules, with $\phi$ nilpotent on $N$ and bijective on $F$. But $E$ is indecomposable, so either $N = 0$ and $\phi$ is bijective, or $F = 0$ and $\phi$ is nilpotent. ∎

**3.2. Krull–Schmidt theorem.** This subsection will be devoted to the proof of a version of the Krull–Schmidt theorem for $(a, b)$-modules. The principal tool in the proof will be Proposition 3.2.

THEOREM 3.7 (Krull–Schmidt for $(a, b)$-modules). *Suppose that we have two decompositions of a regular $(a, b)$-module $E$:*

$$E = \bigoplus_{i=1}^{m} E_i, \qquad E = \bigoplus_{i=1}^{n} F_i,$$

*where $m, n \in \mathbb{N}$ and all $E_i$ and $F_i$ are indecomposable $(a, b)$-modules. Then $m = n$ and up to reindexing, $E_i$ is isomorphic to $F_i$ for all $1 \leq i \leq n$.*

For the proof we need a couple of lemmas:

LEMMA 3.8. *Let $E$ be a regular indecomposable $(a, b)$-module and $\phi$ an automorphism of $E$. Suppose moreover that $\phi = \phi_1 + \phi_2$. Then at least one of $\phi_1$, $\phi_2$ is an isomorphism.*

*Proof.* By applying $\phi^{-1}$ to both terms, we can assume without loss of generality that $\phi = \mathrm{Id}$ is the identity.

The endomorphisms $\phi_1$ and $\phi_2$ commute:

$$\phi_1\phi_2 - \phi_2\phi_1 = \phi_1(\phi_1 + \phi_2) - (\phi_2 + \phi_1)\phi_1 = \phi_1 - \phi_1 = 0.$$

By Lemma 3.2 each $\phi_i$ is either nilpotent or an isomorphism. If they were both nilpotent, their sum would be nilpotent, which is absurd. Hence the result. ∎

REMARK 3.9. By iterating the previous lemma, we can extend the result to sums of more than two endomorphisms.

LEMMA 3.10. *Let $E$ and $F$ be indecomposable regular $(a,b)$-modules and let $\alpha : E \to F$ and $\beta : F \to E$ be $(a,b)$-linear morphisms. Suppose that $\beta \circ \alpha$ is an isomorphism. Then $\alpha$ and $\beta$ are also isomorphisms.*

*Proof.* Let us prove that $F = \operatorname{Im}\alpha \oplus \operatorname{Ker}\beta$. If $\alpha(x) \in \operatorname{Ker}\beta$, we have $\beta \circ \alpha(x) = 0$, hence $x = 0$, and therefore

$$\operatorname{Im}\alpha \cap \operatorname{Ker}\beta = \{0\}.$$

Choose now any $x \in F$ and let

$$y = \alpha \circ (\beta \circ \alpha)^{-1} \circ \beta(x).$$

We have

$$\beta(x-y) = \beta(x) - \beta(y) = \beta(x) - (\beta \circ \alpha) \circ (\beta \circ \alpha)^{-1} \circ \beta(x) = \beta(x) - \beta(x) = 0.$$

Hence $x = y + (x - y)$ with $y \in \operatorname{Im}\alpha$ and $x - y \in \operatorname{Ker}\beta$. This implies $F = \operatorname{Im}\alpha \oplus \operatorname{Ker}\beta$.

Now since $\beta \circ \alpha$ is injective, so must be $\alpha$, and $\operatorname{Im}\alpha$ cannot be 0. As $F$ is indecomposable, we must have $\operatorname{Im}\alpha = F$ and $\operatorname{Ker}\beta = 0$. It follows that $\alpha$ is bijective and hence so is $\beta = (\beta \circ \alpha) \circ \alpha^{-1}$. ∎

*Proof of the Krull–Schmidt theorem for $(a,b)$-modules.* We use induction on $m$.

If $m = 1$, then $E$ is indecomposable, we must have $n = 1$ and $E_1 \simeq F_1$.

In the general case consider the morphisms $q_i = \pi_i \circ p_1$, where $\pi_i$ is the projection on $F_i$ and $p_j$ is the projection on $E_j$. The sum

$$\sum_i p_1 \circ q_i = p_1 \circ \sum_i \pi_i \circ p_1 = p_1 \circ p_1 = p_1$$

is the identity on $E_1$. By Lemma 3.2, there is an $i$ such that $p_1 \circ q_i|_{E_1} : E_1 \to E_1$ is an isomorphism. Suppose, without loss of generality, it is $p_1 \circ q_1$; then by Lemma 3.10, $q_1|_{E_1} = \pi_1 : E_1 \to F_1$ is an isomorphism.

To apply the induction hypothesis, set $G = \sum_{i=2}^m F_i$. We want to show that $E_1 \oplus G$ is equal to $E = F_1 \oplus G$. Since $\pi_1$ is an isomorphism of $E_1$ onto $F_1$ and its kernel is $G$, we need to show

$$E_1 \cap G = \{0\}.$$

Indeed, if $x \in E_1 \cap G$, then $\pi_1(x) = 0$, but $\pi_1$ restricted to $E_1$ is injective, so $x = 0$. On the other hand, every element of $E$ can be written as $v + w$ with $v \in F_1$ and $w \in G$. If $y \in E_1$ is such that $\pi_1(y) = v$, then

$$v + w = y + \pi_1(y) - y + w,$$

and $\pi_1(y) - y \in W$ by definition of $\pi_1$. Hence $E = E_1 \oplus G = E_1 + \sum_{i=2}^m E_i$.

We have immediately $E/E_1 \simeq G \simeq \sum_{i=2}^m E_i$ and we can apply the induction hypothesis to $G$. ∎

We now focus on finding hermitian isomorphisms of an $(a,b)$-module $E$ with its adjoint $\check{E}^*$. The Krull–Schmidt theorem will be useful to show the following decomposition:

PROPOSITION 3.11. *Let $E$ be a regular self-adjoint $(a,b)$-module. Then*

$$E \simeq \bigoplus_{i=1}^{r}(F_i^{\oplus\alpha_i}) \oplus \bigoplus_{i=1}^{s}(G_i \oplus \check{G}_i^*)^{\oplus\beta_i}$$

*where $r$, $s$, $\alpha_i$ and $\beta_i$ are positive integers. The $F_i$ are self-adjoint $(a,b)$-modules and the $G_i$ are non-self-adjoint $(a,b)$-modules. The isomorphism classes of the $F_i$, $G_i$ and $\check{G}_i^*$ are all distinct.*

*Proof.* Consider a decomposition $E = \sum_i E_i$ into indecomposable $(a,b)$-modules. Since $E$ is self-adjoint we have another decomposition

$$E \simeq \check{E}^* = \sum_i \check{E}_i^*.$$

The Krull–Schmidt theorem ensures that the summands are unique up to a permutation. So we can divide the $E_i$ into two groups.

In the first group we put the self-adjoint components $F_i$ with their multiplicities.

In the second one we put the non-self-adjoint components $G_i$ with the respective multiplicities. Since the two decompositions $\sum_i E_i$ and $\sum_i \check{E}_i^*$ contain the same modules up to a permutation, the multiplicities of the $G_i$ and of the $\check{G}_i^*$ must be equal. ∎

REMARK 3.12. From the definition above we can immediately see that the non-self-adjoint part of the decomposition always admits a hermitian non-degenerate form. In fact if we consider the module $G_i \oplus \check{G}_i^*$, a hermitian form can be given by

$$\Phi : G_i \oplus \check{G}_i^* \to (\widecheck{G_i \oplus \check{G}_i^*})^* = \check{G}_i^* \oplus G_i, \qquad (x,y) \mapsto (y,x).$$

If the multiplicity of a self-adjoint term $F_i$ is even, we fall into the same situation.

The case of odd multiplicity of a self-adjoint component is far more interesting and we will study it in the next subsection.

**3.3. Hermitian forms on indecomposable $(a,b)$-modules.** As already noted in the previous subsection, the existence of hermitian forms on an indecomposable self-adjoint $(a,b)$-module is not always guaranteed. We have in fact the following theorem:

THEOREM 3.13. *Let $E \neq \{0\}$ be a regular indecomposable self-adjoint $(a,b)$-module. Then it admits a hermitian non-degenerate form or an anti-hermitian one.*

*Proof.* Let $\Phi : E \to \check{E}^*$ be any isomorphism and set $M = \Phi^{-1}\check{\Phi}^*$. Consider now the two endomorphisms of $E$ given by

$$\mathrm{Id} + M \quad \text{and} \quad \mathrm{Id} - M.$$

They commute and can be either isomorphisms or nilpotent, since $E$ is indecomposable. But if they were both nilpotent, their sum $2\,\mathrm{Id}$ would be nilpotent too, which is absurd.

If $\mathrm{Id} + M$ is an isomorphism, so is $S = \Phi + \check{\Phi}^*$, which is associated to a non-degenerate hermitian form. The bijectivity of $\mathrm{Id} - M$ on the other hand gives us an isomorphism $A = \Phi - \check{\Phi}^*$, which comes from an anti-hermitian form. ∎

Note that both cases of the previous theorem can occur.

Example 3.14. The simplest example of a regular self-adjoint and indecomposable $(a, b)$-module which admits only a hermitian form is the elementary $(a, b)$-module $E_0$ with the isomorphism that sends the generator $e$ to its adjoint $\check{e}^*$.

Example 3.15. To obtain only an anti-hermitian form, we can consider, for given $\lambda, \mu \in \mathbb{C}$ such that $\pm\lambda\pm\mu \notin \mathbb{Z}$ for all choices of signs, the $(a, b)$-module $E$ of rank 4 generated by $\{e_1, e_2, e_3, e_4\}$ which satisfies

$$\begin{aligned} ae_1 &= \lambda b e_1, \\ ae_2 &= \mu b e_2 + e_1, \\ ae_3 &= -\mu b e_3 + e_1, \\ ae_4 &= -\lambda b e_4 + e_2 - e_3, \end{aligned} \tag{3.1}$$

whose adjoint basis satisfies

$$\begin{aligned} a \cdot \check{e}_4^* &= \lambda b \check{e}_4^*, \\ a \cdot \check{e}_3^* &= \mu b \check{e}_3^* - \check{e}_4^*, \\ a \cdot \check{e}_2^* &= -\mu b \check{e}_2^* + \check{e}_4^*, \\ a \cdot \check{e}_1^* &= -\lambda b \check{e}_1^* + \check{e}_3^* + \check{e}_2^*. \end{aligned}$$

It is easy to show by calculation that the only isomorphism between $E$ and $\check{E}^*$ is, up to mutliplication by a complex number, the one that sends $e_1$, $e_2$, $e_3$ and $e_4$ to $\check{e}_4$, $-\check{e}_3$, $\check{e}_2$ and $-\check{e}_1$ respectively.

This isomorphism is anti-hermitian and since there are no other isomorphisms, $E$ is also indecomposable.

Example 3.16. The regular $(a, b)$-module $E_0 \oplus E_0$ admits both a hermitian and an anti-hermitian form.

**4. Duality of geometric $(a, b)$-modules.** In the study of the Brieskorn lattice K. Saito introduced the concept of "higher residue pairings" (cf. [Sai83]), which can be defined using a set of axiomatic properties.

Using the theory of $(a,b)$-modules R. Belgrade showed the existence of a duality isomorphism between an $(a,b)$-module associated to a germ of a holomorphic function in $\mathbb{C}^{n+1}$ with an isolated singularity at the origin and its $(n+1)$-dual. We will prove (as already noticed by R. Belgrade in [Bel01]) that the concepts of "higher residue pairings" and self-adjoint $(a,b)$-module are linked.

$D$ will always denote the Brieskorn module associated to a holomorphic function in $\mathbb{C}^{n+1}$ with an isolated singularity, and $E$ its $b$-adic completion considered as an $(a,b)$-module.

The following theorem of R. Belgrade gives a relationship between $E$ and its $(n+1)$-dual.

Theorem 4.1 (Belgrade). *Let $E$ be the $(a,b)$-module associated to a germ of holomorphic function $f:\mathbb{C}^{n+1}\to\mathbb{C}$. Then there is a natural isomorphism between $E$ and its $(n+1)$-dual:*

$$\Delta:\ E\simeq \check{E}^*\otimes_{(a,b)}E_{n+1}.$$

We can obtain from this isomorphism a series $\Delta_k : E\times E\to\mathbb{C}$ of bilinear forms defined as follows:

$$[\Delta(y)](x)=(n+1)!\sum_{k=0}^{\infty}\Delta_k(x,y)b^k e_{n+1}$$

for $x,y\in E$.

**5. "Higher residue pairings" of K. Saito.** K. Saito introduced in [Sai83] a series of pairings on the Brieskorn lattice $D$ which are called "higher residue pairings":

$$K^{(k)}:\ D\times D\to\mathbb{C},\qquad k\in\mathbb{N},$$

characterized by the following properties:

(i) $K^{(k)}(\omega_1,\omega_2)=K^{(k+1)}(b\omega_1,\omega_2)=-K^{(k+1)}(\omega_1,b\omega_2)$.
(ii) $K^{(k)}(a\omega_1,\omega_2)-K^{(k)}(\omega_1,a\omega_2)=(n+k)K^{(k-1)}(\omega_1,\omega_2)$.
(iii) $K^{(0)}(D,bD)=K^{(0)}(bD,D)=0$ and $K^{(0)}$ induces Grothendieck's residue on the quotient $D/bD$.
(iv) $K^{(k)}$ is $(-1)^k$-symmetric.

Remark 5.1. We notice that from properties (i) and (iii) above we can deduce that $K^{(k)}(D,b^{k+1}D)=K^{(k)}(b^{k+1}D,D)=0$, so we can consider the pairings $K^{(k)}$ as being defined on $D/b^{k+1}D$.

In the following section we will show the following result:

Proposition 5.2. *The $\Delta_k$ have properties* (i)–(iii) *of "higher residue pairings" of K. Saito.*

## 6. Proof of Proposition 5.2

**6.1. Proof of (i).** We use the $b$-linearity of $\Delta(y)$ to obtain

$$\sum_k (n+1)!\Delta_k(bx,y)b^k e_{n+1} = [\Delta(y)](bx) = b[\Delta(y)](x)$$
$$= \sum_k (n+1)!\Delta_k(x,y)b^{k+1}e_{n+1},$$

which gives $\Delta_k(x,y) = \Delta_{k+1}(bx,y)$. Similarly, by using the $b$-linearity of $\Delta$ and the adjoint morphism, we obtain

$$\Delta(by)(x) = \check{\Delta}^*(x)(by) = -b\check{\Delta}^*(x)(y) = -b\Delta(y)(x),$$

and therefore

$$(n+1)!\sum_k \Delta_k(x,by)b^k e_{n+1} = \Delta(by)(x) = -b\Delta(y)(x)$$
$$= (n+1)!\sum_k -\Delta_k(x,y)b^{k+1}e_{n+1},$$

which implies $\Delta_k(bx,y) = -\Delta_{k+1}(x,by)$.

**6.2. Proof of (ii).** Since $\Delta$ is an isomorphism we have $\Delta(ay) = a\cdot_{\check{E}^*\otimes E_\delta} [\Delta(y)]$ and

$$(n+1)!\sum_k \Delta_k(x,ay)b^k e_{n+1} = \Delta(ay)(x) = a\cdot[\Delta(y)](x)$$
$$= \Delta(y)(ax) - a[\Delta(y)(x)] = (n+1)!\sum_k \big(\Delta_k(ax,y)b^k e_{n+1} - \Delta_k(x,y)ab^k e_{n+1}\big).$$

The definitions of $(a,b)$-module and of $E_{n+1}$ ($ae_{n+1} = (n+1)be_{n+1}$) give

$$ab^k e_{n+1} = b^k a e_{n+1} + kb^{k+1}e_{n+1} = (n+k+1)b^{k+1}e_{n+1}$$

hence

$$\Delta_k(ax,y) - \Delta_k(x,ay) = (n+k)\Delta_{k-1}(x,y).$$

**6.3. Grothendieck's residue.** We now have to show that the pairing $\Delta_0$ induces Grothendieck's residue on $D/bD \simeq \Omega^{n+1}/df\wedge\Omega^n$.

**Proof of (iv)**. From the definition of $\Delta_0$ and the $b$-linearity of $\Delta$ it is easy to see that $\Delta_0(D,bD) = \Delta_0(bD,D) = 0$. We can hence consider $\Delta_0$ as a pairing on $D/bD$.

Grothendieck's residue is defined as follows:

$$\operatorname{Res}(g,h) := \lim_{\varepsilon_j\to 0,\forall j} \int_{|\partial f/\partial z_j|=\varepsilon_j} \frac{gh\,dz}{\partial f/\partial z_1\cdots\partial f/\partial z_{n+1}}$$

where $g,h\in\mathcal{O}$ and $dz = dz_1\wedge\cdots\wedge dz_{n+1}$.

The morphism $\Delta$ is defined as the composition of six $(a, b)$-module morphisms ([Bel01]) shown in the following graph:

$$\begin{tikzcd}
E \arrow[r, "\alpha"] & F_1 \arrow[r, "\beta"] & F_2 & \\
& & & F_3 \arrow[ul, "\gamma"] \arrow[dl, "\delta"] \\
\check{E}_{n+1} & F_5 \arrow[l, "\zeta"] & F_4 \arrow[l, "\eta"] &
\end{tikzcd}$$

These morphisms pass to the quotient by the action of $b$, giving a decomposition of the morphism $\Delta_0$:

$$\begin{tikzcd}
E/bE \arrow[r, "\tilde{\alpha}"] & F_1/bF_1 \arrow[r, "\tilde{\beta}"] & F_2/bF_2 & \\
& & & F_3/bF_3 \arrow[ul, "\tilde{\gamma}"] \arrow[dl, "\tilde{\delta}"] \\
\frac{\check{E}^* \otimes E_{n+1}}{b(\check{E}^* \otimes E_{n+1})} & F_5/bF_5 \arrow[l, "\tilde{\zeta}"] & F_4/bF_4 \arrow[l, "\tilde{\eta}"] &
\end{tikzcd}$$

We have to verify that the image of $[g\,dz]$ under $\Delta_0$ is $\operatorname{Res}(g, \cdot)$, where $gdz$ is an element of $\Omega^{n+1}$. We will accomplish this in several steps using the decomposition above.

STEP 1: *$E$, $F_1$ and $F_2$.* We have the isomorphisms

$$\frac{F_1}{bF_1} \simeq \frac{\Omega^{n+1}}{df \wedge \Omega^n}, \qquad \frac{F_2}{bF_2} \simeq \frac{\mathcal{D}b^{n+1}}{(\bar{\partial} - df\wedge)\mathcal{D}b^n},$$

the morphism $\tilde{\alpha}$ coincides with the identity on $\Omega^{n+1}/df \wedge \Omega^n$, and $\tilde{\beta}$ is induced by the inclusion $i : \Omega^{n+1} \to \mathcal{D}b^{n+1}$. We deduce that $\tilde{\beta} \circ \tilde{\alpha}([g\,dz]) = [i(g\,dz)]$. Let us write $T \in \mathcal{D}b^{n+1,0}$ for the current $i(g\,dz)$.

STEP 2: *path between $F_2$ and $F_3$.* By using the description of Lemma 3.4.2 of [Bel01] we see that

$$\frac{F_3}{bF_3} = \frac{\operatorname{Ker}(\mathcal{D}b^{0,n+1} \xrightarrow{df\wedge} \mathcal{D}b^{1,n+1})}{\bar{\partial}\operatorname{Ker}(\mathcal{D}b^{0,n} \xrightarrow{df\wedge} \mathcal{D}b^{1,n})}$$

and the isomorphism $\tilde{\gamma}$ is induced by the inclusion $\mathcal{D}b^{0,n+1} \subset \mathcal{D}b^{n+1}$. In order to find $S := \tilde{\gamma}^{-1}(T)$ we have to solve the following system:

$$\begin{aligned} T &= df \wedge \alpha^{n,0}, \\ \bar{\partial}\alpha^{n,0} &= df \wedge \alpha^{n-1,1}, \\ &\dots \\ \bar{\partial}\alpha^{1,n-1} &= df \wedge \alpha^{0,n}, \\ \bar{\partial}\alpha^{0,n} &= S, \end{aligned}$$

where $\alpha^{p,q} \in \mathcal{D}b^{p,q}$. This system has a solution, since the complex $(\mathcal{D}b^{\bullet,q}; df\wedge)$ is acyclic in degree $\neq 0$ for all $q \in \{0, \dots, n+1\}$, and the solution satisfies $[S] = [T]$ where $[\cdot]$ is the equivalence class in $F_2/bF_2$. We have

$$(\bar{\partial} - df\wedge) \sum_{k=0}^{n} \alpha^{k,n-k} = \bar{\partial}\alpha^{0,n} - df \wedge \alpha^{n,0} = S - T.$$

We can compute this solution explicitly. Let $(p,q) \in \mathbb{N}^2$ and $\phi^{p,q}$ a $C^\infty$ test form with compact support and of type $(p,q)$. The action of $T$ over $\phi^{0,n+1}$ is given by

$$\langle T, \phi^{0,n+1} \rangle = \int \phi^{0,n+1} \wedge g\, dz.$$

Then the current $\alpha^{n,0}$ defined by

$$\langle \alpha^{n,0}, \phi^{1,n+1} \rangle = \lim_{\varepsilon_1 \to 0} \int\limits_{|\partial_1 f| \geq \epsilon_1} \frac{\phi^{1,n+1} \wedge g\, dz_2 \wedge \dots \wedge dz_{n+1}}{\partial_1 f}$$

satisfies $T = df \wedge \alpha^{n,0}$: in fact,

$$\begin{aligned} \langle df \wedge \alpha^{n,0}, \phi^{0,n+1} \rangle &= \lim_{\varepsilon_1 \to 0} \int\limits_{|\partial_1 f| \geq \epsilon_1} \frac{\phi^{0,n+1} \wedge df \wedge g\, dz_2 \wedge \dots \wedge dz_{n+1}}{\partial_1 f} \\ &= \int \phi^{0,n+1} \wedge g\, dz, \end{aligned}$$

and thanks to the Stokes theorem,

$$\begin{aligned} \langle \bar{\partial}\alpha^{n,0}, \phi^{1,n} \rangle = -\langle \alpha^{n,0}, \bar{\partial}\phi^{1,n} \rangle &= \lim_{\varepsilon_1 \to 0} - \int\limits_{|\partial_1 f| \geq \epsilon_1} \frac{\bar{\partial}\phi^{1,n} \wedge g\, dz_2 \wedge \dots \wedge dz_{n+1}}{\partial_1 f} \\ &= \lim_{\varepsilon_1 \to 0} \int\limits_{|\partial_1 f| = \epsilon_1} \frac{\phi^{1,n} \wedge g\, dz_2 \wedge \dots \wedge dz_{n+1}}{\partial_1 f}. \end{aligned}$$

We will remark that the currents $\alpha_k^{n,0}$ defined for $1 \leq k \leq n+1$ by

$$\langle \alpha_k^{n,0}, \phi^{1,n+1} \rangle = \lim_{\varepsilon_k \to 0} \int\limits_{|\partial_k f| \geq \epsilon_k} \frac{(-1)^{k+1} \phi^{1,n+1} \wedge g\, dz_1 \wedge \cdots \widehat{dz_k} \cdots \wedge dz_{n+1}}{\partial_k f}$$

also satisfy $df \wedge \alpha_k^{n,0} = T$. Moreover, $[\bar{\partial}\alpha^{n,0}] = [\bar{\partial}\alpha_k^{n,0}]$ in $F_2/bF_2$: in fact, $(\bar{\partial} - df\wedge)(\alpha^{n,0} - \alpha_k^{n,0}) = \bar{\partial}\alpha^{n,0} - \bar{\partial}\alpha_k^{n,0}$.

For all $k \in 0,\dots,n$ and $1 \le i_1 < \dots < i_{k+1} \le n+1$ define

$$\alpha^{n-k,k}_{i_1,\dots,i_{k+1}} = \frac{1}{(k+1)!} \lim_{\substack{\epsilon_{i_q}\to 0\\ \forall 1\le q\le k+1}} \int\limits_{\substack{|\partial_{i_1} f|\ge \epsilon_{i_1}\\ |\partial_{i_q} f| = \epsilon_{i_q}}} \frac{(-1)^{\sum_q i_q+1} g \bigwedge_{l\ne i_1,\dots,i_{k+1}} dz_l}{\partial_{i_1} f \dots \partial_{i_{k+1}} f}$$

and let $\alpha^{n-k,k} := \alpha^{n-k,k}_{1,2,\dots,k+1}$. A simple computation gives

$$\langle df \wedge \alpha^{n-k,k}_{i_1,\dots,i_{k+1}}, \phi^{k,n-k+1}\rangle = \left\langle \frac{1}{k+1}\sum_{q=1}^{k+1} \bar\partial \alpha^{n-k+1,k-1}_{i_1,\dots,\widehat{i_q},\dots,i_{k+1}}, \phi^{k,n-k+1}\right\rangle.$$

Using this formula, we can prove by induction on $k$ that the class of the current $\alpha^{n-k,k}_{i_1,\dots,i_{k+1}}$ does not depend on the $i_q$s. This gives

$$[df \wedge \alpha^{n-k,k}] = [\bar\partial \alpha^{n-k+1,k-1}].$$

In particular $\bar\partial\alpha^{0,n}$ acts on the test function $\phi^{n+1,0}$ in the following way:

$$\langle \bar\partial \alpha^{0,n}, \phi^{n+1,0}\rangle = \frac{1}{(n+1)!} \lim_{\substack{\epsilon_k\to 0\\ \forall k}} \int\limits_{\substack{|\partial_k f| = \epsilon_k\\ \forall k}} \frac{\phi^{n+1,0} g}{\partial_1 f \dots \partial_{n+1} f}.$$

Step 3: *from $F_3/bF_3$ to $(D/bD)^*$.* Notice that $S$ is a current of type $(0,n+1)$ with support at the origin.

We have the isomorphism

$$\frac{F_4}{bF_4} \simeq \operatorname{Ker}\big(\mathcal{H}^{n+1}_0(X,\mathcal{O}) \xrightarrow{df\wedge} \mathcal{H}^{n+1}_0(X,\Omega^1)\big),$$

the isomorphism between $F_3/bF_3$ and $F_4/bF_4$ is the natural one, and

$$\frac{F_5}{bF_5} \simeq \left(\frac{\Omega^{n+1}}{df\wedge \Omega^n}\right)^*.$$

From Steps 1–3 we deduce that $\Delta_0$ induces Grothendieck's residue.

**6.4. Property (iv).** We will show that the isomorphism given by R. Belgrade can be easily transformed into one that satisfies (iv).

Let $\Delta : E \to \check{E}^* \otimes_{(a,b)} E_{n+1}$ be Belgrade's isomorphism. By tensoring with $E_{(n+1)/2}$ we can show that the isomorphisms between $E$ and $\check{E}^* \otimes_{(a,b)} E_{n+1}$ are in bijection with the isomorphisms between $E \otimes_{(a,b)} E_{-(n+1)/2}$ and its adjoint, through the map that sends an isomorphism $\Phi$ to $\Phi \otimes_{(a,b)} \mathrm{Id}_{E_{-(n+1)/2}}$.

By an easy calculation we can prove the following lemma:

Lemma 6.1. *Let $\Delta : E \to \check{E}^* \otimes E_{n+1}$ be an isomorphism and*

$$\Delta(y)(x) = (n+1)! \sum_k \Delta_k(x,y) b^k e_{n+1}$$

*for $x,y \in E$. Then the $\Delta_k$ satisfy Saito's condition* (iv) *if and only if the isomorphism $\Delta \otimes_{(a,b)} \mathrm{Id}_{E_{-(n+1)/2}}$ is hermitian.*

*Proof.* $\Delta \otimes_{(a,b)} \mathrm{Id}_{E_{-(n+1)/2}}$ is self-adjoint iff

$$\Delta \otimes_{(a,b)} \mathrm{Id}_{E_{-(n+1)/2}}(y \otimes e_{-(n+1)/2})(x \otimes e_{-(n+1)/2}) = \sum_k S_k b^k e_0$$
$$\Leftrightarrow\ \Delta \otimes_{(a,b)} \mathrm{Id}_{E_{-(n+1)/2}}(x \otimes e_{-(n+1)/2})(y \otimes e_{-(n+1)/2}) = \sum_k S_k (-b)^k e_0$$

for all $x, y \in E$. On the other hand,

$$\Delta \otimes_{(a,b)} \mathrm{Id}_{E_{-(n+1)/2}}(y \otimes e_{-(n+1)/2})(x \otimes e_{-(n+1)/2}) = \sum_k S_k b^k e_0$$
$$\Leftrightarrow\ \Delta(y)(x) = \sum_k S_k b^k e_{n+1}. \ \blacksquare$$

By combining the previous equivalence with the results on the existence of hermitian forms, we can extend Belgrade's result:

THEOREM 6.2. *Let $E$ be a regular $(a,b)$-module associated to a holomorphic function from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ with an isolated singularity. Then there exists an isomorphism $\Phi : E \to \check{E}^* \otimes_{(a,b)} E_{n+1}$ with*

$$\Phi(y)(x) = (n+1)! \sum_k \Phi_k(x,y) b^k e_{n+1}$$

*for all $x$ and $y$ such that the $\mathbb{C}$-bilinear forms $\Phi_k$ have all four properties of Saito's "higher residue pairings".*

*Proof.* Let $\Delta$ be Belgrade's isomorphism and define $\Delta_k$ as at the beginning of this section. Consider the isomorphism

$$\check{\Delta}^* \otimes_{(a,b)} \mathrm{Id}_{E_{n+1}} : E \to \check{E}^* \otimes_{(a,b)} E_{n+1}$$

and let $\Phi = (\Delta + \check{\Delta}^* \otimes_{(a,b)} \mathrm{Id}_{E_{n+1}})/2$.

It is easy to see that the $\Phi_k$ satisfy (i) and (ii). Moreover since $\Delta_0$ is symmetric (Grothendieck's residue) and $\check{\Delta}^* \otimes_{(a,b)} \mathrm{Id}_{E_{n+1}}$ induces the transpose of $\Delta_0$ on $E/bE$, we have

$$\Phi_0 = (\Delta_0 + {}^t\Delta_0)/2 = \Delta_0.$$

We have also

$$\begin{aligned}\Phi \otimes_{(a,b)} \mathrm{Id}_{E_{-(n+1)/2}} &= (\widetilde{\Phi \otimes_{(a,b)} \mathrm{Id}_{E_{-(n+1)}}})^* = \check{\Phi}^* \otimes_{(a,b)} \mathrm{Id}_{E_{(n+1)/2}} \\ &= \Phi \otimes_{(a,b)} \mathrm{Id}_{E_{-(n+1)/2}},\end{aligned}$$

therefore the $\Phi_k$ satisfy (iv).

We just have to show that $\Phi \otimes_{(a,b)} \mathrm{Id}_{E_{-(n+1)/2}}$ is an isomorphism. Since there exists an isomorphism between $E \otimes_{(a,b)} E_{-(n+1)/2}$ and its adjoint $\Delta \otimes_{(a,b)} \mathrm{Id}_{E_{-(n+1)/2}}$, we can apply Proposition 3.5 and restrict ourselves to proving the injectivity of $\Phi \otimes_{(a,b)} \mathrm{Id}_{E_{-(n+1)/2}}$. But if $\Phi \otimes_{(a,b)} \mathrm{Id}_{E_{-(n+1)/2}}$ were not injective, $\Phi$ would induce a degenerate form on $E/bE$, which is absurd. $\blacksquare$

The existence of a hermitian form on $E \otimes_{(a,b)} E_{-(n+1)/2}$ gives us an interesting restriction on the kind of $(a,b)$-module associated with Brieskorn lattices:

COROLLARY 6.3. *Let $E$ be a regular $(a,b)$-module associated to a holomorphic function from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ with an isolated singularity. Then $E \otimes_{(a,b)} E_{-(n+1)/2}$ is a hermitian $(a,b)$-module.*

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