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Stepanov-like square-mean pseudo almost automorphic solutions to partial neutral stochastic functional differential equations

by Zuomao Yan, Xiu-Mei Jia and Xingxue Yan (Zhangye)

Abstract. We obtain the existence and uniqueness of square-mean pseudo almost automorphic mild solutions to first-order partial neutral stochastic functional differential equations with Stepanov-like almost automorphic coefficients in a real separable Hilbert space.

1. Introduction. Recently, the theory of almost automorphic functions has been developed extensively. In [XL], Xiao et al. introduced the concept of pseudo almost automorphic functions, which generalizes the one of almost automorphic functions. Moreover, they obtained sufficient conditions for the existence and uniqueness of pseudo almost automorphic mild solutions to some semilinear differential equations, in abstract spaces. For other contributions concerning pseudo almost automorphic solutions to differential equations; see, for example, [A], [CT], [E], [EF], [DM], [LN], [LZ], [XZ] and the references therein.

On the other hand, the notion of Stepanov-like almost automorphic functions was introduced by Casarino in [CA] and developed by N'Guérékata and Pankov [NP]. The latter authors also established the existence and uniqueness theorems for Stepanov-like almost automorphic solutions to parabolic evolution equations.

As a natural generalization of the concept of Stepanov-like almost automorphic functions as well as the one of pseudo automorphic functions, Stepanov-like pseudo almost automorphic functions were introduced by Diagana [DI], who studied the existence and uniqueness of Stepanov-like pseudo almost automorphic mild solutions for some abstract differential equations.

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One can refer to [DN], [FL], [FV], [HJ], [LA] for more recent results in the Stepanov-like almost automorphic theory.

More recently, Fu et al. [FL] introduced the new concept of square-mean almost automorphic stochastic processes, which generalized the almost automorphic theory to the stochastic setting. The paper dealt with the existence and uniqueness of square-mean almost automorphic mild solutions for stochastic differential equations in Hilbert spaces. In [CZ] the existence and uniqueness of Stepanov-like almost automorphic mild solution to a class of nonlinear stochastic differential equations was investigated. The results obtained extend some known ones to the framework of square-mean almost automorphic processes.

Chen and Lin [CL] introduced the concept of square-mean pseudo almost automorphy for a stochastic process and proved the existence, uniqueness and global stability of square-mean pseudo almost automorphic solutions for a general class of stochastic evolution equations. Their results are more general and complicated than those concerning almost periodic solutions or pseudo almost periodic solutions to stochastic differential equations. We refer the reader to the papers [AT], [BA], [CY], [PT], [T] and the references therein. Bezandry and Diagana [B], [BD], [BE] studied the existence and uniqueness of square-mean almost periodic mild solutions for various classes of semilinear stochastic evolution equations. The existence and uniqueness of Stepanov (quadratic-mean) almost periodic solutions to stochastic evolution equations is studied in [BN]. Further, in [BZ], these authors obtained new existence theorems for pth mean pseudo almost automorphic mild solutions to some nonautonomous stochastic differential equations in Hilbert spaces.

In this paper, we investigate the existence and uniqueness of Stepanov-like square-mean pseudo almost automorphic solutions to the following partial neutral stochastic functional differential equations:

$$\begin{aligned} d[x(t)-q(t,x(t-r))] &= Ax(t)\,dt+h(t,x(t-r))\,dt \\ &\quad + f(t,x(t-r))\,dW(t), \qquad t\in\mathbb{R}, \end{aligned} \tag{1.1}$$

where $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\mathcal{F}_t)$ with $\mathcal{F}_t=\sigma\{W(u)-W(v);\ u,v\le t\}$; A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\ge 0}$ on $L^2(\mathbb{P},\mathbb{H})$; $r\ge 0$ is a fixed constant; and q,h,f are appropriate functions to be specified later.

Existence results concerning almost automorphic solutions for different kinds of abstract partial neutral differential equations have been considered in many publications, such as [D], [DH], [EN], [HM]. However, the existence of square-mean pseudo almost automoprhic solutions to partial neutral stochastic functional differential equations of the form (1.1) in the case when the forcing terms q,h,f are Stepanov-like almost automorphic coefficients is an untreated topic and constitutes the main motivation of the present paper. For this reason, we introduce and study the notion of Stepanov-like square-mean pseudo almost automorphic functions for stochastic processes, which generalizes the above-mentioned concepts, in particular, the notion of square-mean pseudo almost automorphic processes. As an application, we prove the existence and uniqueness of Stepanov-like square-mean pseudo almost automorphic mild solutions to a neutral stochastic functional differential equation. Our result generalizes most of the known results on the existence of square-mean almost automorphic (respectively, pseudo almost automorphic) solutions to (1.1).

The paper is organized as follows. In Section 2, we briefly recall some basic notations, definitions, and lemmas relating to stochastic systems and Stepanov-like square-mean pseudo almost automorphic processes. In Section 3, we prove the existence and uniqueness of Stepanov-like square-mean pseudo almost automorphic solutions for (1.1). Finally in Section 4, we give an example to illustrate the abstract results.

2. Preliminaries. In this section, we introduce some basic definitions, notations and lemmas which are used throughout this paper.

We assume that $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ is a real separable Hilbert space, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, and $L^2(\mathbb{P}, \mathbb{H})$ stands for the space of all $\mathbb{H}$-valued random variables x such that $E\|x\|^2 = \int_\Omega \|x\|^2 \, d\mathbb{P} < \infty$, which is a Banach space with the norm $\|x\|_2 = (\int_\Omega \|x\|^2 \, d\mathbb{P})^{1/2}$. It is routine to check that $L^2(\mathbb{P}, \mathbb{H})$ is a Hilbert space equipped with the norm $\|\cdot\|_2$. We let $(\mathbb{K}, \|\cdot\|_{\mathbb{K}}, \langle\cdot,\cdot\rangle_{\mathbb{K}})$ denote a real separable Hilbert space continuously embedded into $\mathbb{H}$. The notation $L(\mathbb{K}, \mathbb{H})$ stands for the space of all bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$, equipped with the usual operator norm $\|\cdot\|_{L(\mathbb{K},\mathbb{H})}$; in particular, we write $L(\mathbb{H})$ when $\mathbb{K} = \mathbb{H}$. In addition, $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{W(u) - W(v);\, u, v \le t\}$.

Let $C(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$ and $BC(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$ stand for the collection of all continuous functions from $\mathbb{R}$ into $L^2(\mathbb{P}, \mathbb{H})$, respectively the Banach space of all bounded continuous functions from $\mathbb{R}$ into $L^2(\mathbb{P}, \mathbb{H})$, equipped with the sup norm $\|\cdot\|_\infty$. Similarly, $C(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}))$ and $BC(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H}))$ stand, respectively, for the class of all jointly continuous functions from $\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H})$ into $L^2(\mathbb{P}, \mathbb{H})$ and the collection of all jointly bounded continuous functions from $\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H})$ into $L^2(\mathbb{P}, \mathbb{H})$.

In this paper, the operator A is the infinitesimal generator of an exponentially stable C_0-semigroup $(T(t))_{t\ge 0}$ on $L^2(\mathbb{P}, \mathbb{H})$, that is, there exist $M, \delta > 0$ such that

$$\|T(t)\| \le Me^{-\delta t} \quad \text{for all } t \ge 0.$$

2.1. Square-mean pseudo almost automorphy

DEFINITION 2.1 ([CL]). A stochastic process $x(\cdot) : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is said to be *stochastically bounded* if there exists $\widehat{M} > 0$ such that $E\|x(t)\| \le \widehat{M}$ for all $t \in \mathbb{R}$.

DEFINITION 2.2 ([CL]). A stochastic process $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is said to be *stochastically continuous* if

$$\lim_{t\to s} E\|x(t) - x(s)\|^2 = 0.$$

Denote by $BC(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$ the collection of all stochastically bounded and continuous processes.

REMARK 2.3 ([CL]). $BC(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$ is a linear space.

REMARK 2.4 ([CL]). $BC(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$ is a Banach space with the norm

$$\|x\|_\infty := \sup_{t\in\mathbb{R}} (E\|x(t)\|^2)^{1/2}$$

for $E\|x(t)\|^2 = (\int_\Omega \|x(t)\|^2 \, d\mathbb{P})^{1/2}$.

DEFINITION 2.5 ([FL]). A stochastically continuous process $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is said to be *square-mean almost automorphic* if for every sequence $(s'_n)_{n\in\mathbb{N}}$ of real numbers there is a subsequence $(s_n)_{n\in\mathbb{N}}$ and a stochastic process $y : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ such that

$$\lim_{n\to\infty} E\|x(t+s_n) - y(t)\|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} E\|y(t-s_n) - x(t)\|^2 = 0$$

for each $t \in \mathbb{R}$. This means that

$$\lim_{m\to\infty} \lim_{n\to\infty} E\|x(t+s_n-s_m) - x(t)\|^2 = 0$$

for each $t \in \mathbb{R}$. Denote the set of all such processes by $AA(L^2(\mathbb{P}, \mathbb{H}))$.

REMARK 2.6 ([FL, Lemma 2.3]). If $x \in AA(L^2(\mathbb{P}, \mathbb{H}))$, then x is bounded, that is, $\|x\|_\infty < \infty$.

LEMMA 2.7 ([N], [NG], [GN]). *If $f, f_1, f_2 \in AA(L^2(\mathbb{P}, \mathbb{H}))$, then*

- (i) $f_1 + f_2 \in AA(L^2(\mathbb{P}, \mathbb{H}))$;
- (ii) $\lambda f \in AA(L^2(\mathbb{P}, \mathbb{H}))$ *for any scalar* λ;
- (iii) $f_\tau, \hat{f} \in AA(L^2(\mathbb{P}, \mathbb{H}))$, *where* $f_\tau(t) = f(t+\tau)$ *and* $\hat{f}(t) := f(-t)$;
- (iv) *the range* $\mathcal{R}_f := \{f(t) : t \in \mathbb{R}\}$ *is relatively compact in* $L^2(\mathbb{P}, \mathbb{H})$, *thus f is bounded in norm;*
- (v) *if $f_n \to f$ uniformly on $\mathbb{R}$ where each f_n is in $AA(L^2(\mathbb{P}, \mathbb{H}))$, then $f \in AA(L^2(\mathbb{P}, \mathbb{H}))$ too;*
- (vi) *if $g \in L^1(\mathbb{R})$, then $f * g \in AA(\mathbb{R})$, where $*$ denotes convolution.*

We set

$$PAP_0(L^2(\mathbb{P},\mathbb{H})) = \Big\{f \in BC(\mathbb{R}, L^2(\mathbb{P},\mathbb{H})) : \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E\|x(t)\|^2\,dt = 0\Big\}.$$

DEFINITION 2.8 ([CL]). A stochastically continuous process $f(\cdot) : \mathbb{R} \to L^2(\mathbb{P},\mathbb{H})$ is said to be *square-mean pseudo almost automorphic* if it can be decomposed as $f = g+\varphi$, where $g \in AA(L^2(\mathbb{P},\mathbb{H}))$ and $\varphi \in PAP_0(L^2(\mathbb{P},\mathbb{H}))$. Denote by $PAA(L^2(\mathbb{P},\mathbb{H}))$ the set of all such processes.

REMARK 2.9 ([CL]). $PAA(L^2(\mathbb{P},\mathbb{H}))$ is a closed linear subspace of $BC(\mathbb{R}, L^2(\mathbb{P},\mathbb{H}))$.

REMARK 2.10 ([CL]). $PAA(L^2(\mathbb{P},\mathbb{H}))$ is a Banach space with the norm $\|\cdot\|_\infty$.

DEFINITION 2.11 ([FL]). A jointly continuous function $f : \mathbb{R}\times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$ is said to be *square-mean almost automorphic* in $t \in \mathbb{R}$ for each $x \in L^2(\mathbb{P},\mathbb{H})$ if for every sequence $(s'_n)_{n\in\mathbb{N}}$ of real numbers there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $\tilde{f} : \mathbb{R}\times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$ such that

$$\lim_{n\to\infty} E\|f(t+s_n,x) - \tilde{f}(t,x)\|^2 = 0, \qquad \lim_{n\to\infty} E\|\tilde{f}(t-s_n) - f(t,x)\|^2 = 0$$

for each $t \in \mathbb{R}$ and each $x \in L^2(\mathbb{P},\mathbb{H})$. Denote the set of all such functions f by $AA(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}))$.

LEMMA 2.12 ([FL]). *Let $f : \mathbb{R}\times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$ be square-mean almost automorphic in $t \in \mathbb{R}$ for each $x \in L^2(\mathbb{P},\mathbb{H})$, and assume that f satisfies a Lipschitz condition in the following sense:*

$$E\|f(t,\phi) - f(t,\psi)\|^2 \le \tilde{M}E\|\phi-\psi\|^2$$

for all $\phi,\psi \in L^2(\mathbb{P},\mathbb{H})$ and $t \in \mathbb{R}$, where $\tilde{M} > 0$ is independent of t. Then for any square-mean almost automorphic process $x : \mathbb{R} \to L^2(\mathbb{P},\mathbb{H})$, the stochastic process $F : \mathbb{R} \to L^2(\mathbb{P},\mathbb{H})$ given by $F(\cdot) = f(\cdot, x(\cdot))$ is square-mean almost automorphic.

Denote

$$AA_0(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H})) = \Big\{f \in BC(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}), L^2(\mathbb{P},\mathbb{H})) : \lim_{T\to\infty} \int_{-T}^{T} E\|f(t,x)\|^2\,dt = 0\Big\}.$$

DEFINITION 2.13 ([CL]). A jointly continuous function $f : \mathbb{R}\times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$ is said to be *square-mean pseudo almost automorphic* in t for any $x \in L^2(\mathbb{P},\mathbb{H})$ if it can be decomposed as $f = g+\varphi$, where $g \in AA(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}))$ and $\varphi \in AA_0(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}))$. Denote the set of all such functions f by $PAA(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}))$.

2.2. Stepanov-like square-mean almost automorphy

Definition 2.14 ([BN]). The Bochner transform $x^b(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$, of a stochastic process $x : \mathbb{R} \to L^2(\mathbb{P},\mathbb{H})$ is defined by

$$x^b(t,s) := x(t+s).$$

Remark 2.15 ([BN]). A stochastic process $\psi(t,s), t \in \mathbb{R}, s \in [0,1]$, is the Bochner transform of a certain stochastic process f,

$$\psi(t,s) = x^b(t,s),$$

if and only if

$$\psi(t+\tau, s-\tau) = \psi(s,t) \quad \text{for all } t \in \mathbb{R}, s \in [0,1] \text{ and } \tau \in [s-1,s].$$

Definition 2.16 ([NP]). The *Bochner transform* $F^b(t,s,u)$, $t \in \mathbb{R}$, $s \in [0,1]$, $u \in L^2(\mathbb{P},\mathbb{H})$, of a function $F : \mathbb{R} \times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$ is defined by

$$F^b(t,s,u) := F(t+s,u) \quad \text{for each } u \in L^2(\mathbb{P},\mathbb{H}).$$

Definition 2.17 ([BN]). The space $BS^2(L^2(\mathbb{P},\mathbb{H}))$ of all *Stepanov bounded stochastic processes* consists of all measurable stochastic processes $x : \mathbb{R} \to L^2(\mathbb{P},\mathbb{H})$ such that

$$x^b \in L^\infty(\mathbb{R}; L^2(0,1; L^2(\mathbb{P},\mathbb{H}))).$$

This is a Banach space with the norm

$$\|x\|_{S^2} = \|x^b\|_{L^\infty(\mathbb{R};L^2)} = \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} E\|x(\tau)\|^2 \, d\tau \Big)^{1/2}.$$

Definition 2.18 ([CZ]). A stochastic process $x \in BS^2(L^2(\mathbb{P},\mathbb{H}))$ is called *Stepanov-like square-mean almost automorphic* (or S^2-*almost automorphic*) if $x^b \in AA(\mathbb{R}; L^2(0,1; L^2(\mathbb{P},\mathbb{H})))$.

In other words, a stochastic process $x \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{P},\mathbb{H}))$ is Stepanov-like almost automorphic if its Bochner transform $x^b : \mathbb{R} \to L^2(0,1; L^2(\mathbb{P},\mathbb{H}))$ is square-mean almost automorphic in the sense that for every sequence $(s'_n)_{n\in\mathbb{N}}$ of real numbers, there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ and a stochastic process $y \in L^2_{\mathrm{log}}(\mathbb{R}, L^2(\mathbb{P},\mathbb{H}))$ such that

$$\int_t^{t+1} E\|x(s+s_n) - y(s)\|^2 \, ds \to 0, \qquad \int_t^{t+1} E\|y(s-s_n) - x(s)\|^2 \, ds \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}$. Write $AS^2(L^2(\mathbb{P},\mathbb{H}))$ for the set of all such processes.

Remark 2.19 ([CZ]). It is clear that, if $x : \mathbb{R} \to L^2(\mathbb{P},\mathbb{H})$ is a square-mean almost automorphic stochastic process, then x is S^2-almost automorphic, that is, $AA(L^2(\mathbb{P},\mathbb{H})) \subset AS^2(L^2(\mathbb{P},\mathbb{H}))$.

2.3. Stepanov-like square-mean pseudo almost automorphy

DEFINITION 2.20. A stochastic process $f \in BS^2(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$ is said to be Stepanov-like square-mean pseudo almost automorphic (or S^2-pseudo almost automorphic) if it can be decomposed as $f = h + \varphi$, where $h^b \in AA(L^2(0,1; L^2(\mathbb{P}, \mathbb{H})))$ and $\varphi^b \in PAP_0(L^2(0,1; L^2(\mathbb{P}, \mathbb{H})))$. Denote the set of all such processes by $PAA^2(L^2(\mathbb{P}, \mathbb{H}))$.

In other words, a stochastic process $f \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$ is said to be Stepanov-like square-mean pseudo almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^2(0,1; L^2(\mathbb{P}, \mathbb{H}))$ is square-mean pseudo almost automorphic in the sense that there exist two functions $h, \varphi : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ such that $f = h + \varphi$, where $h^b \in AA(L^2(0,1; L^2(\mathbb{P}, \mathbb{H})))$ and $\varphi^b \in PAP_0(L^2(0,1; L^2(\mathbb{P}, \mathbb{H})))$.

Obviously, the following inclusions hold:

$$AP(L^2(\mathbb{P}, \mathbb{H})) \subset AA(L^2(\mathbb{P}, \mathbb{H})) \subset PAA(L^2(\mathbb{P}, \mathbb{H})) \subset PAA^2(L^2(\mathbb{P}, \mathbb{H})),$$

where $AP(L^2(\mathbb{P}, \mathbb{H}))$ stands for the collection of all $L^2(\mathbb{P}, \mathbb{H})$-valued almost periodic functions.

DEFINITION 2.21. A function $f : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{K}) \to L^2(\mathbb{P}, \mathbb{H})$ with $f(\cdot, u) \in L^2(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H}))$ for each $u \in L^2(\mathbb{P}, \mathbb{K})$, is said to be *Stepanov-like square-mean pseudo almost automorphic* (or *S^2-pseudo almost automorphic*) if it can be decomposed as $f = h + \varphi$, where $h^b \in AA(\mathbb{R} \times L^2(0,1; L^2(\mathbb{P}, \mathbb{K})))$ and $\varphi^b \in PAP_0(\mathbb{R} \times L^2(0,1; L^2(\mathbb{P}, \mathbb{K})))$. Denote the set of all such functions f by $PAA^2(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{K}))$.

LEMMA 2.22. *Assume $f \in PAA^2(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}))$. Suppose that $f(t,u)$ is Lipschitz in $u \in L^2(\mathbb{P}, \mathbb{H})$ uniformly in $t \in \mathbb{R}$, in the sense that there exists $L > 0$ such that*

$$E\|f(t,u) - f(t,v)\|^2 \le LE\|u - v\|^2$$

for all $t \in \mathbb{R}$ and $u, v \in L^2(\mathbb{P}, \mathbb{H})$. If $\phi(\cdot) \in PAA^2(L^2(\mathbb{P}, \mathbb{H}))$, then $f(\cdot, \phi(\cdot)) \in PAA^2(L^2(\mathbb{P}, \mathbb{H}))$.

Lemma 2.22 can be proved by using Definitions 2.20 and 2.21 and Lemmas 2.12; for more details one may refer to Theorem 3.5 in [DI].

3. Existence results. In this section, we prove that there is a unique mild solution for the problem (1.1). For that, we make the following hypotheses:

(H1) The function $s \mapsto AT(t-s)$ from $(-\infty, t)$ into $L(\mathbb{K}, \mathbb{H})$ is strongly measurable and there exist a non-increasing function $\phi : [0, \infty) \to [0, \infty)$ and $\gamma > 0$ with $s \mapsto e^{-\gamma s}\phi(s) \in L^1[0,\infty) \cap L^2[0,\infty)$ such that

$$\|AT(s)\|_{L(\mathbb{K},\mathbb{H})} \le e^{-\gamma s}\phi(s), \quad s > 0.$$

(H2) The function $q \in PAA^2(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}))\cap C(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}), L^2(\mathbb{P},\mathbb{H}))$, q is $L^2(\mathbb{P},\mathbb{K})$-valued, $q:\mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\to L^2(\mathbb{P},\mathbb{K})$ is continuous and there exists $L_q\in(0,1)$ such that

$$E\|q(t,x)-q(t,y)\|_{\mathbb{K}}^2 \le L_q E\|x-y\|^2$$

for all $t\in\mathbb{R}$ and each $x,y\in L^2(\mathbb{P},\mathbb{H})$.

(H3) $h,f\in PAA^2(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}))\cap C(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}),L^2(\mathbb{P},\mathbb{H}))$ and there exist continuous functions $L_h,L_f:\mathbb{R}\to(0,\infty)$ satisfying $L_h:=\sup_{t\in\mathbb{R}}L_h(t)<1, L_f:=\sup_{t\in\mathbb{R}}L_f(t)<1$ such that

$$E\|h(t,x)-h(t,y)\|^2 \le L_h(t)E\|x-y\|^2,$$
$$E\|f(t,x)-f(t,y)\|^2 \le L_f(t)E\|x-y\|^2$$

for all $t\in\mathbb{R}$ and each $x,y\in L^2(\mathbb{P},\mathbb{H})$.

Also, we need a few preliminary results.

LEMMA 3.1. *Under assumption* (H1), *for $q\in PAA^2(L^2(\mathbb{P},\mathbb{K}))$ define*

$$Q(t):=\int_{-\infty}^{t} AT(t-\tau)q(\tau)\,d\tau \quad \text{for } t\in\mathbb{R} \tag{3.1}$$

and suppose

$$\varphi_{\gamma,\phi}:=\sum_{k=1}^{\infty}\int_{k-1}^{k} e^{-2\gamma\tau}\phi^2(\tau)\,d\tau<\infty.$$

Then $Q\in PAA(L^2(\mathbb{P},\mathbb{H}))$.

Proof. Since $q\in PAA^2(L^2(\mathbb{P},\mathbb{K}))$, write

$$q=q_1+q_2,$$

where $q_1^b\in AA(L^2(0,1;L^2(\mathbb{P},\mathbb{K})))$ and $q_2^b\in PAP_0(L^2(0,1;L^2(\mathbb{P},\mathbb{K})))$. Then

$$\begin{aligned}Q(t)&=\int_{-\infty}^{t} AT(t-\tau)q_1(\tau)\,d\tau+\int_{-\infty}^{t} AT(t-\tau)q_2(\tau)\,d\tau\\ &:=Q_1(t)+Q_2(t).\end{aligned}$$

Next we show that $Q_1\in AA(L^2(\mathbb{P},\mathbb{H}))$ and $Q_2\in PAP_0(L^2(\mathbb{P},\mathbb{H}))$.

To prove that $Q_1\in AA(L^2(\mathbb{P},\mathbb{H}))$, we consider

$$Q_{1,k}(t)=\int_{t-k}^{t-k+1} AT(t-\tau)q_1(\tau)\,d\tau=\int_{k-1}^{k} AT(\tau)q_1(t-\tau)\,d\tau$$

for each $t \in \mathbb{R}$ and $k = 1, 2, \dots$. From Bochner's criterion of integrability and the estimate

$$E\Big\| \int_{t-k}^{t-k+1} AT(t-\tau)q_1(\tau)d\tau \Big\|^2 \leq \int_{t-k}^{t-k+1} \|AT(t-\tau)\|^2_{L(\mathbb{K},\mathbb{H})} E\|q_1(\tau)\|^2_{\mathbb{K}}\, d\tau$$
$$\leq \int_{t-k}^{t-k+1} e^{-2\gamma(t-\tau)}\phi^2(t-\tau)E\|q_1(\tau)\|^2_{\mathbb{K}}\, d\tau,$$

it follows that the function $\tau \mapsto AT(t-\tau)q_1(\tau)$ is integrable over $(-\infty, t)$ for each $t \in \mathbb{R}$, by assumption (H1). Then, by Hölder's inequality,

$$\begin{aligned}
E\|Q_{1,k}(t)\|^2 &\leq E\Big(\int_{t-k}^{t-k+1} \|AT(t-\tau)\|_{L(\mathbb{K},\mathbb{H})}\|q_1(\tau)\|_{\mathbb{K}}\, d\tau\Big)^2 \\
&\leq E\Big(\int_{t-k}^{t-k+1} e^{-\gamma(t-\tau)}\phi(t-\tau)\|q_1(\tau)\|_{\mathbb{K}}\, d\tau\Big)^2 \\
&\leq \Big(\int_{t-k}^{t-k+1} e^{-2\gamma(t-\tau)}\phi^2(t-\tau)\, d\tau\Big)\Big(\int_{t-k}^{t-k+1} E\|q_1(\tau)\|^2_{\mathbb{K}}\, d\tau\Big) \\
&\leq \Big(\int_{k-1}^{k} e^{-2\gamma\tau}\phi^2(\tau)\, d\tau\Big)\|q_1\|^2_{S^2}.
\end{aligned}$$

Since $\varphi_{\gamma,\phi} := \sum_{k=1}^{\infty} \int_{k-1}^{k} e^{-2\delta\gamma\tau}\phi^2(\tau)\, d\tau < \infty$, we deduce from the well-known Weierstrass test that the series $\sum_{k=1}^{\infty} Q_{1,k}(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$Q_1(t) = \int_{-\infty}^{t} AT(t-\tau)q_1(\tau)\, d\tau = \sum_{k=1}^{\infty} Q_{1,k}(t).$$

Let us take a sequence $(s'_n)_{n\in\mathbb{N}}$ and show that it has a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$\lim_{m\to\infty} \lim_{n\to\infty} E\|Q_{1,k}(t+s_n-s_m) - Q_{1,k}(t)\|^2 = 0$$

for each $t \in \mathbb{R}$. Let $\varepsilon > 0$, $N_\varepsilon > 0$. As $q_1^b \in AA(L^2(0,1; L^2(\mathbb{P},\mathbb{K})))$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ of $(s'_n)_{n\in\mathbb{N}}$ such that, for each $t \in \mathbb{R}$,

$$\int_{t}^{t+1} E\|q_1(s+s_n-s_m) - q_1(s)\|^2_{\mathbb{K}}\, ds < \varepsilon \tag{3.2}$$

for all $n, m \geq N_\varepsilon$. On the other hand, using (3.2), exponential stability of $(T(t))_{t\geq 0}$ and Hölder's inequality, we obtain

$$\begin{aligned}
&E\|Q_{1,k}(t+s_n-s_m)-Q_{1,k}(t)\|^2\\
&\quad\leq E\Big(\int_{k-1}^{k}\|AT(\tau)\|_{L(\mathbb{K},\mathbb{H})}\|q_1(t+s_n-s_m-\tau)-q_1(t-\tau)\|_{\mathbb{K}}\,d\tau\Big)^2\\
&\quad\leq E\Big(\int_{k-1}^{k}e^{-\gamma\tau}\phi(\tau)\|q_1(t+s_n-s_m-\tau)-q_1(t-\tau)\|_{\mathbb{K}}\,d\tau\Big)^2\\
&\quad\leq \Big(\int_{k-1}^{k}e^{-2\gamma\tau}\phi^2(\tau)\,d\tau\Big)\Big(\int_{k-1}^{k}E\|q_1(t+s_n-s_m-\tau)-q_1(t-\tau)\|_{\mathbb{K}}^2\,d\tau\Big)\\
&\quad\leq \chi_{\gamma,\phi}\Big(\int_{t-k}^{t-k+1}E\|q_1(s+s_n-s_m)-q_1(s)\|_{\mathbb{K}}^2\,ds\Big)<\chi_{\gamma,\phi}\varepsilon,
\end{aligned}$$

where $\chi_{\gamma,\phi}=\sup_k\int_{k-1}^{k}e^{-2\delta\tau}\phi^2(\tau)\,d\tau<\infty$, as $\varphi_{\delta,\phi}<\infty$. Thus, we immediately see that

$$\lim_{m\to\infty}\lim_{n\to\infty}E\|Q_{1,k}(t+s_n-s_m)-Q_{1,k}(t)\|^2=0$$

for each $t\in\mathbb{R}$. Therefore, $Q_{1,k}\in AA(L^2(\mathbb{P},\mathbb{H}))$. Applying Lemma 2.7, we deduce that the uniform limit $Q_1(t)=\sum_{k=1}^{\infty}Q_{1,k}(t)\in AA(L^2(\mathbb{P},\mathbb{H}))$.

Next, we will prove that $Q_2\in PAP_0(L^2(\mathbb{P},\mathbb{H}))$. It is obvious that $Q_2\in BC(\mathbb{R},L^2(\mathbb{P},\mathbb{H}))$; it remains to show that

$$\lim_{r\to\infty}\frac{1}{2T}\int_{-T}^{T}E\|Q_2(t)\|^2\,dt=0.$$

For this, we consider

$$Q_{2,k}(t)=\int_{t-k}^{t-k+1}AT(t-\tau)q_2(\tau)\,d\tau=\int_{k-1}^{k}AT(\tau)q_2(t-\tau)\,d\tau$$

for each $t\in\mathbb{R}$ and $k=1,2,\ldots$. From Bochner's integrability criterion and the estimate

$$\begin{aligned}
E\Big\|\int_{t-k}^{t-k+1}AT(t-\tau)q_2(\tau)\,d\tau\Big\|^2&\leq\int_{t-k}^{t-k+1}\|AT(t-\tau)\|_{L(\mathbb{K},\mathbb{H})}^2E\|q_2(\tau)\|_{\mathbb{K}}^2\,d\tau\\
&\leq\int_{t-k}^{t-k+1}e^{-2\gamma(t-\tau)}\phi^2(t-\tau)E\|q_2(\tau)\|_{\mathbb{K}}^2\,d\tau,
\end{aligned}$$

it follows that the function $\tau\mapsto AT(t-\tau)q_2(\tau)$ is integrable over $(-\infty,t)$

for each $t \in \mathbb{R}$, by assumption (H1). Then, by Hölder's inequality,

$$\begin{aligned} E\|Q_{2,k}(t)\|^2 &\le E\Big(\int_{t-k}^{t-k+1} \|AT(t-\tau)\|_{L(\mathbb{K},\mathbb{H})}\|q_2(\tau)\|_{\mathbb{K}}\,d\tau\Big)^2 \\ &\le E\Big(\int_{t-k}^{t-k+1} e^{-\gamma(t-\tau)}\phi(t-\tau)\|q_2(\tau)\|_{\mathbb{K}}\,d\tau\Big)^2 \\ &\le \Big(\int_{t-k}^{t-k+1} e^{-2\gamma(t-\tau)}\phi^2(t-\tau)\,d\tau\Big)\Big(\int_{t-k}^{t-k+1} E\|q_2(\tau)\|^2_{\mathbb{K}}\,d\tau\Big) \\ &\le \Big(\int_{k-1}^{k} e^{-2\gamma\tau}\phi^2(\tau)\,d\tau\Big)\Big(\int_{t-k}^{t-k+1} E\|q_2(\tau)\|^2_{\mathbb{K}}\,d\tau\Big). \end{aligned}$$

Since $q_2^b \in PAP_0(L^2(0,1;L^2(\mathbb{P},\mathbb{K})))$, the above inequality leads to $Q_{2,k} \in PAP_0(L^2(\mathbb{P},\mathbb{H}))$. The inequality also leads to

$$E\|Q_{2,k}(t)\|^2 \le \Big(\int_{k-1}^{k} e^{-2\gamma\tau}\phi^2(\tau)\,d\tau\Big)\|q_2\|^2_{S^2}.$$

Since $\varphi_{\gamma,\phi} := \sum_{k=1}^{\infty}\int_{k-1}^{k} e^{-2\gamma\tau}\phi^2(\tau)\,d\tau < \infty$, the series $\sum_{k=1}^{\infty} Q_{2,k}(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$Q_2(t) = \int_{-\infty}^{t} AT(t-\tau)q_2(\tau)\,d\tau = \sum_{k=1}^{\infty} Q_{2,k}(t).$$

Applying $Q_{2,k} \in PAP_0(L^2(\mathbb{P},\mathbb{H}))$ and the inequality

$$\begin{aligned} &\frac{1}{2T}\int_{-T}^{T} E\|Q_2(t)\|^2\,dt \\ &\quad\le \frac{1}{2T}\int_{-T}^{T} 2\Big[E\Big\|Q_2(t)-\sum_{k=1}^{n} Q_{2,k}(t)\Big\|^2 + E\Big\|\sum_{k=1}^{n} Q_{2,k}(t)\Big\|^2\Big]\,dt \\ &\quad\le 2\Big[\frac{1}{2T}\int_{-T}^{T} E\Big\|Q_2(t)-\sum_{k=1}^{n} Q_{2,k}(t)\Big\|^2\,dt + n\sum_{k=1}^{n}\frac{1}{2T}\int_{-T}^{T} E\|Q_{2,k}(t)\|^2\,dt\Big] \end{aligned}$$

shows that the uniform limit $Q_2(t) = \sum_{k=1}^{\infty} Q_{2,k}(t)$ is in $PAP_0(L^2(\mathbb{P},\mathbb{H}))$. ∎

LEMMA 3.2. *If $h \in PAA^2(L^2(\mathbb{P},\mathbb{H})) \cap C(\mathbb{R},L^2(\mathbb{P},\mathbb{H}))$ and if H is the function defined by*

$$H(t) := \int_{-\infty}^{t} T(t-\tau)h(\tau)\,d\tau \tag{3.3}$$

for each $t \in \mathbb{R}$, then $H \in PAA(L^2(\mathbb{P},\mathbb{H}))$.

Proof. Since $h \in PAA^2(L^2(\mathbb{P},\mathbb{H})) \cap C(\mathbb{R}, L^2(\mathbb{P},\mathbb{H}))$, write $h = h_1 + h_2$, where

$$\begin{aligned} &h_1^b \in AA(L^2(0,1;L^2(\mathbb{P},\mathbb{H}))) \cap C(\mathbb{R}, L^2(0,1;L^2(\mathbb{P},\mathbb{H}))),\\ &h_2^b \in PAP_0(L^2(0,1;L^2(\mathbb{P},\mathbb{H}))) \cap C(\mathbb{R}, L^2(0,1;L^2(\mathbb{P},\mathbb{H}))). \end{aligned}$$

Then

$$H(t) = \int_{-\infty}^{t} T(t-\tau)h_1(\tau)\,d\tau + \int_{-\infty}^{t} T(t-\tau)h_2(\tau)\,d\tau =: H_1(t) + H_2(t).$$

Next we show that $H_1 \in AA(L^2(\mathbb{P},\mathbb{H}))$ and $H_2 \in PAP_0(L^2(\mathbb{P},\mathbb{H}))$.

To prove that $H_1 \in AA(L^2(\mathbb{P},\mathbb{H}))$, we consider

$$H_{1,k}(t) = \int_{t-k}^{t-k+1} T(t-\tau)h_1(\tau)\,d\tau = \int_{k-1}^{k} T(\tau)h_1(t-\tau)\,d\tau$$

for each $t \in \mathbb{R}$ and $k = 1, 2, \ldots$. Using exponential stability of $(T(t))_{t\geq 0}$ and Hölder's inequality, it follows that

$$\begin{aligned} E\|H_{1,k}(t)\|^2 &\leq E\Big(\int_{t-k}^{t-k+1} \|T(t-\tau)\|\,\|h_1(\tau)\|\,d\tau\Big)^2\\ &\leq M^2 E\Big(\int_{t-k}^{t-k+1} e^{-\delta(t-\tau)}\|h_1(\tau)\|\,d\tau\Big)^2\\ &\leq M^2\Big(\int_{t-k}^{t-k+1} e^{-2\delta(t-\tau)}\,d\tau\Big)\Big(\int_{t-k}^{t-k+1} E\|q_1(\tau)\|^2\,d\tau\Big)\\ &\leq M^2\Big(\int_{k-1}^{k} e^{-2\delta\tau}\,d\tau\Big)\|h_1\|_{S^2}^2 \leq \frac{M^2}{2\delta}e^{-2\delta k}(e^{2\delta}-1)\|h_1\|_{S^2}^2. \end{aligned}$$

Since $\frac{M^2}{2\delta}(e^{2\delta}-1)\|h_1\|_{S^2}^2 \sum_{k=1}^{\infty} e^{-2\delta k} < \infty$, the series $\sum_{k=1}^{\infty} H_{1,k}(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$H_1(t) = \int_{-\infty}^{t} T(t-\tau)h_1(\tau)\,d\tau = \sum_{k=1}^{\infty} H_{1,k}(t).$$

Let us take a sequence $(s_n')_{n\in\mathbb{N}}$ and show that it has a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$\lim_{m\to\infty}\lim_{n\to\infty} E\|H_{1,k}(t+s_n-s_m) - H_{1,k}(t)\|^2 = 0$$

for each $t \in \mathbb{R}$. Let $\varepsilon > 0, N_\varepsilon > 0$. As $h_1^b \in AA(L^2(0,1;L^2(\mathbb{P},\mathbb{H})))$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ of $(s'_n)_{n\in\mathbb{N}}$ such that, for each $t \in \mathbb{R}$,

$$\int_t^{t+1} E\|h_1(s+s_n-s_m)-h_1(s)\|^2\,ds < \varepsilon \tag{3.4}$$

for all $n, m \geq N_\varepsilon$. On the other hand, using (3.4), exponential stability of $(T(t))_{t\geq 0}$ and Hölder's inequality, we obtain

$$\begin{aligned}
&E\|H_{1,k}(t+s_n-s_m)-H_{1,k}(t)\|^2\\
&\quad\leq E\Big\|\int_{k-1}^{k} T(\tau)[h_1(t+s_n-s_m-\tau)-h_1(t-\tau)]\,d\tau\Big\|^2\\
&\quad\leq E\Big(\int_{k-1}^{k} e^{-\delta\tau}\|h_1(t+s_n-s_m-\tau)-h_1(t-\tau)\|\,d\tau\Big)^2\\
&\quad\leq M^2\Big(\int_{k-1}^{k} e^{-2\delta\tau}\,d\tau\Big)\Big(\int_{k-1}^{k} E\|h_1(t+s_n-s_m-\tau)-h_1(t-\tau)\|^2\,d\tau\Big)\\
&\quad\leq \frac{M^2}{2\delta}e^{-2\delta k}(e^{2\delta}-1)\Big(\int_{t-k}^{t-k+1} E\|h_1(s+s_n-s_m)-h_1(s)\|^2\,ds\Big)\\
&\quad< \frac{M^2}{2\delta}e^{-2\delta k}(e^{2\delta}-1)\varepsilon.
\end{aligned}$$

Thus, we immediately see that

$$\lim_{m\to\infty}\lim_{n\to\infty} E\|H_{1,k}(t+s_n-s_m)-H_{1,k}(t)\|^2 = 0$$

for each $t \in \mathbb{R}$. Therefore, $H_{1,k} \in AA(L^2(\mathbb{P},\mathbb{H}))$. Applying Lemma 2.7, we deduce that the uniform limit $H_1(t) = \sum_{k=1}^{\infty} H_{1,k}(t)$ is in $AA(L^2(\mathbb{P},\mathbb{H}))$.

Next, we will prove that $H_2 \in PAP_0(L^2(\mathbb{P},\mathbb{H}))$. It is obvious that $H_2 \in BC(\mathbb{R}, L^2(\mathbb{P},\mathbb{H}))$; it remains to show that

$$\lim_{r\to\infty}\frac{1}{2T}\int_{-T}^{T} E\|H_2(t)\|^2\,dt = 0.$$

For this, we consider

$$H_{2,k}(t) = \int_{t-k}^{t-k+1} T(t-\tau)h_2(\tau)\,d\tau = \int_{k-1}^{k} T(\tau)h_2(t-\tau)\,d\tau$$

for each $t \in \mathbb{R}$ and $k = 1, 2, \ldots$. Then, by exponential stability of $(T(t))_{t\geq 0}$ and Hölder's inequality,

$$
\begin{aligned}
E\|H_{2,k}(t)\|^2 &\leq E \int_{t-k}^{t-k+1} e^{-\delta(t-\tau)}\|h_2(\tau)\|\,d\tau \\
&\leq M^2\Big(\int_{t-k}^{t-k+1} e^{-2\delta(t-\tau)}\,d\tau\Big)\Big(\int_{t-k}^{t-k+1} E\|h_2(\tau)\|^2\,d\tau\Big) \\
&\leq M^2\Big(\int_{k-1}^{k} e^{-2\delta\tau}\,d\tau\Big)\Big(\int_{t-k}^{t-k+1} E\|h_2(\tau)\|^2\,d\tau\Big) \\
&\leq \frac{M^2}{2\delta}e^{-2\delta k}(e^{2\delta}-1)\Big(\int_{t-k}^{t-k+1} E\|h_2(\tau)\|^2\,d\tau\Big).
\end{aligned}
$$

Since $h_2^b \in PAP_0(L^2(0,1;L^2(\mathbb{P},\mathbb{H})))$, the above inequality leads to $H_2 \in PAP_0(L^2(\mathbb{P},\mathbb{H}))$. It also leads to

$$
E\|H_{2,k}(t)\|^2 \leq \frac{M^2}{2\delta}e^{-2\delta k}(e^{2\delta}-1)\|h_2\|_{S^2}^2.
$$

Since $\frac{M^2}{2\delta}(e^{2\delta}-1)\sum_{k=1}^{\infty}\|h_2\|_{S^2}^2 e^{-2\delta k} < \infty$, the series $\sum_{k=1}^{\infty} H_{2,k}(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$
H_2(t) = \int_{-\infty}^{t} AT(t-\tau)h_2(\tau)\,d\tau = \sum_{k=1}^{\infty} H_{2,k}(t).
$$

Applying $H_{2,k} \in PAP_0(L^2(\mathbb{P},\mathbb{H}))$ and the inequality

$$
\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T} E\|H_2(t)\|^2\,dt \\
&\quad\leq \frac{1}{2T}\int_{-T}^{T} 2\Big[E\Big\|H_2(t)-\sum_{k=1}^{n}H_{2,k}(t)\Big\|^2 + E\Big\|\sum_{k=1}^{n}H_{2,k}(t)\Big\|^2\Big]\,dt \\
&\quad\leq 2\Big[\frac{1}{2T}\int_{-T}^{T} E\Big\|H_2(t)-\sum_{k=1}^{n}H_{2,k}(t)\Big\|^2\,dt + n\sum_{k=1}^{n}\frac{1}{2T}\int_{-T}^{T}E\|H_{2,k}(t)\|^2\,dt\Big]
\end{aligned}
$$

shows that the uniform limit $H_2(t) = \sum_{k=1}^{\infty} H_{2,k}(t)$ is in $PAP_0(L^2(\mathbb{P},\mathbb{H}))$. ∎

LEMMA 3.3. *If $f \in PAA^2(L^2(\mathbb{P},\mathbb{H})) \cap C(\mathbb{R}, L^2(\mathbb{P},\mathbb{H}))$ and if F is the function defined by*

$$
F(t) := \int_{-\infty}^{t} T(t-\tau)f(\tau)\,dW(\tau) \tag{3.5}
$$

for each $t \in \mathbb{R}$, then $F \in PAA(L^2(\mathbb{P},\mathbb{H}))$.

Proof. Since $f \in PAA^2(L^2(\mathbb{P},\mathbb{H})) \cap C(\mathbb{R}, L^2(\mathbb{P},\mathbb{H}))$, write $f = f_1 + f_2$, where

$$f_1^b \in AA(L^2(0,1;L^2(\mathbb{P},\mathbb{H}))) \cap C(\mathbb{R}, L^2(0,1;L^2(\mathbb{P},\mathbb{H}))),$$
$$f_2^b \in PAP_0(L^2(0,1;L^2(\mathbb{P},\mathbb{H}))) \cap C(\mathbb{R}, L^2(0,1;L^2(\mathbb{P},\mathbb{H}))).$$

Then

$$F(t) = \int_{-\infty}^{t} T(t-\tau) f_1(\tau)\, dW(\tau) + \int_{-\infty}^{t} T(t-\tau) f_2(\tau)\, dW(\tau) =: F_1(t) + F_2(t).$$

Next we show that $F_1 \in AA(L^2(\mathbb{P},\mathbb{H}))$ and $F_2 \in PAP_0(L^2(\mathbb{P},\mathbb{H}))$.

To prove that $F_1 \in AA(L^2(\mathbb{P},\mathbb{H}))$, we consider

$$F_{1,k}(t) = \int_{t-k}^{t-k+1} T(t-\tau) f_1(\tau)\, dW(\tau) = \int_{k-1}^{k} T(\tau) f_1(t-\tau)\, dW(\tau)$$

for $t \in \mathbb{R}$ and $k = 1, 2, \ldots$. By an estimate on the Ito integral established in [I],

$$\begin{aligned} E\|F_{1,k}(t)\|^2 &\le M^2 \int_{t-k}^{t-k+1} e^{-2\delta(t-\tau)} E\|f_1(\tau)\|^2\, d\tau \\ &\le M^2 \int_{k-1}^{k} e^{-2\delta\tau} E\|f_1(t-\tau)\|^2\, d\tau \\ &\le M^2 \sup_{\tau\in[k-1,k]} e^{-2\delta\tau} \int_{k-1}^{k} E\|f_1(t-\tau)\|^2\, d\tau \\ &\le M^2 e^{-2\delta k} e^{2\delta} \|f_1\|_{S^2}^2. \end{aligned}$$

Since $M^2 e^{2\delta}\|f_1\|_{S^2}^2 \sum_{k=1}^{\infty} e^{-2\delta k} < \infty$, the series $\sum_{k=1}^{\infty} F_{1,k}(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$F_1(t) = \int_{-\infty}^{t} T(t-\tau) f_1(\tau)\, dW(\tau) = \sum_{k=1}^{\infty} F_{1,k}(t).$$

Take a sequence $(s_n')_{n\in\mathbb{N}}$. Let $\varepsilon > 0, N_\varepsilon > 0$. As $f_1^b \in AA(L^2(0,1;L^2(\mathbb{P},\mathbb{H})))$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ of $(s_n')_{n\in\mathbb{N}}$ such that, for each $t \in \mathbb{R}$,

$$\int_{t}^{t+1} E\|f_1(t + s_n - s_m) - f_1(t)\|^2 < \varepsilon \tag{3.6}$$

for all $n, m \geq N_\varepsilon$. On the other hand, using (3.6), exponential stability of $(T(t))_{t\geq 0}$ and the properties of the Ito integral, we obtain

$$\begin{aligned}
&E\|F_{1,k}(t+s_n-s_m)-F_{1,k}(t)\|^2 \\
&\quad = E\Big\| \int_{k-1}^{k} T(\tau)[f_1(t+s_n-s_m-\tau)-f_1(t-\tau)]\,dW(\tau)\Big\|^2 \\
&\quad \leq M^2 \int_{k-1}^{k} e^{-2\delta\tau} E\|f_1(t+s_n-s_m-\tau)-f_1(t-\tau)\|^2\,d\tau \\
&\quad \leq M^2 \sup_{\tau\in[k-1,k]} e^{-2\delta\tau} \int_{k-1}^{k} E\|f_1(t+s_n-s_m-\tau)-f_1(t-\tau)\|^2\,d\tau \\
&\quad \leq M^2 e^{-2\delta k} e^{2\delta} \int_{t-k}^{t-k+1} E\|f_1(s+s_n-s_m)-f_1(s)\|^2\,ds < M^2 e^{-2\delta k} e^{2\delta}\varepsilon.
\end{aligned}$$

Thus,

$$\lim_{m\to\infty}\lim_{n\to\infty} E\|F_{1,k}(t+s_n-s_m)-F_{1,k}(t)\|^2 = 0$$

for each $t \in \mathbb{R}$. Therefore, $F_{1,k} \in AA(L^2(\mathbb{P},\mathbb{H}))$. Applying Lemma 2.7, we deduce that the uniform limit $F_1(t) = \sum_{k=1}^{\infty} F_{1,k}(t)$ is in $AA(L^2(\mathbb{P},\mathbb{H}))$.

Next, we will prove that $F_2 \in PAP_0(L^2(\mathbb{P},\mathbb{H}))$. It is obvious that $F_2 \in BC(\mathbb{R}, L^2(\mathbb{P},\mathbb{H}))$; it remains to show that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E\|F_2(t)\|^2\,dt = 0.$$

For this, we consider

$$F_{2,k}(t) = \int_{t-k}^{t-k+1} T(t-\tau) f_2(\tau)\,d\tau = \int_{k-1}^{k} T(\tau) f_2(t-\tau)\,d\tau$$

for $t \in \mathbb{R}$ and $k = 1, 2, \ldots$. By exponential stability of $(T(t))_{t\geq 0}$ and the properties of the Ito integral, it follows that

$$\begin{aligned}
E\|F_{2,k}(t)\|^2 &\leq M^2 \int_{t-k}^{t-k+1} e^{-2\delta(t-\tau)} E\|f_2(\tau)\|^2\,d\tau \\
&\leq M^2 \int_{k-1}^{k} e^{-2\delta\tau} E\|f_2(t-\tau)\|^2\,d\tau \\
&\leq M^2 \sup_{\tau\in[k-1,k]} e^{-2\delta\tau} \int_{k-1}^{k} E\|f_2(t-\tau)\|^2\,d\tau \leq M^2 e^{-2\delta k} e^{2\delta} \|f_2\|_{S^2}^2.
\end{aligned}$$

Since $f_2^b \in PAP_0(L^2(0,1;L^2(\mathbb{P},\mathbb{H})))$, this yields $F_2 \in PAP_0(L^2(\mathbb{P},\mathbb{H}))$ and

$$E\|F_{2,k}(t)\|^2 \le M^2 e^{-2\delta k} e^{2\delta} \|f_2\|_{S^2}^2.$$

Since $M^2 e^{2\delta} \sum_{k=1}^{\infty} \|f_2\|_{S^2}^2 e^{-2\delta k} < \infty$, the series $\sum_{k=1}^{\infty} F_{2,k}(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$F_2(t) = \int_{-\infty}^{t} T(t-\tau) f_2(\tau)\, d\tau = \sum_{k=1}^{\infty} F_{2,k}(t).$$

Applying $F_{2,k} \in PAP_0(L^2(\mathbb{P},\mathbb{H}))$ and the inequality

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T} E\|F_2(t)\|^2\, dt \\
&\quad\le \frac{1}{2T}\int_{-T}^{T} 2\Big[E\Big\|F_2(t) - \sum_{k=1}^{n} F_{2,k}(t)\Big\|^2 + E\Big\|\sum_{k=1}^{n} F_{2,k}(t)\Big\|^2\Big]\, dt \\
&\quad\le 2\Big[\frac{1}{2T}\int_{-T}^{T} E\Big\|F_2(t) - \sum_{k=1}^{n} F_{2,k}(t)\Big\|^2 dt + n\sum_{k=1}^{n} \frac{1}{2T}\int_{-T}^{T} E\|F_{2,k}(t)\|^2\, dt\Big],
\end{aligned}$$

we deduce that $F_2(t) = \sum_{k=1}^{\infty} F_{2,k}(t) \in PAP_0(L^2(\mathbb{P},\mathbb{H}))$. ■

Next, we establish the existence and uniqueness of pseudo almost automorphic mild solutions to the stochastic evolution equation (1.1).

Definition 3.4. An $\mathcal{F}_t$-progressively measurable stochastic process $\{x(t)\}_{t\in\mathbb{R}}$ is called a *mild solution* of problem (1.1) on $\mathbb{R}$ if the function $s \mapsto AT(t-s)q(s,x(s-r))$ is integrable on $(-\infty,t)$ for each $t \in \mathbb{R}$, and $x(t)$ satisfies the corresponding stochastic integral equation

$$\begin{aligned}
x(t) &= T(t-s)[x(s) - q(s,x(s-r))] + q(t,x(t-r)) \\
&\quad + \int_s^t AT(t-\tau) q(\tau, x(\tau-r))\, d\tau \\
&\quad + \int_s^t T(t-\tau) h(\tau, x(\tau-r))\, d\tau \\
&\quad + \int_s^t T(t-\tau) f(\tau, x(\tau-r))\, dW(\tau)
\end{aligned} \tag{3.7}$$

for all $t \ge s$ and all $s \in \mathbb{R}$.

Lemma 3.5. *If $x(\cdot) \in PAA(L^2(\mathbb{P},\mathbb{H}))$, then $x(\cdot - r) \in PAA(L^2(\mathbb{P},\mathbb{H}))$ for any fixed constant $r \ge 0$.*

The proof is similar to the proof of [XZ, Theorem 2.6], we omit the details.

Theorem 3.6. *Assume that* (H1)–(H3) *hold. If*

$$(3.8)\quad 4\Big[L_q + L_q \sup_{t\in\mathbb{R}}\Big(\int_{-\infty}^{t} e^{-\gamma(t-s)}\phi(t-s)\,ds\Big)^2 + \frac{M^2}{\delta^2}L_h + \frac{M^2}{2\delta}L_f\Big] < 1,$$

then (1.1) *admits a unique pseudo almost automorphic mild solution on* $\mathbb{R}$.

Proof. Consider the nonlinear operator on $BC(\mathbb{R}, L^2(\mathbb{P},\mathbb{H}))$ defined by

$$(3.9)\quad \begin{aligned}(\Psi x)(t) = q(t, x(t-r)) &+ \int_{-\infty}^{t} AT(t-s)q(s, x(s-r))\,ds \\ &+ \int_{-\infty}^{t} T(t-s)h(s, x(s-r))\,ds \\ &+ \int_{-\infty}^{t} T(t-s)f(s, x(s-r))\,dW(s), \quad t\in\mathbb{R}.\end{aligned}$$

From the previous assumptions and the properties of $(T(t))_{t\ge 0}$ one can easily see that Ψx is well-defined and continuous. Let $x(\cdot) \in PAA(L^2(\mathbb{P},\mathbb{H})) \subset PAA^2(L^2(\mathbb{P},\mathbb{H}))$. From Lemma 3.5 it is clear that

$$x(\cdot - s) \in PAA(L^2(\mathbb{P},\mathbb{H})) \subset PAA^2(L^2(\mathbb{P},\mathbb{H})).$$

Using (H2), (H3) and the composition theorem for Stepanov-like pseudo almost automorphic functions, we deduce that $q(\cdot, x(\cdot - r)) \in PAA^2(L^2(\mathbb{P},\mathbb{K}))$, $h(\cdot, x(\cdot - r)), f(\cdot, x(\cdot - r)) \in PAA^2(L^2(\mathbb{P},\mathbb{H}))$. It is easy to check that $q(\cdot, x(\cdot - r)) \in C(\mathbb{R}, L^2(\mathbb{P},\mathbb{K}))$, $h(\cdot, x(\cdot - r)), f(\cdot, x(\cdot - r)) \in C(\mathbb{R}, L^2(\mathbb{P},\mathbb{H}))$. Applying Lemmas 3.1–3.3 for $q(\cdot) = q(\cdot, x(\cdot - r)), h(\cdot) = h(\cdot, x(\cdot - r)), f(\cdot) = f(\cdot, x(\cdot - r))$, it follows that Ψ maps $PAA^2(L^2(\mathbb{P},\mathbb{H}))$ into $PAA(L^2(\mathbb{P},\mathbb{H}))$. In particular, it maps

$$PAA(L^2(\mathbb{P},\mathbb{H})) \subset PAA^2(L^2(\mathbb{P},\mathbb{H})) \quad \text{into} \quad PAA(L^2(\mathbb{P},\mathbb{H}))$$

whenever $x \in PAA(L^2(\mathbb{P},\mathbb{H}))$, that is, Ψ maps $PAA(L^2(\mathbb{P},\mathbb{H}))$ into itself. Next, we prove that Ψ is a strict contraction on $PAA(L^2(\mathbb{P},\mathbb{H}))$.

Let $x, y \in PAA(L^2(\mathbb{P},\mathbb{H}))$. Then

$$\begin{aligned}E\|(\Psi x)(t) - (\Psi y)(t)\|^2 \le 4E\|q(t, x(t-r)) - q(t, y(t-r))\|_{\mathbb{K}}^2 \\ + 4E\Big\|\int_{-\infty}^{t} AT(t-s)[q(s, x(s-r)) - q(s, y(s-r))]\,ds\Big\|^2 \\ + 4E\Big\|\int_{-\infty}^{t} T(t-s)[h(s, x(s-r)) - h(s, y(s-r))]\,ds\Big\|^2\end{aligned}$$

$$+ 4E\Big\| \int_{-\infty}^{t} T(t-s)[f(s,x(s-r)) - f(s,y(s-r))]\, dW(s) \Big\|^2.$$

By using the Cauchy–Schwarz inequality, we first estimate the first three terms of the right-hand side:

$$\begin{aligned} 4E\|q(t,x(t-r)) - q(t,y(t-r))\|_{\mathbb{K}}^2 &\le 4L_q E\|x(s-r) - y(s-r)\|^2 \\ &\le 4L_q \sup_{s\in\mathbb{R}} E\|x(s) - y(s)\|^2 \le 4L_q\|x-y\|_\infty^2, \end{aligned}$$

$$\begin{aligned} &4E\Big\| \int_{-\infty}^{t} AT(t-s)[q(s,x(s-r)) - q(s,y(s-r))]\, ds \Big\|^2 \\ &\le 4E\Big(\int_{-\infty}^{t} \|AT(t-s)\|_{L(\mathbb{K},\mathbb{H})} \|q(s,x(s-r)) - q(s,y(s-r))\|_{\mathbb{K}}\, ds \Big)^2 \\ &\le 4E\Big(\int_{-\infty}^{t} e^{-\gamma(t-s)}\phi(t-s) \|q(s,x(s-r)) - q(s,y(s-r))\|_{\mathbb{K}}\, ds \Big)^2 \\ &\le 4\Big(\int_{-\infty}^{t} e^{-\gamma(t-s)}\phi(t-s)\, ds \Big) \\ &\qquad\qquad \times\Big(\int_{-\infty}^{t} e^{-\gamma(t-s)}\phi(t-s) E\|q(s,x(s-r)) - q(s,y(s-r))\|_{\mathbb{K}}^2\, ds \Big) \\ &\le 4L_q\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}\phi(t-s)\, ds \Big)\Big(\int_{-\infty}^{t} e^{-\gamma(t-s)}\phi(t-s) E\|x(s-r)-y(s-r)\|^2\, ds \Big) \\ &\le 4L_q\Big(\int_{-\infty}^{t} e^{-\gamma(t-s)}\phi(t-s)\, ds \Big)^2 \sup_{s\in\mathbb{R}} E\|x(s) - y(s)\|^2 \\ &\le 4L_q \sup_{t\in\mathbb{R}} \Big(\int_{-\infty}^{t} e^{-\gamma(t-s)}\phi(t-s)\, ds \Big)^2 \|x-y\|_\infty^2, \end{aligned}$$

and

$$\begin{aligned} &4E\Big\| \int_{-\infty}^{t} T(t-s)[h(s,x(s-r)) - h(s,y(s-r))]\, ds \Big\|^2 \\ &\le 4M^2 E\Big(\int_{-\infty}^{t} e^{-\delta(t-s)} \|h(s,x(s-r)) - h(s,y(s-r))\|\, ds \Big)^2 \\ &\le 4M^2\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}\, ds \Big)\Big(\int_{-\infty}^{t} e^{-\delta(t-s)} E\|h(s,x(s-r)) - h(s,y(s-r))\|^2\, ds \Big) \end{aligned}$$

$$\leq 4M^2 L_h\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}\,ds\Big)\Big(\int_{-\infty}^{t} e^{-\delta(t-s)} E\|x(s-r)-y(s-r)\|^2\,ds\Big)$$

$$\leq 4M^2 L_h\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}\,ds\Big)^2 \sup_{s\in\mathbb{R}} E\|x(s)-y(s)\|^2 \leq \frac{4M^2}{\delta^2} L_h \|x-y\|_\infty^2.$$

As to the last term, by the properties of the Ito integral, we get

$$4E\Big\|\int_{-\infty}^{t} T(t-s)[f(s,x(s-r))-f(s,y(s-r))]\,dW(s)\Big\|^2$$

$$\leq 4M^2 \int_{-\infty}^{t} e^{-2\delta(t-s)} E\|f(s,x(s-r))-f(s,y(s-r))\|^2\,ds$$

$$\leq 4M^2 L_f \int_{-\infty}^{t} e^{-2\delta(t-s)} E\|x(s-r)-y(s-r)\|^2\,ds$$

$$\leq 4M^2 L_f \int_{-\infty}^{t} e^{-2\delta(t-s)}\,ds \sup_{s\in\mathbb{R}} E\|x(s)-y(s)\|^2 \leq \frac{4M^2}{2\delta} L_f \|x-y\|_\infty^2.$$

Altogether, it follows that, for each $t\in\mathbb{R}$,

$$E\|(\Psi x)(t)-(\Psi y)(t)\|^2 \leq 4\Big[L_q + L_q \sup_{t\in\mathbb{R}}\Big(\int_{-\infty}^{t} e^{-\delta(t-s)}\phi(t-s)\,ds\Big)^2 + \frac{M^2}{\delta^2}L_h + \frac{M^2}{2\delta}L_f\Big]\|x-y\|_\infty^2.$$

Hence,

$$\|\Psi x - \Psi y\|_\infty \leq \sqrt{L_0}\,\|x-y\|_\infty,$$

where $L_0 = 4[L_q + L_q \sup_{t\in\mathbb{R}}(\int_{-\infty}^{t} e^{-\gamma(t-s)}\phi(t-s)\,ds)^2 + \frac{M^2}{\delta^2}L_h + \frac{M^2}{2\delta}L_f] < 1$, so Ψ is a strict contraction. By the Banach contraction principle, there exists a unique fixed point $x(\cdot)$ for Ψ in $PAA(L^2(\mathbb{P},\mathbb{H}))$, that is,

$$x(t) = q(t,x(t-r)) + \int_{-\infty}^{t} AT(t-s)q(s,x(s-r))\,ds + \int_{-\infty}^{t} T(t-s)h(s,x(s-r))\,ds + \int_{-\infty}^{t} T(t-s)f(s,x(s-r))\,dW(s) \tag{3.10}$$

for all $t\in\mathbb{R}$.

To prove that x satisfies (3.7) for all $t \geq s$ and all $s \in \mathbb{R}$, we write

$$x(s) = q(s, x(s-r)) + \int_{-\infty}^{s} AT(s-\tau)q(\tau, x(\tau-r))\,d\tau + \int_{-\infty}^{s} T(s-\tau)h(\tau, x(\tau-r))\,d\tau + \int_{-\infty}^{s} T(s-\tau)f(\tau, x(\tau-r))\,dW(\tau), \quad s \in \mathbb{R}. \tag{3.11}$$

Multiply both sides of (3.11) by $T(t-s)$ for all $t \geq s$ to obtain

$$\begin{aligned}
&T(t-s)x(s)\\
&= T(t-s)q(s, x(s-r)) + \int_{-\infty}^{s} AT(t-\tau)q(\tau, x(\tau-r))\,d\tau\\
&\quad + \int_{-\infty}^{s} T(t-\tau)h(\tau, x(\tau-r))\,d\tau + \int_{-\infty}^{s} T(t-\tau)f(\tau, x(\tau-r))\,dW(\tau)\\
&= T(t-s)q(s, x(s-r)) + \int_{-\infty}^{t} AT(t-\tau)q(\tau, x(\tau-r))\,d\tau\\
&\quad - \int_{s}^{t} AT(t-\tau)q(\tau, x(\tau-r))\,d\tau + \int_{-\infty}^{t} T(t-\tau)h(\tau, x(\tau-r))\,d\tau\\
&\quad - \int_{s}^{t} T(t-\tau)h(\tau, x(\tau-r))\,d\tau + \int_{-\infty}^{t} T(t-\tau)f(\tau, x(\tau-r))\,dW(\tau)\\
&\quad - \int_{s}^{t} T(t-\tau)f(\tau, x(\tau-r))\,dW(\tau)\\
&= x(t) + T(t-s)q(s, x(s-r)) - q(t, x(t-r))\\
&\quad - \int_{s}^{t} AT(t-\tau)q(\tau, x(\tau-r))\,d\tau - \int_{s}^{t} T(t-\tau)h(\tau, x(\tau-r))\,d\tau\\
&\quad - \int_{s}^{t} T(t-\tau)f(\tau, x(\tau-r))\,dW(\tau).
\end{aligned}$$

Hence x is a mild solution to (3.7) and $x(\cdot) \in PAA(L^2(\mathbb{P}, \mathbb{H}))$. It is clear that $x(t)$ is the unique mild solution to (1.1). ∎

REMARK 3.7. The condition (3.8) is satisfied if L^* is small enough, where $L^* = \max\{L_q, L_h, L_f\}$.

4. Application. Let $\Gamma \subset \mathbb{R}^N (N \geq 1)$ be an open bounded subset with C^2 regular boundary $\partial\Gamma$ and let $\mathbb{H} = L^2(\Gamma)$ be equipped with its natural topology $\|\cdot\|_{L^2(\Gamma)}$. We study the existence of Stepanov-like pseudo almost automorphic solutions to the following neutral stochastic partial functional differential equations:

$$\begin{aligned}(4.1)\quad & d[z(t,x) - \mu_1(t, z(t-r,x))] = \Delta z(t,x)\,dt \\ & \qquad + \mu_2(t, z(t-r,x))\,dt + \mu_3(t, z(t-r,x))\,dW(t), \qquad (t,x) \in \mathbb{R}\times\Gamma,\\ (4.2)\quad & z(t,x) = 0, \qquad (t,x) \in \mathbb{R}\times\partial\Gamma,\end{aligned}$$

where $\Delta = \sum_{i=1}^N \partial^2/\partial x_i^2$ is the Laplace operator on Γ and $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$. In this system, μ_i, $i = 1,2,3$, are Stepanov-like pseudo almost automorphic continuous functions.

Let A be the linear operator given by

$$Au = \Delta u \quad \text{ for all } u \in D(A) = H_0^1(\Gamma) \cap H^2(\Gamma).$$

The operator A is sectorial and hence is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$. One can define the fractional power $(-A)^\alpha = (-\Delta)^\alpha$, $0 < \alpha \leq 1$ of A, as a closed linear operator over its domain $D((-A)^\alpha)$. If $\mathbb{H}_\alpha$ denotes the space $D((-A)^\alpha)$ endowed with the graph norm $\|\cdot\|_\alpha$, then $\mathbb{H}_\alpha$ is a Banach space. Moreover, $\mathbb{H}_\alpha \to \mathbb{H}_\beta$ is continuous for $0 < \beta \leq \alpha \leq 1$ and there exist constants $C_\alpha, \delta_\alpha > 0$ such that $\|T(t)\|_{L(\mathbb{H}_\alpha,\mathbb{H})} \leq C_\alpha e^{-\delta_\alpha t}/t^\alpha$ for $t > 0$.

Take $\alpha = 1/2$ and $\mathbb{K} = [D((-A)^{1/2})]$. We require the following assumption:

- (Ha) $\mu_1, \mu_2, \mu_3 : \mathbb{R}\times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$ are Stepanov-like pseudo almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in L^2(\mathbb{P},\mathbb{H})$, μ_1 is $L^2(\mathbb{P},\mathbb{K})$-valued, and $\mu_1 : \mathbb{R}\times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H}_{1/2})$ is continuous.
- (Hb) There exist constants $L_i \in (0,1)$, $i = 1,2,3$, such that

$$\begin{aligned}E\|\mu_1(t,u) - \mu_1(t,v)\|_{1/2}^2 &\leq L_1\|u-v\|_{L^2(\Gamma)}^2,\\ E\|\mu_2(t,u) - \mu_2(t,v)\|_{L^2(\Gamma)}^2 &\leq L_2\|u-v\|_{L^2(\Gamma)}^2,\\ E\|\mu_3(t,u) - \mu_3(t,v)\|_{L^2(\Gamma)}^2 &\leq L_3\|u-v\|_{L^2(\Gamma)}^2\end{aligned}$$

  for all $t \in \mathbb{R}$ and each $u, v \in L^2(\mathbb{P}, L^2(\Gamma))$.

For $x \in \Gamma$ and $t \in \mathbb{R}$, we can define

$$\begin{aligned}q(t,u)(x) &= \mu_1(t, u(t-r)(x)),\\ h(t,u)(x) &= \mu_2(t, u(t-r)(x)),\\ f(t,u)(x) &= \mu_3(t, u(t-r)(x)).\end{aligned}$$

Then the above equation can be written in the abstract form as the system (1.1).

Consequently all assumptions (H1)–(H3) are satisfied, and Theorem 3.6 yields the following result.

PROPOSITION 4.1. *Under the above assumption, if moreover*

$$4\left[L_1 + L_1(2C_{1/2} + \delta_{1/2}^{-1})^2 + \frac{M^2}{\delta^2}L_2 + \frac{M^2}{2\delta}L_3\right] < 1,$$

then (4.1)–(4.2) *has a unique pseudo almost automorphic solution on* $\mathbb{R}$.

The functions $\mu_1, \mu_2, \mu_3 : \mathbb{R} \times L^2(\mathbb{P}, L^2(\varGamma)) \to L^2(\mathbb{P}, L^2(\varGamma))$ mentioned above are given as follows:

$$\begin{aligned}
\mu_1(t, z(t-r,x)) &= \frac{L^* a(t)}{1 + |z(t-r,x)|},\\
\mu_2(t, z(t-r,x)) &= \frac{L^* b(t)}{1 + |z(t-r,x)|},\\
\mu_3(t, z(t-r,x)) &= \frac{L^* c(t)}{1 + |z(t-r,x)|},
\end{aligned}$$

where the functions $a, b, c : \mathbb{R} \to \mathbb{R}$ are Stepanov-like pseudo almost automorphic in $t \in \mathbb{R}$.

In this particular case, the corresponding stochastic equation, that is,

$$\begin{aligned}
d\left[z(t,x) - \frac{L^* a(t)}{1 + |z(t-r,x)|}\right] &= \varDelta z(t,x)\,dt + \frac{L^* b(t)}{1 + |z(t-r,x)|}\,dt\\
&\quad + \frac{L^* c(t)}{1 + |z(t-r,x)|} dW(t),\ \ (t,x) \in \mathbb{R} \times \varGamma,\\
z(t,x) = 0,\quad (t,x) &\in \mathbb{R} \times \partial\varGamma,
\end{aligned}$$

has a unique pseudo almost automorphic solution $z \in L^2(\mathbb{P}, H_0^1(\varGamma) \cap H^2(\varGamma))$ whenever L^* is small enough.

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Zuomao Yan, Xiu-Mei Jia, Xingxue Yan
Department of Mathematics
Hexi University
Zhangye, Gansu
734000, P.R. China
E-mail: yanzuomao@163.com

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