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Some properties of solutions of complex q-shift difference equations

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Abstract. Combining difference and q-difference equations, we study the properties of meromorphic solutions of q-shift difference equations from the point of view of value distribution. We obtain lower bounds for the Nevanlinna lower order for meromorphic solutions of such equations. Our results improve and extend previous theorems by Zheng and Chen and by Liu and Qi. Some examples are also given to illustrate our results.

1. Introduction and main results. The purpose of this paper is to study some growth properties of solutions of complex q-shift difference equations. The fundamental results and standard notations of Nevanlinna value distribution theory of meromorphic functions will be used (see Hayman [6], Yang [16]). Moreover, for a meromorphic function f, $S(r,f)$ denotes any quantity satisfying $S(r,f) = o(T(r,f))$ for all r outside a possible exceptional set E of finite logarithmic measure, $\lim_{r\to\infty} \int_{[1,r)\cap E} dt/t < \infty$.

In recent years, there has been an increasing interest in difference equations, difference products and q-differences in the complex plane; a number of papers (including [3, 4, 9, 12, 19, 20]) have focused on the growth of solutions of difference equations, value distribution and uniqueness of difference analogues of Nevanlinna's theory. Chiang and Feng [3] and Halburd and Korhonen [5] independently established a difference analogue of the Logarithmic Derivative Lemma, and then applied it to prove a number of results on meromorphic solutions of complex difference equations. About the same time, Barnett, Halburd, Korhonen and Morgan [1] also established an analogue of the Logarithmic Derivative Lemma for q-difference operators.

In 2001, Heittokangas et al. [8] investigated the growth of transcendental meromorphic solutions of complex difference equations and obtained the following results.

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THEOREM 1.1 (see [8, Theorem 10]). *Let* $c_1, \dots, c_n \in \mathbb{C} \setminus \{0\}$ *and let* $m \geq 2$. *Suppose that* y *is a transcendental meromorphic solution of the difference equation*

$$\sum_{i=1}^{m} a_i(z) y(z + c_i) = \sum_{i=0}^{m} b_i(z) y(z)^i \tag{1}$$

with rational coefficients $a_i(z)$, $b_i(z)$. *Denote* $C = \max\{|c_1|, \dots, |c_n|\}$.

(i) *If* y *is entire or has finitely many poles, then there exist constants* $K > 0$ *and* $r_0 > 0$ *such that* $\log M(r, y) \geq K m^{r/C}$ *for all* $r \geq r_0$.
(ii) *If* y *has infinitely many poles, then there exist constants* $K > 0$ *and* $r_0 > 0$ *such that* $n(r, y) \geq K m^{r/C}$ *for all* $r \geq r_0$.
(iii) *Thus, all transcendental meromorphic solutions of* (1) *have infinite lower order.*

THEOREM 1.2 (see [8, Theorem 11]). *Let* $c_1, \dots, c_n \in \mathbb{C} \setminus \{0\}$ *and* y *be a transcendental meromorphic solution of the difference equation*

$$\sum_{i=1}^{n} d_i(z) y(z + c_i) = \frac{a_0(z) + a_1(z) y(z) + \cdots + a_p(z) y(z)^p}{b_0(z) + b_1(z) y(z) + \cdots + b_t(z) y(z)^t}, \tag{2}$$

where all coefficients are $o(T(r, f))$ *without an exceptional set as* $r \to \infty$, *and* d_i*'s are non-vanishing. If* $d = \max\{p, t\} > n$, *then for any* $0 < \varepsilon < \frac{d-n}{d+n}$, *there exists an* $r_0 > r$ *such that*

$$T(r, y) \geq K \left(\frac{d}{n} \left(\frac{1 - \varepsilon}{1 + \varepsilon} \right) \right)^{r/C}$$

for all $r \geq r_0$, *where* $C = \max\{|c_1|, \dots, |c_n|\}$ *and* $K > 0$ *is a constant.*

In 2005, Laine, Rieppo and Silvennoinen [11] further investigated the growth of transcendental meromorphic solution of complex difference equations and obtained the following theorem.

THEOREM 1.3 (see [11, Theorem 2.2]). *Let* $c_1, \dots, c_n$ *be non-zero complex constants and suppose that* f *is a transcendental meromorphic solution of*

$$\sum_{\{J\}} \alpha_J(z) \Big(\prod_{j \in J} f(z + c_j) \Big) = R(z, f),$$

where the coefficients α_J *are small relative to* f, *and where* R *is rational in* f *with coefficients small relative to* f. *If* $\deg_f R > n$, *then the lower order* $\mu(f)$ *is* ∞.

In 2002, Gundersen et al. [4] studied the growth of meromorphic solutions of q-difference equations and obtained the following result.

Theorem 1.4 (see [4, Theorem 3.2]). *Suppose that f is a transcendental meromorphic solution of an equation of the form*

$$f(cz) = R(z, f(z)) = \frac{\sum_{j=0}^{p} a_j(z) f(z)^j}{\sum_{j=0}^{q} b_j(z) f(z)^j}$$

with meromorphic coefficients $a_j(z), b_j(z)$, and a constant $c \neq 0$ ($|c| > 1$), where we assume that $d := \max\{p, q\} \geq 1$, $a_p(z) \neq 0$, $b_q(z) \neq 0$, and $R(z, f(z))$ is irreducible in f. Then $\rho(f) = \log d / \log |c|$.

In 2010, Zheng and Chen [20] further studied the growth of meromorphic solutions of q-difference equations and obtained some results which extended the theorems of Heittokangas et al. [8].

Theorem 1.5 (see [20, Theorem 1]). *Suppose that f is a transcendental meromorphic solution of the equation*

$$\sum_{j=1}^{n} a_j(z) f(q^j z) = \sum_{i=0}^{d} b_i(z) f(z)^i,$$

where $q \in \mathbb{C}$, $|q| > 1$, $d \geq 2$ and the coefficients $a_j(z)$, $b_i(z)$ are rational functions.

(i) *If f is entire or has finitely many poles, then there exist constants $K > 0$ and $r_0 > 0$ such that $\log M(r, f) \geq K d^{\log r/(n \log |q|)}$ for all $r \geq r_0$.*
(ii) *If f has infinitely many poles, then there exist constants $K > 0$ and $r_0 > 0$ such that $n(r, f) \geq K d^{\log r/(n \log |q|)}$ for all $r \geq r_0$.*
(iii) *Thus, all transcendental meromorphic solutions f satisfy $\mu(f) \geq \log d/(n \log |q|)$.*

Theorem 1.6 (see [20, Theorem 2]). *Suppose that f is a transcendental meromorphic solution of the equation*

$$\sum_{j=1}^{n} a_j(z) f(q^j z) = R(z, f(z) = \frac{P(z, f(z))}{Q(z, f(z))},$$

where $q \in \mathbb{C}$, $|q| > 1$, the coefficients $a_j(z)$, $b_i(z)$ are rational functions and P, Q are relatively prime polynomials in f over the field of rational functions satisfying $p = \deg_f P$, $t = \deg_f Q$, $d = p - t \geq 2$. If f has infinitely many poles, then for sufficiently large r, $n(r, f) \geq K d^{\log r/(n \log |q|)}$ for some constant $K > 0$. Thus, the lower order of f which has infinitely many poles satisfies $\mu(f) \geq \log d/(n \log |q|)$.

In 2011, Liu and Qi [13] investigated the properties of meromorphic solutions of q-shift difference equation and obtained the following result.

THEOREM 1.7 (see [13, Theorem 4.1]). *Suppose that f is a transcendental meromorphic solution of*

$$f(cz+\eta)=\frac{\sum_{j=0}^{p} a_j(z)f(z)^j}{\sum_{j=0}^{q} b_j(z)f(z)^j},$$

where the coefficients $a_j(z)$, $b_j(z)$ are rational, and $|c|>1$. Assume $R(z,f(z))$ is irreducible in f and $a_p(z)b_q(z)\not\equiv 0$. If $p>q+1$ and $m:=p-q$, then $\lambda(1/f)\geq \log|m|/\log|c|$, provided that f has infinitely many poles.

It is a natural question to ask what will happen when the functions $y(z+c)$, $f(z+c)$, $f(cz)$ and $f(qz)$ are replaced by $f(qz+c)$ in Theorems 1.1–1.6, where $q\in\mathbb{C}\setminus\{0\}$, $c\in\mathbb{C}$.

In this paper, we will investigate this question and obtain the following theorems.

THEOREM 1.8. *Suppose that f is a transcendental meromorphic solution of the equation*

$$\sum_{j=1}^{n} a_j(z)f(q^j z+c_j)=\sum_{i=0}^{d} b_i(z)f(z)^i, \tag{3}$$

where the coefficients $a_j(z)$, $b_i(z)$ are rational functions, $q\in\mathbb{C}\setminus\{0\}$, $|q|>1$, $d\geq 2$ and $c_j\in\mathbb{C}$. If f is entire or has finitely many poles, then there exist constants $K>0$ and $r_0>0$ such that $\log M(r,f)\geq Kd^{\log r/(n\log|q|)}$ for all $r\geq r_0$.

EXAMPLE 1.1. The function $f(z)=e^z/z$ satisfies the equation

$$\sum_{j=1}^{n}\frac{2^j z+c_j}{e^{c_j}z^{2^j}}f(2^j z+c_j)=\sum_{i=1}^{n} f(z)^{2^j}$$

with rational coefficients, where $|q|=2>1$, $d=2^n$ and $c_j\in\mathbb{C}$. Since $n<2^n=d$ for all $n\in N$, we have $\log M(r,f)=r-\log r\geq \frac{1}{2}r=\frac{1}{2}d^{\log r/(n\log|q|)}$ $(r\to\infty)$ and $\mu(f)=\sigma(f)=1=\log d/(n\log|q|)$. This shows that in the inequality $\mu(f)\geq \log d/(n\log|q|)$ following from Theorem 1.8, equality can be attained.

EXAMPLE 1.2. The function $f(z)=z$ satisfies the equation

$$\sum_{j=1}^{n} f(2^j z+c_j)=\Big(2(2^n-1)z+\sum_{j=1}^{n} c_j\Big)+\frac{2}{z}f(z)-\frac{1}{z^2}f(z)^2-\frac{1}{z^3}f(z)^3$$

with rational coefficients, where $q=2$, $d=3>2$ and $c_j\in\mathbb{C}$. This shows that (3) in Theorem 1.8 may have non-transcendental solutions.

Theorem 1.9. *Suppose that f is a transcendental meromorphic solution of the equation*

$$\sum_{j=1}^{n} a_j(z)f(q^j z + c_j) = R(z, f(z)) = \frac{P(z, f(z))}{Q(z, f(z))}, \tag{4}$$

where the coefficients $a_j(z)$, $b_i(z)$ are rational functions, $q \in \mathbb{C} \setminus \{0\}$, $|q| > 1$, $d \geq 2$ and $c_j \in \mathbb{C}$ and P, Q are relatively prime polynomials in f over the field of rational functions satisfying $s = \deg_f P$, $t = \deg_f Q$, $d = s - t \geq 2$. If f has infinitely many poles, then for sufficiently large r, $n(r, f) \geq K d^{\log r/(n \log |q|)}$ for some constant $K > 0$. Thus, the lower order of f which has infinitely many poles satisfies $\mu(f) \geq \log d/(n \log |q|)$.

Remark 1.1. For Theorem 1.9, the equation (3) is a special form of (4). So, we get the following conclusions.

Under the assumptions of Theorem 1.8, if f is a solution of (3) with infinitely many poles, then there exist constants $K > 0$ and $r_0 > 0$ such that $n(r, f) \geq K d^{\log r/(n \log |q|)}$ for all $r \geq r_0$. Thus, all transcendental meromorphic solutions f satisfy $\mu(f) \geq \log d/(n \log |q|)$.

Theorem 1.10. *Suppose that f is a transcendental meromorphic solution of the equation*

$$\begin{aligned}\frac{\sum_{\lambda \in I} d_\lambda(z) f(qz + c_1)^{i_{\lambda_1}} f(q^2 z + c_2)^{i_{\lambda_2}} \cdots f(q^n z + c_n)^{i_{\lambda_n}}}{\sum_{\mu \in J} e_\mu(z) f(qz + c_1)^{j_{\mu_1}} f(q^2 z + c_2)^{j_{\mu_2}} \cdots f(q^n z + c_n)^{j_{\mu_n}}} \\ = \frac{\sum_{j=0}^{s} a_j(z) f(z)^j}{\sum_{j=0}^{t} b_j(z) f(z)^j},\end{aligned} \tag{5}$$

where $I = \{(i_{\lambda_1}, \ldots, i_{\lambda_n})\}$, $J = \{j_{\mu_1}, \ldots, j_{\mu_n}\}$ are finite index sets satisfying

$$\max_{\lambda,\mu}\{i_{\lambda_1} + \cdots + i_{\lambda_n}, j_{\mu_1} + \cdots + j_{\mu_n}\} = \sigma,$$

$q \in \mathbb{C} \setminus \{0\}$, $|q| > 1$, $d \geq 2$ and $c_j \in \mathbb{C}$, and all coefficients of (5) are of growth $S(r, f)$. If $d = \max\{s, t\} > 2n\sigma$, then for sufficiently large r,

$$T(r, f) \geq K \left(\frac{d}{2n\sigma} \right)^{\frac{\log r}{n \log |q|}},$$

where $K > 0$ is a constant. Thus, the lower order of f satisfies

$$\mu(f) \geq \frac{\log d - \log 2n\sigma}{n \log |q|}.$$

EXAMPLE 1.3. The function $f(z) = e^{z^2}$ satisfies the equation

$$\frac{f(4z+c_2) + f(2z+c_1)f(4z+c_2)}{f(2z+c_1)} = e^{-8c_2z-c_2^2}f(z)^{16} + e^{(4c_1-8c_2)z+(c_1^2-c_2^2)}f(z)^{12}$$

with small function coefficients, where $q = n = \sigma = 2$, $d = 16 > 2n\sigma$ and $c_1, c_2 \in \mathbb{C}$. It is clear that $\mu(f) = \sigma(f) = 2$, showing that the conclusion of Theorem 1.10 can hold.

THEOREM 1.11. *Suppose that f is a transcendental meromorphic solution of the equation*

$$\sum_{j=1}^{n} a_j(z)f(q^jz+c_j) = R(z, f(z)) = \frac{P(z, f(z))}{Q(z, f(z))}, \tag{6}$$

where the coefficients $a_j(z)$ are non-vanishing small functions relative to f and P, Q are relatively prime polynomials in f over the field of small functions relative to f. Moreover, assume that $t = \deg_f Q > 0$,

$$n = \max\{s, t\} := \max\{\deg_f P, \deg_f Q\}$$

and that, without restricting generality, Q is a monic polynomial. If there exists $\alpha \in [0, n)$ such that for all sufficiently large r,

$$\overline{N}\Big(r, \sum_{j=1}^{n} a_j(z)f(q^jz+c_j)\Big) \le \alpha\overline{N}(|q|^n r + C, f(z)) + S(r, f), \tag{7}$$

where $q \in \mathbb{C} \setminus \{0\}$, $|q| > 1$, $d \ge 2$ and $c_j \in \mathbb{C}$, $C := \max\{|c_1|, \dots, |c_n|\}$, then either $\rho(f) > 0$, or $Q(z, f(z)) \equiv (f(z) + h(z))^t$, where $h(z)$ is a small meromorphic function.

REMARK 1.2. The ideas and formulations of Theorems 1.8–1.11 come from [17] and [13], respectively, with the q-difference $f(qz)$ replaced by the q-shift difference $f(qz+c)$. However, the case of q-shift difference is more intricate than the cases of $f(qz)$ and $f(z+c)$.

REMARK 1.3. In this paper, we combine difference and q-difference equations, and only study the properties of meromorphic solutions of q-shift difference equations from the point of view of value distribution theory. However, the existence of meromorphic solutions to these equations remains an open problem.

2. Some lemmas

LEMMA 2.1 (Valiron–Mohon'ko, see [10]). *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z,f(z))=\frac{\sum_{i=0}^{m}a_i(z)f(z)^i}{\sum_{j=0}^{n}b_j(z)f(z)^j},$$

with meromorphic coefficients $a_i(z)$, $b_j(z)$, *the characteristic function of* $R(z,f(z))$ *satisfies*

$$T(r,R(z,f(z)))=dT(r,f)+O(\Psi(r)),$$

where $d=\max\{m,n\}$ *and* $\Psi(r)=\max_{i,j}\{T(r,a_i),T(r,b_j)\}$.

LEMMA 2.2 (see [15]). *Let* f *be a meromorphic function, and let* $\phi=f^n+a_{n-1}f^{n-1}+\cdots+a_0$, *where* $a_0,a_1,\dots,a_{n-1}$ *are small meromorphic functions relative to* f. *Then either*

$$\phi=\left(f+\frac{a_{n-1}}{n}\right)^n \quad \text{or} \quad T(r,f)\le\overline{N}(r,1/\phi)+\overline{N}(r,f)+S(r,f).$$

LEMMA 2.3 (see [11]). *Let* f *be a non-constant meromorphic function and let* $P(z,f)$, $Q(z,f)$ *be polynomials in* f *with meromorphic coefficients small relative to* f. *If* P *and* Q *have no common factors of positive degree in* f *over the field of small functions relative to* f, *then*

$$\overline{N}\left(r,\frac{1}{Q(z,f)}\right)\le\overline{N}\left(r,\frac{P(z,f)}{Q(z,f)}\right)+S(r,f).$$

From Lemma 4 in [7], we can get the following lemma.

LEMMA 2.4 (see [7, 20]). *Let* f *be a non-constant meromorphic function, let* $\beta>1$ *and* $\alpha<1$ *be given constants, and let* $F\subset\mathbb{R}^+$ *be the set of all* r *such that* $\overline{N}(r,f)\le\alpha\overline{N}(\beta r,f)$. *If the logarithmic density of* F *is non-zero, that is,* $\operatorname{logdens}F>0$, *then the exponent of convergence of distinct poles* $\overline{\lambda}(1/f)$ *is non-zero. Thus,* $\rho(f)$ *is non-zero.*

LEMMA 2.5 (see [20]). *Let* $f_1,\dots,f_n$ *be meromorphic functions. Then*

$$T\left(r,\sum_{\lambda\in I}f_1^{i_{\lambda_1}}\cdots f_n^{i_{\lambda_n}}\right)\le\sigma\sum_{i=1}^{n}T(r,f_i)+\log s,$$

where $I=\{(i_{\lambda_1},\dots,i_{\lambda_n})\}$ *is an index set consisting of* s *elements, and* $\sigma=\max_{\lambda\in I}\{i_{\lambda_1}+\cdots+i_{\lambda_n}\}$.

LEMMA 2.6 (see [3, Theorem 2.1]). *Let* $f(z)$ *be a meromorphic function of finite order* ρ, *and* c *a non-zero complex constant. Then, for each* $\varepsilon>0$,

$$T(r,f(z+c))=T(r,f(z))+O(r^{\rho-1+\varepsilon})+O(\log r).$$

LEMMA 2.7 (see [19, Theorems 1.1 and 1.3]). *Let* $f(z)$ *be a transcendental meromorphic function of zero order and* q *be a non-zero complex constant. Then*

$$T(r,f(qz))=T(r,f(z))+S(r,f)$$

on a set of logarithmic density 1.

3. Proofs of Theorems 1.8 and 1.9

3.1. The proof of Theorem 1.8. Since $a_j(z), b_i(z)$ in (3) are rational, we can multiply out the denominators of the coefficients $a_j(z), b_i(z)$ to get the equation

$$\sum_{j=1}^{n} A_j(z) f(q^j z + c_j) = \sum_{i=0}^{d} B_i(z) f(z)^i, \tag{8}$$

where the coefficients $A_j(z), B_i(z)$ are polynomials.

Suppose that f is a transcendental entire function solution of (3) (or (8)). Set $p_j = \deg A_j$ $(j = 1, \ldots, n)$, $q_i = \deg B_i$ $(i = 0, 1, \ldots, d)$ and $C = \max\{|c_1|, \ldots, |c_n|\}$. Take $m = \max\{p_1, \ldots, p_n\} + 1$. Since $|q| > 1$ and since $M(r, f(q^j z + c_j)) \leq M(|q|^j r + |c_j|, f)$, from (8) and $A_j(z), B_i(z)$ are polynomials, we get

$$\begin{aligned} M\Big(r, \sum_{i=0}^{d} B_i(z) f(z)^i\Big) &= M\Big(r, \sum_{j=1}^{n} A_j(z) f(q^j z + c_j)\Big) \\ &\leq n r^m M(|q|^n r + C, f), \end{aligned} \tag{9}$$

when r is sufficiently large. Since B_i $(i = 0, 1, \ldots, d)$ are polynomials and f is a transcendental entire function, we have

$$M\Big(r, \sum_{i=0}^{d-1} B_i(z) f(z)^i\Big) = o(M(r, f(z)^d)).$$

From the above inequality, for sufficiently large r, we have

$$M\Big(r, \sum_{i=0}^{d} B_i(z) f(z)^i\Big) \geq \tfrac{1}{2} M(r, B_d(z) f(z)^d). \tag{10}$$

From (9) and (10), we get

$$\log M(|q|^n r + C, f) \geq d \log M(r, f) + g(r), \tag{11}$$

where $|g(r)| < K \log r$ for some $K > 0$ and sufficiently large r. Iterating (11), we have

$$\begin{aligned} &\log M(|q|^{nk} r + (|q|^{n(k-1)} + \cdots + |q|^n + |q|^0) C, f) \\ &\qquad \geq d^k \log M(r, f) + E_k(r), \end{aligned} \tag{12}$$

where $k \in \mathbb{N}_+$ and

$$\begin{aligned} &|E_k(r)| \\ &= \Big| d^{k-1} g(r) + d^{k-2} g(|q|^n r + C) + \cdots + g\Big(|q|^{n(k-1)} r + C \sum_{j=0}^{k-2} |q|^{jn}\Big) \Big| \end{aligned} \tag{13}$$

$$\leq Kd^{k-1}\sum_{s=0}^{k-1}\frac{\log(|q|^{sn}r+C\sum_{j=0}^{s-1}|q|^{jn})}{d^s}$$
$$\leq Kd^{k-1}\sum_{s=0}^{\infty}\frac{\log(|q|^{sn}r+C\sum_{j=0}^{s-1}|q|^{jn})}{d^s}.$$

Since $|q|>1$ and $\log(r+sC)\leq(\log r)(\log sC)$ for r and s sufficiently large, we have

$$\begin{aligned}(14)\qquad \log\left(|q|^{sn}r+C\sum_{j=0}^{s-1}|q|^{jn}\right) &\leq (\log(|q|^{sn}r))(\log(sC))(\log(|q|^{(s-1)n}))\\ &\leq n^2(\log|q|)^2(\log C)(\log r)s(s-1)\log s,\end{aligned}$$

for sufficiently large r and s. From (13) and (14), we have

$$(15)\qquad |E_k(r)|\leq K'd^{k-1}\log r\sum_{s=0}^{\infty}\frac{s(s-1)\log s}{d^s},$$

where $K_1 = Kn^2(\log|q|)^2(\log C)$. We can easily deduce that the series $\sum_{s=0}^{\infty}s(s-1)(\log s)/d^s$ is convergent when $d\geq 2$. Therefore, from (15),

$$(16)\qquad |E_k(r)|\leq K_2d^k\log r,$$

where $K_2>0$ is a constant. Since f is a transcendental entire function, for sufficiently large r, we have

$$(17)\qquad \log M(r,f)\geq 2K_2\log r.$$

By (12), (16) and (17), we see that there exists an $r_0\geq e$ such that for $r\geq r_0$,

$$(18)\qquad \log M\big(|q|^{nk}r+(|q|^{n(k-1)}+\cdots+|q|^n+|q|^0)C,f\big)\geq K_2d^k\log r.$$

Thus, for each sufficiently large γ, since $|q|>1$, there exists a $k\in\mathbb{N}_+$ such that

$$\gamma\in\Big[|q|^{nk}r_0+C\sum_{s=0}^{k-1}|q|^{sn},|q|^{n(k+1)}r_0+C\sum_{s=0}^{k}|q|^{sn}\Big],$$

i.e.

$$(19)\qquad k>\frac{\log\gamma+\log[(|q|^n-1)(1+o(1))]-\log[|q|^n(r_0(|q|^n-1)+C)]}{n\log|q|}.$$

Hence, from (18) and (19), we have

(20)

$$\log M(\gamma,f)\geq\log M\Big(|q|^{nk}r_0+C\sum_{s=0}^{k-1}|q|^{sn},f\Big)\geq K_2d^k\log r_0\geq K_3d^{\frac{\log\gamma}{n\log|q|}},$$

where

$$K_3 = K_2 \log r_0 d^{\frac{\log[(|q|^n-1)(1+o(1))]-\log[|q|^n(r_0(|q|^n-1)+C)]}{n\log|q|}}.$$

Hence, we have proved the conclusion when f is an entire function.

Suppose that f is a meromorphic solution of (3) (or (8)) with finitely many poles. Then there exists a polynomial $P(z)$ such that $g(z) = P(z)f(z)$ is entire. Substituting $f(z) = g(z)/P(z)$ into (8) and again multiplying away the denominators, we get an equation similar to (8). For $g(z)$, by using the same argument as above, for sufficiently large r, we get

$$\log M(r,f) = \log M(r,g) + O(1) \geq (K_3 - \varepsilon) d^{\frac{\log r}{n\log|q|}} = K_4 d^{\frac{\log r}{n\log|q|}}$$

where $K_4 > 0$ is a constant.

Therefore, the proof of Theorem 1.8 is complete.

3.2. The proof of Theorem 1.9. Since the coefficients of $R(z,f)$ are rational functions and f has infinitely many poles, we can take a sufficiently large constant $R > 0$ such that the coefficients of $R(z,f)$ have no zeros or poles in $\{z \in \mathbb{C} : |z| > R\}$ and that we can choose a pole z_0 of f of multiplicity $\nu \geq 1$ satisfying $|z_0| > R$. Since $d = s - t \geq 2$, and the right hand side of (4) has a pole of multiplicity $d\nu$ at z_0, there exists $j_1 \in \{1,\dots,n\}$ such that $q^{j_1}z_0 + c_{j_1}$ is a pole of f of multiplicity $\nu_1 \geq d\nu$. Replacing z by $q^{j_1}z_0 + c_{j_1}$ in (4), we have

$$\begin{aligned}(21)\qquad \sum_{j=1}^{n} a_j(q^{j_1}z_0 + c_{j_1}) f(q^{j+j_1}z_0 + q^j c_{j_1} + c_j)& \\ = R\big(q^{j_1}z_0 + c_{j_1}, f(q^{j_1}z_0 + c_{j_1})\big).&\end{aligned}$$

Since the coefficients of $R(z,f)$ are rational and have just finitely many zeros and poles, there exists $j_2 \in \{1,\dots,n\}$ such that $q^{j_1+j_2}z_0 + q^{j_2}c_{j_1} + c_{j_2}$ is a pole of f of multiplicity $\nu_2 \geq d\nu_1 \geq d^2\nu$.

We now continue inductively to construct poles

$$\vartheta_k = \prod_{i=1}^{k} q^{j_i} z_0 + \prod_{i=2}^{k} q^{j_i} c_{j_1} + \cdots + q^{j_k} c_{j_{k-1}} + c_{j_k}$$

of f of multiplicity ν_k for all $k \in \mathbb{N}_+$, satisfying $\nu_k \geq d^k\nu \to \infty$ $(k \to \infty)$, where $j_i \in \{1,\dots,n\}$, $i = 1,\dots,k$. Obviously, we have $|\vartheta_k| \to \infty$ as $k \to \infty$. Then there exists a positive integer $k_0 \in \mathbb{N}_+$ such that for sufficiently large $k \geq k_0$,

$$\begin{aligned}(22)\qquad \nu d^k &\leq \nu(1 + d + \cdots + d^k) \leq n(|\vartheta_k|, f) \\ &\leq n\big(|q|^{nk}|z_0| + C\big((|q|^{n(k-1)} + \cdots + |q|^n + |q|^0)\big), f\big),\end{aligned}$$

where $C := \max\{|c_1|,\dots,|c_n|\}$. Thus, for each sufficiently large r, there exists a $k \in \mathbb{N}_+$ such that

$$r \in \Big[|q|^{nk}|z_0| + C\sum_{i=0}^{k-1}|q|^{in}, |q|^{n(k+1)}r_0 + C\sum_{i=0}^{k}|q|^{in}\Big].$$

By the same argument as in the proof of Theorem 1.8, from (22) we get

$$n(r,f) \geq \nu d^k \geq \nu d^{\frac{\log r + \log[(|q|^n-1)(1+o(1)] - n\log|q| - \log[|z_0|(|q|^n-1)+C]}{n\log|q|}} \geq K_5 d^{\frac{\log r}{n\log|q|}},$$

where

$$K_5 = \nu d^{\frac{\log[(|q|^n-1)(1+o(1)] - n\log|q| - \log[|z_0|(|q|^n-1)+C]}{n\log|q|}}.$$

Moreover there exists an $r_0 > 0$ such that for all $r \geq r_0$, we have

$$K_5 d^{\frac{\log r}{n\log|q|}} \leq n(r,f) \leq \frac{1}{\log 2}T(2r,f).$$

From this inequality, we get $\mu(f) \geq \log d/(n\log|q|)$ easily.

This completes the proof of Theorem 1.9.

4. Proof of Theorem 1.10. Since $|q| > 1$, $c_j \in \mathbb{C}$ and $T(r, f(q^j z + c_j)) = T(|q|^j r + |c_j|, f) + O(1)$, from (5) and Lemmas 2.1 and 2.5, for any given $0 < \varepsilon < \frac{d-2n\sigma}{d+2n\sigma}$, we have

$$\begin{aligned} d(1-\varepsilon)T(r,f) &\leq dT(r,f) + S(r,f) \\ &\leq 2\sigma\sum_{j=1}^{n}T(|q|^j r + |c_j|, f) + S(r,f) \\ &\leq 2n\sigma T(|q|^n r + C, f) + S(r,f) \\ &\leq 2n\sigma(1+\varepsilon)T(|q|^n r + C, f), \end{aligned} \tag{23}$$

outside of a possible exceptional set of finite linear measure. Then there exists an $r_0 > 0$ such that

$$T(|q|^n r + C, f) \geq \frac{d(1-\varepsilon)}{2n\sigma(1+\varepsilon)}T(r,f) \tag{24}$$

for all $r > r_0$. Iterating (24), for any $k \in \mathbb{N}_+$, we have

$$T\Big(|q|^{nk}r + C\sum_{i=0}^{k-1}|q|^{in}, f\Big) \geq \left(\frac{d(1-\varepsilon)}{2n\sigma(1+\varepsilon)}\right)^k T(r,f), \quad r \geq r_0. \tag{25}$$

For sufficiently large ϱ, by using the same argument as in the proof of Theorem 1.8, from (25) we get

(26)

$$T(\varrho,f) \geq T(r_0,f)\left(\frac{d(1-\varepsilon)}{2n\sigma(1+\varepsilon)}\right)^{\frac{\log\varrho + \log[(|q|^n-1)(1+o(1)] - n\log|q| - \log[|z_0|(|q|^n-1)+C]}{n\log|q|}}.$$

Letting $\varepsilon \to 0$, from (26) we have

$$
(27)\quad T(\varrho, f) \geq T(r_0, f)\left(\frac{d}{2n\sigma}\right)^{\frac{\log \varrho+\log[(|q|^n-1)(1+o(1)]-n\log|q|-\log[|z_0|(|q|^n-1)+C]}{n\log|q|}} = K\left(\frac{d}{2n\sigma}\right)^{\frac{\log \varrho}{n\log|q|}},
$$

where K is a constant satisfying

$$
K = T(r_0, f)\left(\frac{d}{2n\sigma}\right)^{\frac{\log[(|q|^n-1)(1+o(1)]-n\log|q|-\log[|z_0|(|q|^n-1)+C]}{n\log|q|}} \quad (> 0).
$$

Hence, from (27) we get $\mu(f) \geq \log d - \log 2n\sigma/(n\log|q|)$ easily.

This completes the proof of Theorem 1.10.

5. Proof of Theorem 1.11. Suppose that the assertion $Q(z, f(z)) \equiv (f(z)+h(z))^t$ does not hold; we will prove that then $\rho(f) > 0$. By Lemmas 2.2 and 2.3, we have

$$
(28)\quad T(r,f) \leq \overline{N}\left(r, \frac{1}{Q}\right) + \overline{N}(r,f) + S(r,f) \leq \overline{N}\left(r, \frac{P(z,f)}{Q(z,f)}\right) + \overline{N}(r,f) + S(r,f).
$$

From (6), (7) and (28), we get

$$
\begin{aligned}
(29)\quad T(r,f) - \overline{N}(r,f) &\leq \overline{N}\left(r, \frac{P(z,f)}{Q(z,f)}\right) + S(r,f) \\
&= \overline{N}\Big(r, \sum_{j=1}^{n} a_j(z) f(q^j z + c_j)\Big) + S(r,f) \\
&\leq \alpha\overline{N}(|q|^n r + C, f) + S(r,f).
\end{aligned}
$$

From Lemmas 2.6 and 2.7, we have

$$
(30)\quad T(r, f(q^j z + c_j)) = T(r,f) + S(r,f), \quad j = 1, \ldots, n,
$$

on a set of logarithmic density 1.

If a set is of finite linear measure, then it is of logarithmic density 0. Thus, from (30), we have

$$
(31)\quad S(r, f(q^j z + c_j)) = o(T(r, f(z)), \quad j = 1, \ldots, n,
$$

on a set of logarithmic density 1. From (31), replacing $f(z)$ by $f(q^j z + c_j)$ $(j = 1, \ldots, n)$ in (29), we have

$$(32) \qquad T(r, f(q^j z + c_j)) - \overline{N}(r, f(q^j z + c_j)) \\ \leq \alpha \overline{N}(|q|^n r + C, f(q^j z + c_j)) + o(T(r, f))$$

on a set of logarithmic density 1. Applying Lemma 2.1 on both sides of (6), from (7), we have

$$\begin{aligned}(33) \qquad nT(r, f) &= T\Big(r, \sum_{j=1}^{n} a_j(z) f(q^j z + c_j)\Big) + o(T(r, f)) \\ &= T\Big(r, \sum_{j=1}^{n} a_j(z) f(q^j z + c_j)\Big) - \overline{N}\Big(r, \sum_{j=1}^{n} a_j(z) f(q^j z + c_j)\Big) \\ &\quad + \overline{N}\Big(r, \sum_{j=1}^{n} a_j(z) f(q^j z + c_j)\Big) + o(T(r, f)) \\ &\leq m\Big(r, \sum_{j=1}^{n} a_j(z) f(q^j z + c_j)\Big) + N_1\Big(r, \sum_{j=1}^{n} a_j(z) f(q^j z + c_j)\Big) \\ &\quad + \alpha \overline{N}(|q|^n r + C, f(z)) + o(T(r, f)) \\ &\leq \sum_{j=1}^{n} \big(m(r, f(q^j z + c_j)) + N_1(r, f(q^j z + c_j))\big) + \alpha \overline{N}(|q|^n r + C, f(z)) \\ &\quad + o(T(r, f)) \\ &= \sum_{j=1}^{n} \big(T(r, f(q^j z + c_j)) - \overline{N}(r, f(q^j z + c_j))\big) + \alpha \overline{N}(|q|^n r + C, f(z)) \\ &\quad + o(T(r, f))\end{aligned}$$

on a set of logarithmic density 1, where $N_1(r, f) = N(r, f) - \overline{N}(r, f)$. From (32) and (33), we have

$$\begin{aligned}(34) \qquad & nT(r, f) \\ &\leq \sum_{j=1}^{n} \alpha \overline{N}\big(|q|^n r + C, f(q^j z + c_j)\big) + \alpha \overline{N}(|q|^n r + C, f(z)) + o(T(r, f)) \\ &\leq n\alpha \overline{N}\big(|q|^{2n} r + C(1 + |q|^n), f(z)\big) + \alpha \overline{N}(|q|^n r + C, f(z)) + o(T(r, f)) \\ &\leq (n+1)\alpha \overline{N}\big(|q|^{2n} r + C(1 + |q|^n), f(z)\big) + o(T(r, f)) \\ &\leq (n+1)\alpha \overline{N}\big(|q|^{2n} r + 2C|q|^{2n}, f(z)\big) + o(T(r, f))\end{aligned}$$

on a set of logarithmic density 1. From (34), we have

$$(35) \qquad T(r, f) - \overline{N}(r, f) \leq \frac{n+1}{n} \alpha \overline{N}\big(|q|^{2n} r + 2C|q|^{2n}, f(z)\big) - \overline{N}(r, f) \\ + o(T(r, f))$$

on a set of logarithmic density 1. We now proceed inductively to prove

$$T(r,f)-\overline{N}(r,f)\le \frac{n+m}{n}\alpha\overline{N}(|q|^{2mn}r+2mC|q|^{2mn},f(z)) - m\overline{N}(r,f)+o(T(r,f)) \tag{36}$$

on a set of logarithmic density 1.

From (35), we can see that the case $m=1$ has already been proved. We now turn to the inductive step. Observe that the above reasoning also applies to the functions $f(q^jz+c_j)$, $j=1,\dots,n$, instead of $f(z)$. Therefore, since $|q|>1$, we may apply the inductive assertion to infer by (33) that

$$\begin{aligned} nT(r,f) &\le \sum_{j=1}^{n}\big(T(r,f(q^jz+c_j))-\overline{N}(r,f(q^jz+c_j))\big)+\alpha\overline{N}(|q|^nr+C,f(z)) \\ &\quad +o(T(r,f)) \\ &\le \sum_{j=1}^{n}\frac{n+m}{n}\alpha\overline{N}\big(|q|^{2mn}r+2mC|q|^{2mn},f(q^jz+c_j)\big) \\ &\quad -\sum_{j=1}^{n}m\overline{N}(r,f(q^jz+c_j))+\alpha\overline{N}(|q|^nr+C,f(z))+o(T(r,f)) \\ &\le \sum_{j=1}^{n}\frac{n+m}{n}\alpha\overline{N}\big(|q|^{(2m+1)n}r+(2m+1)C|q|^{2mn},f\big) \\ &\quad -\sum_{j=1}^{n}m\overline{N}(r-C,f)+\alpha\overline{N}(|q|^nr+C,f)+o(T(r,f)) \\ &\le (n+m+1)\alpha\overline{N}\big(|q|^{(2m+1)n}r+(2m+1)C|q|^{2mn},f\big) \\ &\quad -mn\overline{N}(r-C,f)+o(T(r,f)) \end{aligned} \tag{37}$$

on a set of logarithmic density 1.

Since $T(r,f)\le T(r+C,f)$, from (37), we have

$$\begin{aligned} nT(r,f) &\le nT(r+C,f) \\ &\le (n+m+1)\alpha\overline{N}(|q|^{(2m+1)n}r+C|q|^{2mn}[(2m+1)+|q|^n],f) \\ &\quad -mn\overline{N}(r,f)+o(T(r,f)) \\ &\le (n+m+1)\alpha\overline{N}(|q|^{2(m+1)n}r+2(m+1)C|q|^{2(m+1)n},f) \\ &\quad -mn\overline{N}(r,f)+o(T(r,f)) \end{aligned} \tag{38}$$

on a set of logarithmic density 1. From this inequality, we get

$$T(r,f)-\overline{N}(r,f)\le \frac{n+m+1}{n}\alpha\overline{N}\big(|q|^{2(m+1)n}r+2(m+1)C|q|^{2(m+1)n},f(z)\big) - (m+1)\overline{N}(r,f)+o(T(r,f))$$

on a set of logarithmic density 1. This completes the induction.

Thus, since $T(r,f) - o(T(r,f)) \geq 0$, from (36), we immediately get

$$\overline{N}(r,f) \leq \frac{n+m}{n(m-1)}\alpha\overline{N}(|q|^{2mn}r + 2mC|q|^{2mn}, f) \tag{39}$$
$$\leq \frac{n+m}{n(m-1)}\alpha\overline{N}(2m|q|^{2mn}r, f)$$

on a set of logarithmic density 1. Since $|q| > 1$ and $\alpha \in [0,n)$, we see that for sufficiently large m,

$$\frac{n+m}{n(m-1)}\alpha = \left(\frac{1}{m-1} + \frac{1}{n}\frac{m}{m-1}\right)\alpha < 1. \tag{40}$$

Then, from (39), (40) and Lemma 2.4, we get $\rho(f) > 0$, a contradiction.

Thus, the proof of Theorem 1.11 is complete.

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