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Uniqueness theorems for entire functions whose difference polynomials share a meromorphic function of a smaller order

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Abstract. We deal with uniqueness of entire functions whose difference polynomials share a nonzero polynomial CM, which corresponds to Theorem 2 of I. Laine and C. C. Yang [Proc. Japan Acad. Ser. A 83 (2007), 148–151] and Theorem 1.2 of K. Liu and L. Z. Yang [Arch. Math. 92 (2009), 270–278]. We also deal with uniqueness of entire functions whose difference polynomials share a meromorphic function of a smaller order, improving Theorem 5 of J. L. Zhang [J. Math. Anal. Appl. 367 (2010), 401–408], where the entire functions are of finite orders.

1. Introduction and main results. In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notation of the Nevanlinna theory of meromorphic functions as explained in [6], [12] and [19]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h)=o\{T(r,h)\}$ as $r\to\infty$ and $r\notin E$.

Let f and g be two nonconstant meromorphic functions, and let a be a value in the extended plane. We say that f and g *share the value a CM* provided that f and g have the same a-points with the same multiplicities. We say that f and g *share the value a IM* provided that f and g have the same a-points ignoring multiplicities (see [19]). We say that a is a *small function of f* if a is a meromorphic function satisfying $T(r,a)=S(r,f)$ (see [19]). Throughout this paper, we denote by $\rho(f)$ the order of f (see [6], [12] and [19]). We also need the following two definitions.

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Definition 1.1 (see [11, Definition 1]). Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. Then we denote by $N_{p)}(r, 1/(f-a))$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p, and by $\overline{N}_{p)}(r, 1/(f-a))$ the corresponding reduced counting function (ignoring multiplicities). Moreover we denote by $N_{(p}(r, 1/(f-a))$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not less than p, and by $\overline{N}_{(p}(r, 1/(f-a))$ the corresponding reduced counting function (ignoring multiplicities). Finally $N_{p)}(r, 1/(f-a))$, $\overline{N}_{p)}(r, 1/(f-a))$, $N_{(p}(r, 1/(f-a))$ and $\overline{N}_{(p}(r, 1/(f-a))$ mean $N_{p)}(r, f)$, $\overline{N}_{p)}(r, f)$, $N_{(p}(r, f)$ and $\overline{N}_{(p}(r, f)$ respectively if $a = \infty$.

Definition 1.2. Let a be any value in the extended complex plane, and let k be an arbitrary nonnegative integer. We define

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + \cdots + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right).$$

Much research has been devoted to uniqueness of meromorphic functions whose differential polynomials share one nonzero value (for example, see [3], [13], [17] and [18]). Recently the difference variant of Nevanlinna theory has been established in [1], [5] and [4], by Halburd–Korhonen and Chiang–Feng, independently. Using these theories, some Finnish and Chinese mathematicians began to consider uniqueness questions for meromorphic functions sharing values with their shifts (for example, see [9], [8] and [20]). In this paper, we will consider uniqueness of entire functions whose difference polynomials share one nonzero value or a small function of a smaller order.

We recall the following result, proved by Clunie and Hayman.

Theorem A (see [2] and [7]). *Let $f(z)$ be a transcendental entire function, and let $n \geq 1$ be a positive integer. Then $f(z)^n f'(z) - 1$ has infinitely many zeros.*

Regarding Theorem A, it is natural to ask the following question.

Question 1.1. What can be said if $f^n(z)f'(z)$ in Theorem A is replaced with $f^n(z)f(z+\eta)$ for a transcendental entire function $f(z)$ and a nonzero complex number η?

In 2007, Laine and Yang proved the following result.

Theorem B (see [14, Theorem 2]). *Let $f(z)$ be a transcendental entire function of a finite order, and let η be a nonzero complex number. Then $f(z)^n f(z+\eta)$ assumes every finite nonzero value a infinitely often for each $n \geq 2$.*

We recall the following two examples.

Example A (see [13]). Let $f(z) = 1+e^z$. Then $f(z)f(z+\pi i)-1 = -e^{2z}$ has no zeros. This shows that Theorem B does not remain valid if $n = 1$.

Example B (see [15]). Let $f(z) = e^{-e^z}$. Then $f(z)^2 f(z+\eta)-2 = -1$ and $\rho(f) = \infty$, where η is the nonzero constant satisfying $e^\eta = -2$. Evidently, $f(z)^2 f(z+\eta)-2$ has no zeros. This shows that Theorem B does not remain valid if f is of infinite order.

Recently K. Liu and L. Z. Yang proved the following result.

Theorem C (see [15]). *Let $f(z)$ be a transcendental entire function of finite order, let η be a nonzero complex number, and let $n \geq 2$ be an integer. Then $f(z)^n f(z+\eta) - P(z)$ has infinitely many zeros, where $P(z) \not\equiv 0$ is any polynomial.*

We recall the following example.

Example C (see [15]). Let $f(z) = e^{-e^z}$. Then $f(z)^n f(z+\eta) - P(z) = 1 - P(z)$ and $\rho(f) = \infty$, where η is a nonzero constant satisfying $e^\eta = -n$, $P(z)$ is a nonconstant polynomial, and n is a positive integer. Evidently, $f(z)^n f(z+\eta) - P(z)$ has finitely many zeros. This example shows that the condition "$\rho(f) < \infty$" in Theorem C is necessary.

Regarding Theorem C, it is natural to ask the following question.

Question 1.2. What can be said if $f(z)^n f(z+\eta) - P(z)$ and $g(z)^n g(z+\eta) - P(z)$ share 0 CM for two transcendental entire functions f, g and a polynomial $P \not\equiv 0$?

We will prove the following uniqueness theorem which deals with Question 1.2.

Theorem 1.1. *Let f and g be distinct transcendental entire functions of finite orders, and let $P \not\equiv 0$ be a polynomial. Suppose that η is a nonzero complex number and $n \geq 4$ is an integer such that $2\deg(P) < n+1$. Suppose that $f(z)^n f(z+\eta) - P(z)$ and $g(z)^n g(z+\eta) - P(z)$ share 0 CM.*

- (I) *If $n \geq 4$ and $f(z)^n f(z+\eta)/P(z)$ is a Möbius transformation of $g(z)^n g(z+\eta)/P(z)$, then either*
  - (i) *$f = tg$, where $t \neq 1$ is a constant satisfying $t^{n+1} = 1$, or*
  - (ii) *$f = e^Q$ and $g = te^{-Q}$, where P reduces to a nonzero constant c, t is a constant such that $t^{n+1} = c^2$, and Q is a nonconstant polynomial.*
- (II) *If $n \geq 6$, then* (I)(i) *or* (I)(ii) *holds.*

From Theorem 1.1 we get the following corollary.

Corollary 1.1. *Let f and g be distinct nonconstant entire functions of finite orders. Suppose that η is a nonzero complex number and $n \geq 6$ is* *an integer. If* $f(z)^n f(z+\eta) - z$ *and* $g(z)^n g(z+\eta) - z$ *share* 0 *CM, then* $f = tg$*, where* t *is a constant satisfying* $t^{n+1} = 1$ *and* $t \neq 1$.

Recently J. L. Zhang proved the following result.

THEOREM D (see [20]). *Let* f *and* g *be transcendental entire functions of finite orders, and let* α *be a small function relative to* f *and* g*. Suppose that* η *is a nonzero complex number and* $n \geq 7$ *is an integer. If* $f(z)^n(f(z)-1)f(z+\eta) - \alpha(z)$ *and* $g(z)^n(g(z)-1)g(z+\eta) - \alpha(z)$ *share* 0 *CM, then* $f = g$.

We will prove the following result, which improves Theorem D.

THEOREM 1.2. *Let* f *and* g *be transcendental entire functions of finite orders, and let* α *be a meromorphic function such that* $\rho(\alpha) < \rho(f)$ *and* $\alpha \not\equiv 0, \infty$*. Suppose that* η *is a nonzero complex number, and* n *and* m *are positive integers, where* $n \geq m+6$*. If* $f(z)^n(f(z)^m - 1)f(z+\eta) - \alpha(z)$ *and* $g(z)^n(g(z)^m - 1)g(z+\eta) - \alpha(z)$ *share* 0 *CM, then* $f = tg$*, where* t *is a constant satisfying* $t^m = 1$.

2. Some lemmas

LEMMA 2.1 (see [19, proof of Theorem 1.12]). *Let* f *be a nonconstant meromorphic function in the complex plane, and let*

$$P(f) = a_n f(z)^n + a_{n-1} f(z)^{n-1} + \cdots + a_1 f(z) + a_0, \tag{2.1}$$

where $a_0, a_1, \ldots, a_n$ *are constants and* $a_n \neq 0$*. Then*

$$m(r, P(f)) = nm(r, f) + O(1).$$

LEMMA 2.2 (see [1, Corollary 2.5]). *Let* $f(z)$ *be a meromorphic function of order* $\rho(f) < \infty$*, and let* η *be a nonzero complex number. Then*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O(r^{\rho(f)-1+\varepsilon});$$

here and in what follows, ε *is an arbitrary positive number.*

LEMMA 2.3. *Let* $f(z)$ *be a nonconstant meromorphic function of order* $\rho(f) < \infty$*, let* η *be a nonzero complex number, and let* $P(f)$ *be as in* (2.1)*. Suppose that* $F(z) = P(f(z))f(z+\eta)$*. Then*

$$m(r, F(z)) = (n+1)m(r, f(z)) + O(r^{\rho(f)-1+\varepsilon}) + O(\log r).$$

Proof. First of all, by Lemmas 2.1 and 2.2, the assumptions of Lemma 2.3 and the standard Valiron–Mokhon'ko lemma (see [16]) we get

$$\begin{aligned}(n+1)m(r,f(z)) &= m(r,f(z)P(f(z)))+O(1)\\ &\le m\left(r,\frac{f(z)P(f(z))}{F(z)}\right)+m(r,F(z))+O(1)\\ &= m\left(r,\frac{f(z)}{f(z+\eta)}\right)+m(r,F(z))+O(1)\\ &\le m(r,F(z))+O(r^{\rho-1+\varepsilon})+O(1),\end{aligned}$$

i.e.,

$$m(r,F(z)) \ge (n+1)m(r,f(z))+O(r^{\rho-1+\varepsilon})+O(1). \tag{2.2}$$

Next from Lemmas 2.1 and 2.2 and the standard Valiron–Mokhon'ko lemma we get

$$\begin{aligned}m(r,F(z)) &\le m(r,P(f(z)))+m\left(r,f(z)\frac{f(z+\eta)}{f(z)}\right)\\ &\le nm(r,f(z))+m(r,f(z))+m\left(r,\frac{f(z+\eta)}{f(z)}\right)+O(1)\\ &= (n+1)m(r,f(z))+O(r^{\rho-1+\varepsilon})+O(\log r).\end{aligned} \tag{2.3}$$

From (2.2) and (2.3) we get the conclusion of Lemma 2.3.

Lemma 2.4 (see [1, Theorem 2.1]). *Let $f(z)$ be a meromorphic function of order $\rho(f)<\infty$, and let η be a nonzero complex number. Then*

$$T(r,f(z+\eta)) = T(r,f(z))+O(r^{\rho(f)-1+\varepsilon})+O(\log r).$$

Lemma 2.5. *Let f and g be transcendental entire functions of finite orders, and let $P\not\equiv 0$ be a polynomial. Suppose that η is a nonzero complex number and $n\ge 2$ is an integer. If $f(z)^n f(z+\eta)-P(z)$ and $g(z)^n g(z+\eta) - P(z)$ share 0 CM, then $\rho(f)=\rho(g)$.*

Proof. Set

$$F(z)=\frac{f(z)^n f(z+\eta)}{P(z)},\qquad G(z)=\frac{g(z)^n g(z+\eta)}{P(z)} \tag{2.4}$$

for all $z\in\mathbb{C}$. First of all, from (2.4), Lemma 2.3 and the condition that f and g are entire functions we get

$$T(r,F(z)) = (n+1)T(r,f(z))+O(r^{\rho(f)-1+\varepsilon})+O(\log r), \tag{2.5}$$

$$T(r,G(z)) = (n+1)T(r,g(z))+O(r^{\rho(g)-1+\varepsilon})+O(\log r). \tag{2.6}$$

Since f, g are of finite orders, it follows from (2.5) and (2.6) that the same is true for F and G as well. Hence it follows from Lemma 2.4, the assumptions of Lemma 2.5 and the second fundamental theorem that

$$\begin{aligned} T(r, F(z)) &\le \overline{N}(r, F(z)) + \overline{N}\left(r, \frac{1}{F(z)}\right) + \overline{N}\left(r, \frac{1}{F(z)-1}\right) + O(\log r) \\ &\le \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z+\eta)}\right) + \overline{N}\left(r, \frac{1}{G(z)-1}\right) \\ &\quad + O(\log r) \\ &\le 2T(r, f(z)) + T(r, G(z)) + O(r^{\rho(f)-1+\varepsilon}) + O(\log r), \end{aligned}$$

which together with (2.5) and (2.6) gives

$$\begin{aligned} (n+1)T(r, f(z)) \le 2T(r, f(z)) + (n+1)T(r, g(z)) + O(r^{\rho(f)-1+\varepsilon}) \\ + O(r^{\rho(g)-1+\varepsilon}) + O(\log r), \end{aligned}$$

i.e.,

$$\begin{aligned} (n-1)T(r, f(z)) \le (n+1)T(r, g(z)) + O(r^{\rho(f)-1+\varepsilon}) \\ + O(r^{\rho(g)-1+\varepsilon}) + O(\log r). \end{aligned} \tag{2.7}$$

From (2.7) and $n \ge 2$ we get

$$\rho(f) \le \rho(g). \tag{2.8}$$

Similarly

$$\rho(g) \le \rho(f). \tag{2.9}$$

Thus $\rho(f) = \rho(g)$, proving Lemma 2.5.

LEMMA 2.6 (see [10, Lemma 2.2]). *Let $\varphi(r)$ be a nondecreasing, continuous function on $\mathbb{R}^+$. Suppose that*

$$0 < \rho < \limsup_{r\to\infty} \frac{\log \varphi(r)}{\log r},$$

and set

$$G := \{r \in \mathbb{R}^+ \mid \varphi(r) \ge r^\rho\}.$$

Then

$$\overline{\log \operatorname{dens} G} = \limsup_{r\to\infty} \frac{\int_{G\cap[1,r]} \frac{dr}{r}}{\log r} > 0.$$

LEMMA 2.7 (see [19, Lemma 7.1]). *Let F and G be nonconstant meromorphic functions such that G is a Möbius transformation of F. Suppose that there exists a subset $I \subset \mathbb{R}^+$ with linear measure* $\operatorname{mes} I = +\infty$ *such that*

$$\overline{N}(r, 1/F) + \overline{N}(r, F) + \overline{N}(r, 1/G) + \overline{N}(r, G) < (\lambda + o(1))T(r, F)$$

as $r \in I$ and $r \to \infty$, where $\lambda < 1$. If there exists a point $z_0 \in \mathbb{C}$ such that $F(z_0) = G(z_0) = 1$, then $F = G$ or $FG = 1$.

Let F and G be nonconstant meromorphic functions, let $a \in \mathbb{C} \cup \{\infty\}$, and let $\overline{N}_E(r,a)$ "count" those points in $\overline{N}(r, 1/(F-a))$, where a is taken by F and G with the same multiplicity, and each point is counted only once; $\overline{N}(r, 1/(F-\infty))$ means $\overline{N}(r,f)$. We say that F and G *share the value a CM** if

$$\overline{N}\left(r, \frac{1}{F-a}\right) - \overline{N}_E(r,a) = S(r,F),$$
$$\overline{N}\left(r, \frac{1}{G-a}\right) - \overline{N}_E(r,a) = S(r,G).$$

LEMMA 2.8 (see [19, proof of Theorems 1.48 and 7.10]). *Let F and G be nonconstant meromorphic functions that share* $1, \infty$ *CM*. Suppose that there exists a subset $I \subset \mathbb{R}^+$ with* $\operatorname{mes} I = +\infty$ *such that*

$$N_2(r, 1/F) + N_2(r, 1/G) + 2\overline{N}(r,F) < \lambda T(r) + S(r)$$

as $r \to \infty$ and $r \in I$, where $\lambda < 1$, $T(r) = \max\{T(r,F), T(r,G)\}$ and $S(r) = o\{T(r)\}$. Then $F = G$ or $FG = 1$.

3. Proofs of theorems

Proof of Theorem 1.1. First of all, we define F and G by (2.4). From (2.4), Lemma 2.3 and the assumptions of Theorem 1.1 we get (2.5) and (2.6). Suppose that $z_0 \in \mathbb{C}$ is a zero of $F-1$ of multiplicity μ. Then, since $P \not\equiv 0$ is a polynomial, we can see that z_0 is a zero of $f(z)^n f(z+\eta) - P(z)$ of multiplicity $\mu + \nu$, where $\nu \geq 0$ is the multiplicity of z_0 as a zero of $P(z)$. Hence z_0 is a zero of $g(z)^n g(z+\eta) - P(z)$ of multiplicity $\mu+\nu$ by the value sharing assumption. Now (2.4) shows that z_0 is a zero of $G-1$ of multiplicity μ. This also works in the other direction. Therefore, F and G indeed share 1 CM. Since f, g are of finite orders, it follows from (2.5) and (2.6) that so are F and G. We discuss the following two cases.

CASE 1. Suppose that F is a Möbius transformation of G. Then it follows from (2.4) and the standard Valiron–Mokhon'ko lemma that

$$T(r, f(z)^n f(z+\eta)) = T(r, g(z)^n g(z+\eta)) + O(\log r). \tag{3.1}$$

From (2.5), (2.6), Lemmas 2.5, 2.6 and the condition that f, g are transcendental entire functions we deduce that there exists a subset $I \subset \mathbb{R}^+$ with $\operatorname{mes} I = +\infty$ such that $T(r,f) \geq r^{\rho(f)-1+2\varepsilon}$ and $T(r,g) \geq r^{\rho(g)-1+2\varepsilon}$ as $r \to \infty$ and $r \in I$, and moreover

$$\lim_{\substack{r\to\infty \\ r\in I}} \frac{T(r,f)}{T(r,g)} = 1, \qquad \lim_{\substack{r\to\infty \\ r\in I}} \frac{T(r,F)}{T(r,f)} = n+1. \tag{3.2}$$

From Lemma 2.4, the left equality of (2.4) and the condition that f, g are transcendental entire functions we get

$$(3.3)\quad \overline{N}(r,F(z))+\overline{N}\left(r,\frac{1}{F(z)}\right)\le \overline{N}\left(r,\frac{1}{f(z)}\right)+\overline{N}\left(r,\frac{1}{f(z+\eta)}\right) + O(\log r)$$
$$\le T(r,f(z))+T(r,f(z+\eta))+O(\log r)$$
$$\le 2T(r,f(z))+O(r^{\rho(f)-1+\varepsilon})+O(\log r)$$

as $r\to\infty$. Similarly

$$(3.4)\quad \overline{N}(r,G(z))+\overline{N}\left(r,\frac{1}{G(z)}\right)\le 2T(r,g(z))+O(r^{\rho(g)-1+\varepsilon})+O(\log r)$$

as $r\to\infty$. By the property of I introduced in (3.2) we know that

$$r^{\rho(f)-1+\varepsilon}+\log r=r^{\rho(g)-1+\varepsilon}+\log r=o\{T(r,f)\}$$

as $r\to\infty$ and $r\in I$. This together with (3.2)–(3.4) gives

$$(3.5)\ \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)\le \frac{4}{n+1}T(r,F)(1+o(1))$$

as $r\to\infty$ and $r\in I$. From (2.4) and the second fundamental theorem,

$$T(r,F(z))\le \overline{N}(r,F(z))+\overline{N}\left(r,\frac{1}{F(z)}\right)+\overline{N}\left(r,\frac{1}{F(z)-1}\right)+O(\log r)$$
$$\le \overline{N}\left(r,\frac{1}{f(z)}\right)+\overline{N}\left(r,\frac{1}{f(z+\eta)}\right)+\overline{N}\left(r,\frac{1}{F(z)-1}\right) + O(\log r)$$
$$\le 2T(r,f(z))+\overline{N}\left(r,\frac{1}{F(z)-1}\right)+O(r^{\rho(f)-1+\varepsilon})+O(\log r),$$

which together with (2.5) and Lemma 2.6 implies that there exists a subset $I\subset\mathbb{R}^+$ with $\operatorname{mes} I=+\infty$ such that

$$(3.6)\qquad (n-1)T(r,f)\le \overline{N}\left(r,\frac{1}{F-1}\right)+o\{T(r,f)\}$$

as $r\to\infty$ and $r\in I$. From (3.6) and the fact that F, G share 1 CM* we know that there exists $z_0\in\mathbb{C}$ such that $F(z_0)=G(z_0)=1$. Hence from (3.5), Lemma 2.7 and the condition $n\ge 4$ we get $FG=1$ or $F=G$. We discuss the following two subcases.

SUBCASE 1.1. Suppose that $F=G$. Then it follows from (2.4) that

$$(3.7)\qquad f(z)^nf(z+\eta)=g(z)^ng(z+\eta)$$

for all $z\in\mathbb{C}$. Let

$$(3.8)\qquad h=f/g.$$

From (3.7) and (3.8) we get

$$(3.9)\qquad h(z)^nh(z+\eta)=1$$

for all $z \in \mathbb{C}$. First suppose that h is rational. If h has a zero at some point z_0, then h has a pole at $z_0 + \eta$ by (3.9). Continuing, $h(z_0 + 2\eta) = 0$, $h(z_0 + 3\eta) = \infty$, and so on. Therefore, h would have infinitely many zeros and poles, which is impossible. Hence, h has neither zeros nor poles, meaning that it is a constant, say $h = t$. By (3.9), $t^{n+1} = 1$. This together with (3.8) gives the conclusion (I)(i) of Theorem 1.1.

Next suppose that h is transcendental meromorphic. Since f, g are of finite order, the same is true for h as well. Thus it follows from (3.9) and Lemma 2.4 that

$$nT(r,h) = T(r,h) + O(r^{\rho(h)-1+\varepsilon}) + O(\log r),$$

and so

$$(n-1)T(r,h(z)) = O(r^{\rho(h)-1+\varepsilon}) + O(\log r), \tag{3.10}$$

as $r \to \infty$. From (3.10) and the condition $n \geq 4$, we get $\rho(h) \leq \rho(h) - 1$, a contradiction.

SUBCASE 1.2. By substituting (2.4) into $FG = 1$ we get

$$f(z)^n f(z+\eta) g(z)^n g(z+\eta) = P(z)^2 \tag{3.11}$$

for all $z \in \mathbb{C}$. From (3.11) and the condition that f, g are transcendental entire functions, one can immediately see that f, g each have at most finitely many zeros, and so we may write

$$f = Se^U, \quad g = Te^V, \tag{3.12}$$

where S, T, U, V are polynomials, and U, V are nonconstant. Substituting (3.12) into (3.11) we obtain

$$S^n(z)S(z+\eta)T^n(z)T(z+\eta)e^{nU(z)+U(z+\eta)+nV(z)+V(z+\eta)} = P(z)^2 \tag{3.13}$$

for all $z \in \mathbb{C}$. To avoid a contradiction, from (3.13) we must have

$$nU(z) + U(z+\eta) + nV(z) + V(z+\eta) = A \tag{3.14}$$

for all $z \in \mathbb{C}$, where A is a constant. Let

$$U + V = W. \tag{3.15}$$

Then it follows from (3.15) that (3.14) can be rewritten as

$$nW(z) + W(z+\eta) = A \tag{3.16}$$

for all $z \in \mathbb{C}$. From (3.16) we know that $W = B$, where B is a constant. This together with (3.15) gives

$$V = B - U. \tag{3.17}$$

From (3.12) and (3.17) we conclude that $f = Se^U$, $g = Te^Be^{-U}$. Now (3.13) can be rewritten as

$$\{S(z)T(z)\}^n\{S(z+\eta)T(z+\eta)\} = e^A P(z)^2 \tag{3.18}$$

for all $z \in \mathbb{C}$. If ST is not a constant, then the degree of the left side of (3.18) is not less than $n+1$. But the condition $2\deg(P) < n+1$ implies that the degree of the right side of (3.18) is less than $n+1$, which is a contradiction. Hence ST and P reduce to nonzero constants, say $ST = t$ and $P = c$. The assertion (I)(ii) of Theorem 1.1 now follows from (3.12).

Case 2. Suppose that $n \geq 6$. From (2.4), Lemma 2.4 and the assumptions of Theorem 1.1 we get

$$\begin{aligned}(3.19)\quad 2\overline{N}(r, F(z)) + N_2\left(r, \frac{1}{F(z)}\right) &\leq 2\overline{N}\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{f(z+\eta)}\right) \\ &\quad + O(\log r) \\ &\leq 3T(r, f(z)) + O(r^{\rho(f)-1+\varepsilon}) + O(\log r) \\ &= 3T(r, F(z))/(n+1) + O(r^{\rho(f)-1+\varepsilon}) \\ &\quad + O(\log r)\end{aligned}$$

and

$$(3.20)\qquad N_2\left(r, \frac{1}{G(z)}\right) \leq \frac{3}{n+1} T(r, G(z)) + O(r^{\rho(g)-1+\varepsilon}) + O(\log r)$$

as $r \to \infty$. From (3.19), (3.20) and Lemmas 2.5 and 2.6 we know that there exists a subset $I \subset \mathbb{R}^+$ with $\operatorname{mes} I = +\infty$ such that

$$(3.21)\qquad N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 2\overline{N}(r, F) \leq \frac{6}{n+1} T(r) + o\{T(r)\}$$

as $r \to \infty$ and $r \in I$, where $T(r) = \max\{T(r, F), T(r, G)\}$. From (3.21), Lemma 2.8 and the condition $n \geq 6$ we have $FG = 1$ or $F = G$. Next in the same manner as in Subcases 1.1 and 1.2 we get the conclusion (II) of Theorem 1.1. This completes the proof of Theorem 1.1.

Proof of Corollary 1.1. We discuss the following three cases.

Case 1. Suppose that one of f and g is a polynomial, and the other is a transcendental entire function. Without loss of generality, we suppose that f is transcendental and g is a polynomial. Then, on the one hand, Theorem C shows that $f(z)^n f(z+\eta) - z$ has infinitely many zeros in $\mathbb{C}$. On the other hand, as g is a polynomial, so is $g(z)^n g(z+\eta) - z$, and hence it has at most finitely many zeros in $\mathbb{C}$, contrary to the assumption that $f(z)^n f(z+\eta) - z$ and $g(z)^n g(z+\eta) - z$ share 0 CM.

Case 2. Suppose f and g are transcendental entire functions. Then Theorem 1.1 and the assumptions of Corollary 1.1 yield the desired conclusion.

Case 3. Suppose that f and g are nonconstant polynomials. Then

$$(3.22)\qquad f(z)^n f(z+\eta) - z = c\{g(z)^n g(z+\eta) - z\}$$

for all $z \in \mathbb{C}$, where c is some nonzero complex number.

If $c = 1$, then $f(z)^n f(z+\eta) = g(z)^n g(z+\eta)$ for all $z \in \mathbb{C}$; then in the same manner as in Subcase 1.1 of the proof of Theorem 1.1 we get the conclusion.

If $c \neq 1$, then (3.22) can be rewritten as

$$f(z)^n f(z+\eta) - cg(z)^n g(z+\eta) = (1-c)z \tag{3.23}$$

for all $z \in \mathbb{C}$. From (3.23) and the condition $n \geq 6$ we can deduce that $\deg(f) = \deg(g)$. Let

$$f(z) = a_0 z^m + a_1 z^{m-1} + \cdots + a_{m-1} z + a_m, \tag{3.24}$$

$$g(z) = b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m, \tag{3.25}$$

where a_j $(0 \leq j \leq m)$ and b_k $(0 \leq k \leq m)$ are complex numbers, $a_0 \neq 0$ and $b_0 \neq 0$, and

$$\deg(f) = \deg(g) = m. \tag{3.26}$$

By (3.24)–(3.26) and the standard Valiron–Mokhon'ko lemma we have

$$T(r, f(z)) = T(r, g(z)) + O(1) = m \log r + O(1) = \deg(f) \log r + O(1). \tag{3.27}$$

By rewriting (3.23) we get

$$F_1(z) + 1 = G_1(z), \tag{3.28}$$

where

$$F_1(z) = \frac{f(z)^n f(z+\eta)}{(c-1)z}, \qquad G_1(z) = \frac{cg^n(z) g(z+\eta)}{(c-1)z}. \tag{3.29}$$

By (3.29), the condition $n \geq 6$ and the standard Valiron–Mokhon'ko lemma we can deduce that $F_1(z)$ is not a constant. Therefore, from (3.27)–(3.29) and the second fundamental theorem we get

$$\begin{aligned}
n \deg(f) \log r &\leq T(r, F_1(z)) + O(1) \\
&\leq \overline{N}(r, F_1(z)) + \overline{N}\left(r, \frac{1}{F_1(z)}\right) + \overline{N}\left(r, \frac{1}{F_1(z)+1}\right) + O(1) \\
&\leq \overline{N}\left(r, \frac{1}{(c-1)z}\right) + \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z+\eta)}\right) \\
&\quad + \overline{N}\left(r, \frac{1}{G_1}\right) + O(1) \\
&\leq \overline{N}\left(r, \frac{1}{(c-1)z}\right) + \overline{N}\left(r, \frac{1}{f(z)}\right) + \overline{N}\left(r, \frac{1}{f(z+\eta)}\right) \\
&\quad + \overline{N}\left(r, \frac{1}{g(z)}\right) + \overline{N}\left(r, \frac{1}{g(z+\eta)}\right) + O(1) \\
&\leq [4 \deg(f) + 1] \log r + O(1)
\end{aligned}$$

as $r \to \infty$, i.e.,

$$[(n-4)\deg(f)-1]\log r = O(1). \tag{3.30}$$

Since $\deg(f) \geq 1$ and $n \geq 6$, this yields a contradiction.

Corollary 1.1 is thus completely proved.

Proof of Theorem 1.2. First of all, we set

$$\begin{aligned} F(z) &= \frac{f(z)^n(f(z)^m-1)f(z+\eta)}{\alpha(z)}, \\ G(z) &= \frac{g(z)^n(g(z)^m-1)g(z+\eta)}{\alpha(z)} \end{aligned} \tag{3.31}$$

for all $z \in \mathbb{C}$. From Lemma 2.3 and the condition that $\rho(\alpha) < \rho(f)$ we get

$$T(r,F) = (n+m+1)T(r,f) + O(r^{\rho(f)-1+\varepsilon}) + O(r^{\rho(\alpha)+\varepsilon}), \tag{3.32}$$

$$T(r,G) = (n+m+1)T(r,g) + O(r^{\rho(g)-1+\varepsilon}) + O(r^{\rho(\alpha)+\varepsilon}). \tag{3.33}$$

From (3.32) and (3.33) we get

$$\rho(F) \leq \max\{\rho(f),\rho(\alpha)\}, \quad \rho(f) \leq \max\{\rho(F),\rho(\alpha)\}, \tag{3.34}$$

$$\rho(G) \leq \max\{\rho(g),\rho(\alpha)\}, \quad \rho(g) \leq \max\{\rho(G),\rho(\alpha)\}. \tag{3.35}$$

From (3.34) and $\rho(\alpha) < \rho(f)$ we have

$$\rho(F) = \rho(f). \tag{3.36}$$

By Lemma 2.4, the condition $\rho(\alpha) < \rho(f)$ and the standard Valiron–Mokhon'ko lemma we can deduce that F is not a constant. Proceeding as at the beginning of the proof of Theorem 1.1, we can deduce from (3.31) and the assumptions of Theorem 1.2 that F and G share 1 CM. This together with the second fundamental theorem gives

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-1}\right) + O(\log r) \\ &\leq \overline{N}\left(r,\frac{1}{f(z)}\right) + \overline{N}\left(r,\frac{1}{f^m(z)-1}\right) \\ &\quad + \overline{N}\left(r,\frac{1}{f(z+\eta)}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) + O(r^{\rho(\alpha)+\varepsilon}) + O(\log r) \\ &\leq (m+2)T(r,f) + T(r,G) + O(r^{\rho(f)-1+\varepsilon}) + O(r^{\rho(\alpha)+\varepsilon}) + O(\log r), \end{aligned}$$

i.e.,

$$\begin{aligned} T(r,F) \leq{} & (m+2)T(r,f) + T(r,G) \\ & + O(r^{\rho(f)-1+\varepsilon}) + O(r^{\rho(\alpha)+\varepsilon}) + O(\log r). \end{aligned} \tag{3.37}$$

Similarly

$$\begin{aligned} T(r,G) \leq{} & (m+2)T(r,g) + T(r,F) \\ & + O(r^{\rho(g)-1+\varepsilon}) + O(r^{\rho(\alpha)+\varepsilon}) + O(\log r). \end{aligned} \tag{3.38}$$

From (3.32), (3.36), (3.37) and the conditions $n \geq m+6$ and $\rho(\alpha) < \rho(f) < \infty$ we get

$$\rho(F) \leq \rho(G). \tag{3.39}$$

From (3.35), (3.36), (3.39) and the condition $\rho(\alpha) < \rho(f) < \infty$ we get

$$\rho(G) = \rho(g). \tag{3.40}$$

From (3.33), (3.36), (3.38)–(3.40) and the condition $\rho(\alpha) < \rho(f) < \infty$ we get

$$\rho(G) \leq \rho(F). \tag{3.41}$$

From (3.36) and (3.39)–(3.41) we get

$$\rho(f) = \rho(g) = \rho(F) = \rho(G). \tag{3.42}$$

From (3.31)–(3.33), (3.37)-(3.38), Lemma 2.6, the condition $\rho(\alpha) < \rho(f) < \infty$ and the assumptions of Theorem 1.2 we know that there exists a subset $I \subseteq \mathbb{R}^+$ with $\operatorname{mes} I = \infty$ such that

$$O(r^{\rho(\alpha)+\varepsilon}) + O(r^{\rho(f)-1+\varepsilon}) + O(r^{\rho(g)-1+\varepsilon}) = o\{T(r,f)\}, \tag{3.43}$$

$$O(r^{\rho(\alpha)+\varepsilon}) + O(r^{\rho(f)-1+\varepsilon}) + O(r^{\rho(g)-1+\varepsilon}) = o\{T(r,g)\}, \tag{3.44}$$

$$\overline{N}\left(r, \frac{1}{F-1}\right) - \overline{N}_E(r,1) = 0, \tag{3.45}$$

$$\overline{N}\left(r, \frac{1}{G-1}\right) - \overline{N}_E(r,1) = 0, \tag{3.46}$$

as $r \to \infty$ and $r \in I$, and such that

$$\begin{aligned} N_2\left(r, \frac{1}{F}\right) &+ 2\overline{N}(r,F) \\ &\leq 2\overline{N}\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{f(z)^m - 1}\right) \\ &\quad + N\left(r, \frac{1}{f(z+\eta)}\right) + o\{T(r,f)\} \\ &\leq (m+2)T(r,f(z)) + T(r,f(z+\eta)) + o\{T(r,f)\} \\ &= (m+3)T(r,f(z)) + O(r^{\rho(f)-1+\varepsilon}) + o\{T(r,f)\} \\ &= (m+3)T(r,f(z)) + o\{T(r,f)\} \\ &= \frac{m+3}{m+n+1}T(r,F(z)) + o\{T(r,F(z))\} \end{aligned} \tag{3.47}$$

and

$$N_2\left(r, \frac{1}{G}\right) \leq \frac{m+3}{m+n+1}T(r,G) + o\{T(r,G)\} \tag{3.48}$$

as $r\to\infty$ and $r\in I$. From (3.47) and (3.48) we get

$$N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+2\overline{N}(r,F)\le\frac{2m+6}{m+n+1}T(r)+o\{T(r)\} \tag{3.49}$$

as $r\to\infty$ and $r\in I$, where $T(r)=\max\{T(r,F),T(r,G)\}$. From (3.49), Lemma 2.8 and the condition $n\ge m+6$ we have $F=G$ or $FG=1$. We discuss the following two cases.

Case 1. Suppose that $F=G$. Then it follows from (3.31) that

$$f(z)^n(f(z)^m-1)f(z+\eta)=g(z)^n(g(z)^m-1)g(z+\eta) \tag{3.50}$$

for all $z\in\mathbb{C}$. Let h be as in (3.8). From (3.8) and (3.50) we get

$$\{h(z)^{m+n}h(z+\eta)-1\}g(z)^m=h(z)^nh(z+\eta)-1 \tag{3.51}$$

for all $z\in\mathbb{C}$. First suppose that h is rational. If $h(z)^{m+n}h(z+\eta)-1\not\equiv 0$, then (3.51) can be rewritten as

$$g(z)^m=\frac{h(z)^nh(z+\eta)-1}{h(z)^{m+n}h(z+\eta)-1} \tag{3.52}$$

for all $z\in\mathbb{C}$. From (3.52) and the above supposition we know that g is a polynomial, which is impossible. Hence $h(z)^{m+n}h(z+\eta)-1\equiv 0$; this together with (3.51) gives $h(z)^nh(z+\eta)-1\equiv 0$, and so $h^m=1$, which by (3.8) yields the conclusion of Theorem 1.2.

Next suppose that h is transcendental meromorphic. Since f, g are of finite order, the same is true for h as well. If $h(z)^{m+n}h(z+\eta)-1\equiv 0$, then from Lemma 2.4, Lemma 2.6 and the standard Valiron–Mokhon'ko lemma we get

$$(m+n)T(r,h(z))=T(r,h(z))+S(r,h) \tag{3.53}$$

as $r\to\infty$ and $r\in I$, where $I\subset\mathbb{R}^+$ is a subset with $\operatorname{mes} I=\infty$. From (3.53) we have $T(r,h)=S(r,h)$ as $r\to\infty$ and $r\in I$, and so h is a constant, which is impossible. Thus $h(z)^{m+n}h(z+\eta)-1\not\equiv 0$, and so (3.51) can be rewritten as (3.52). Set

$$H(z)=h(z)^{m+n}h(z+\eta) \tag{3.54}$$

for all $z\in\mathbb{C}$. From (3.52) and the condition that g is an entire function we know that $h(z)^{m+n}h(z+\eta)-1=0$ implies $h(z)^nh(z+\eta)-1=0$, and so $h(z)^m=1$. Since h is of finite order, it follows from Lemma 2.4 that the same is true for H as well. Hence from (3.54), Lemma 2.4 and the second fundamental theorem we get

$$T(r,H) \le \overline{N}(r,H) + \overline{N}\Big(r,\frac{1}{H}\Big) + \overline{N}\Big(r,\frac{1}{H-1}\Big) + O(\log r) \tag{3.55}$$
$$\le \overline{N}(r,h(z)) + \overline{N}(r,h(z+\eta)) + \overline{N}\Big(r,\frac{1}{h(z)}\Big)$$
$$+ \overline{N}\Big(r,\frac{1}{h(z+\eta)}\Big) + \overline{N}\Big(r,\frac{1}{h(z)^m-1}\Big) + O(\log r)$$
$$\le (m+4)T(r,h(z)) + O(r^{\rho(h)-1+\varepsilon}) + O(\log r)$$

as $r \to \infty$. From Lemma 2.4 and the standard Valiron–Mokhon'ko lemma,

$$(m+n+1)T(r,h(z)) = T(r,h(z)^{m+n+1}) + O(1)$$
$$\le T(r,H(z)) + T\Big(r,\frac{h(z)^{m+n+1}}{H(z)}\Big) + O(1)$$
$$= T(r,H(z)) + T\Big(r,\frac{h(z)}{h(z+\eta)}\Big) + O(1)$$
$$\le T(r,H(z)) + 2T(r,h(z)) + O(r^{\rho(h)-1+\varepsilon}) + O(\log r)$$

as $r \to \infty$, which together with (3.55) gives

$$(n-5)T(r,h) \le O(r^{\rho(h)-1+\varepsilon}) + O(\log r) \tag{3.56}$$

as $r \to \infty$. From (3.56) and the condition $n \ge m+6$ we get $\rho(h) \le \rho(h)-1$, which is impossible.

Case 2. Suppose that $FG = 1$ and $F \not\equiv G$. Then it follows from (3.31) that

$$f(z)^n(f(z)^m-1)f(z+\eta)g(z)^n(g(z)^m-1)g(z+\eta) = \alpha(z)^2 \tag{3.57}$$

for all $z \in \mathbb{C}$. From the condition $\rho(\alpha) < \rho(f)$ and Lemma 2.6 we know that there exists a subset $I \subseteq \mathbb{R}^+$ with $\operatorname{mes} I = \infty$ such that

$$T(r,\alpha) = o\{T(r,f)\} \tag{3.58}$$

as $r \to \infty$ and $r \in I$. By rewriting (3.57) we have

$$f(z)^n(f(z)^m-1)f(z+\eta) = \frac{\alpha(z)^2}{g(z)^n(g(z)^m-1)g(z+\eta)} \tag{3.59}$$

for all $z \in \mathbb{C}$. Since f, g are entire functions, from (3.58) and (3.59) we get

$$\overline{N}\Big(r,\frac{1}{f}\Big) + \overline{N}\Big(r,\frac{1}{f^m-1}\Big) = \overline{N}\Big(r,\frac{1}{f}\Big) + \sum_{j=1}^{m}\overline{N}\Big(r,\frac{1}{f-\omega_j}\Big) \tag{3.60}$$
$$\le 2\overline{N}\Big(r,\frac{1}{\alpha}\Big) = o\{T(r,f)\}$$

as $r \to \infty$ and $r \in I$, where ω_j's stand for the roots of $\omega^m = 1$. From (3.60) and the second fundamental theorem we have

$$mT(r,f) \leq \overline{N}\left(r,\frac{1}{f}\right) + \sum_{j=1}^{m} \overline{N}\left(r,\frac{1}{f-\omega_j}\right) + O(\log r) = o\{T(r,f)\}$$

as $r \to \infty$ and $r \in I$, which is impossible.

Theorem 1.2 is thus completely proved.

4. Concluding remarks. Now we give the following example.

Example 4.1. Let $f(z) = e^z$ and $g(z) = e^{-z}$. Then $f(z)^j f(z+\pi i) = -e^{(j+1)z}$ and $g(z)^j g(z+\pi i) = -e^{-(j+1)z}$ for $1 \leq j \leq 5$, and $\rho(f) = \rho(g) = 1$. Moreover, we can verify that $f(z)^j f(z+\pi i)$ and $g(z)^j g(z+\pi i)$ share 1 CM.

From Example 4.1 we know that Theorem 1.1 holds possibly for $1 \leq n \leq 5$, so we give the following conjecture.

Conjecture 4.1. The conclusion (I) of Theorem 1.1 still holds for $1 \leq n \leq 3$, and the conclusion (II) of Theorem 1.1 still holds for $1 \leq n \leq 5$.

Regarding Theorem 1.2, we pose the following question.

Question 4.1. What can be said if the condition "$n \geq m+6$" in Theorem 1.2 is replaced with "$1 \leq n \leq m+5$"?

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