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Value distribution and uniqueness of difference polynomials and entire solutions of difference equations

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Abstract. This paper is devoted to value distribution and uniqueness problems for difference polynomials of entire functions such as $f^n(f-1)f(z+c)$. We also consider sharing value problems for $f(z)$ and its shifts $f(z+c)$, and improve some recent results of Heittokangas et al. [J. Math. Anal. Appl. 355 (2009), 352–363]. Finally, we obtain some results on the existence of entire solutions of a difference equation of the form $f^n + P(z)(\Delta_c f)^m = Q(z)$.

1. Introduction and main results. A meromorphic function means meromorphic in the whole complex plane. We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna theory [8, 16]. As usual, the abbreviation CM stands for "counting multiplicities", while IM means "ignoring multiplicities". For f meromorphic in $\mathbb{C}$, denote by $S(f)$ the family of all meromorphic functions $a(z)$ that satisfy $T(r,a) = o(T(r,f))$ for $r \to \infty$ outside a possible exceptional set of finite logarithmic measure. In addition, we define difference operators by $\Delta_c f = f(z+c) - f(z)$ where c is a non-zero constant. If $c = 1$, we use the usual difference notation $\Delta_c f = \Delta f$.

Let f be a transcendental meromorphic function, and let n be a positive integer. Concerning the value distribution of $f^n f'$, Hayman [6, Corollary to Theorem 9] proved that $f^n f'$ takes every non-zero complex value infinitely often if $n \geq 3$. Mues [14, Satz 3] proved that $f^2 f' - 1$ has infinitely many zeros. Later on, Bergweiler and Eremenko [1, Theorem 2] showed that $ff' - 1$ has infinitely many zeros as well. Corresponding to these results, Fang [3] considered the number of zeros of $(f^n(f-1))^{(k)} - 1$:

Theorem A ([3, Proposition 1]). *Let f be a transcendental entire function, and let n, k be positive integers with $n \geq k+2$. Then $(f^n(f-1))^{(k)} - 1$ has infinitely many zeros.*

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Corresponding to the value distribution of $f^n f'$, Laine and Yang [9] investigated the value distribution of difference products of entire functions, and obtained the following:

Theorem B ([9, Theorem 2]). *Let f be a transcendental entire function of finite order, and let c be a non-zero complex constant. Then for $n \geq 2$, $f(z)^n f(z+c)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*

Some improvements of Theorem B can be found in [13]. In the present paper, we consider the value distribution of $f(z)^n(f(z)-1)f(z+c)$, which can be seen as a difference analogue of Theorem A in the case $k=1$.

Theorem 1. *Let f be a transcendental meromorphic function of finite order $\sigma(f)$, let $a \neq 0$ be a small function with respect to f, and let c be a non-zero complex constant. If the exponent of convergence of the poles of f satisfies $\lambda(1/f) < \sigma(f)$ and $n \geq 2$, then $f(z)^n(f(z)-1)f(z+c)-a$ has infinitely many zeros.*

Corollary 1. *Let f be a transcendental entire function of finite order, and let c be a non-zero complex constant. Then for $n \geq 2$, $f(z)^n(f(z)-1) \cdot f(z+c)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*

Remark. The restriction on the order in Theorem 1 cannot be deleted. This can be seen by taking $f(z)=e^{e^z}$, $e^c=-n$ $(n \geq 2)$ and $a=-1$. Then f is of infinite order, while $f(z)^n(f(z)-1)f(z+c)+1=e^{e^z}$ has no zeros.

Concerning the uniqueness problems related to Theorem A, some results have been obtained by Fang [3, Theorem 2] and Lin and Yi [11]. One of them can be stated as follows.

Theorem C ([11, Theorem 1]). *Let f and g be non-constant entire functions, and let $n \geq 7$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.*

The following result is a difference analogue of Theorem C.

Theorem 2. *Let f and g be transcendental entire functions of finite order, let c be a non-zero complex constant, and let $n \geq 7$ be an integer. If $f(z)^n(f(z)-1)f(z+c)$ and $g(z)^n(g(z)-1)g(z+c)$ share a CM, where $a \in S(f) \cap S(g) \setminus \{0\}$, then $f(z) \equiv g(z)$.*

Remark. Very recently, Zhang [18, Theorem 6] has obtained the same result of Theorem 2. However, our proof is different, being based on Lemma 5 of Section 2, while Zhang does not use that lemma.

Similarly to the above situations, one may also consider sharing value problems for $f(z)$ and its shifts $f(z+c)$. Next, we recall a result which may be understood as a "1 CM + 1 IM" theorem for differences:

Theorem D ([7, Corollary 3]). *Let f be an entire function of finite order, let $c \in \mathbb{C}$, and let $a, b \in S(f)$ be distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a CM and b IM, then $f(z) \equiv f(z+c)$ for all $z \in \mathbb{C}$.*

Next, we show that "1 CM + 1 IM" in Theorem D can be replaced by "2 IM".

Theorem 3. *Let f be an entire function of finite order, let $c \in \mathbb{C}$, and let $a, b \in S(f)$ be distinct periodic functions with period c. If $f(z)$ and $f(z+c)$ share a IM and b IM, then $f(z) \equiv f(z+c)$ for all $z \in \mathbb{C}$.*

Remark. Theorem 3 is the best possible, in the sense that "2 IM" cannot be replaced by "1 CM". Indeed, let $f = e^z$ and $f(z+c) = e^{z+c}$, where $c \neq 2n\pi i$, n an integer. It is easy to see that $f(z)$ and $f(z+c)$ share 0 CM, but $f(z) \not\equiv f(z+c)$. Let $f = e^z + 1$ and $f(z+c) = e^{z+c} + 1$, where $c \neq 2n\pi i$, n is an integer. Clearly, $f(z)$ and $f(z+c)$ share 1 CM, but $f(z) \not\equiv f(z+c)$. The proof of Theorem 3 is based on some ideas that Li and Yang used to prove a result of different nature (see [10, Theorem 2.1]).

We investigate the existence of entire solutions of an equation of the form

$$f^n + P(z)(\Delta_c f)^m = Q(z). \tag{1.1}$$

If $m = n$ and $P(z) = Q(z) = 1$, then we rewrite (1.1) as

$$f^n + (\Delta_c f)^n = 1. \tag{1.2}$$

It is well known that (1.2) has no entire solutions when $n \geq 3$ (see [4, Theorem 3]). Recently, Liu [12, Proposion 5.3] proved that (1.2) has no non-constant finite order entire solutions when $n = 2$. Clearly, if $n = 1$, there are no non-constant solutions of (1.2). Thus, *there are no non-constant finite order entire solutions of the equation* (1.2).

Recently, Yang and Laine [15] considered the existence of finite order solutions of a certain type of non-linear difference equation.

Theorem E ([15, Theorem 3.4]). *Let P, Q be polynomials. Then the non-linear difference equation*

$$f(z)^2 + P(z)f(z+1) = Q(z)$$

has no transcendental entire solutions of finite order.

Theorem F ([15, Theorem 3.5]). *The non-linear difference equation*

$$f(z)^3 + P(z)f(z+1) = c \sin bz \tag{1.3}$$

where $P(z)$ is a non-constant polynomial and $b, c \in \mathbb{C}$ are non-zero constants, does not admit entire solutions of finite order. If $P(z) = p$ is a non-zero constant, then (1.3) *has three distinct entire solutions of finite order whenever $b = 3n\pi$ and $p^3 = (-1)^{n+1}\frac{27}{4}c^2$ for a non-zero integer n.*

Replacing $f(z+c)$ with $\Delta_c f$ in Theorems E and F, we get the following results.

THEOREM 4. *Let P, Q be polynomials, and let n and m be integers satisfying $n > m \geq 0$. Then equation* (1.1) *has no transcendental entire solutions of finite order.*

REMARK. The conclusion of Theorem 4 is not true if $m > n$. In the special case of

$$f(z) - (\Delta_{-1/4} f)^2 = z - 1/16$$

a finite order entire solution is $f(z) = 4e^{8\pi iz} - e^{4\pi iz} + z$.

The reasoning used in proving Theorem 4 yields the following result, which can be seen as an improvement of Theorem E.

COROLLARY 2. *Let P, Q be polynomials, and let n, m be distinct positive integers. Then the equation*

$$f^n + P(z)f(z+c)^m = Q(z)$$

has no transcendental entire solutions of finite order.

If $m \neq 1$ and $Q \neq 0$, then Theorem 4 can be improved.

THEOREM 5. *If $n, m \neq 1$ are positive integers such that $n > m/(m-1)$, and P, $Q \neq 0$ are polynomials, then equation* (1.1) *has no transcendental entire solutions.*

In connection with Theorems 4 and 5, we consider equation (1.1) in the case $m = 1$, that is,

$$f^n + P(z)\Delta_c f = Q(z). \tag{1.4}$$

We get the following result.

THEOREM 6. *Equation* (1.4) *has no entire solutions of infinite order if $\overline{N}(r, 1/\Delta_c f) \leq T(r, f)$, $n \geq 3$ and $P(z)$, $Q(z) \not\equiv 0$ are polynomials.*

REMARK. (1) Clearly, if $n = 1$ and $P(z) \equiv 1$, then (1.4) has no entire solutions of infinite order. However, if $P(z) \not\equiv 1$, there may exist such solutions. Indeed, $f(z) = e^z e^{e^{2z}} + 1$ is an entire function of infinite order and satisfies $f + \frac{1}{2}\Delta_c f = 1$, where $c = \pi i$.

(2) If $n = 2$, then (1.4) may have an infinite order entire solution. Indeed, $f(z) = e^{e^z} - 1/2$ is an entire function of infinite order and satisfies $f^2 - \Delta_c f = 1/4$, where $e^c = 2$.

THEOREM 7. *Let P be a non-constant polynomial, and let $b, c \in \mathbb{C}$ be non-zero constants. Then the equation*

$$f(z)^3 + P(z)\Delta f = c \sin bz \tag{1.5}$$

has no transcendental non-periodic entire solutions of finite order. In particular, if $P(z) = p$ *is a non-zero constant, then* (1.5) *has three distinct entire solutions of finite order whenever* $b = 3k\pi$ *and* $p^3 = \frac{27}{32}c^2$ *for an odd number* k*.*

The proof of Theorem 7 is similar to the proof of Theorem F. In fact, one has to apply Lemmas 2 and 4 below, instead of Remark of [15, Lemma 3.2], and use an elementary computation. We omit the details.

2. Some lemmas. The first lemma is a difference analogue of the logarithmic derivative lemma, given by Halburd–Korhonen [5]. Chiang and Feng have obtained similar estimates for the logarithmic difference [2, Corollary 2.5], and their work is independent of [5].

Lemma 1 ([5, Theorem 2.1]). *Let* f *be a meromorphic function of finite order, and let* $c \in \mathbb{C}$ *and* $\delta \in (0, 1)$*. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = o\left(\frac{T(r,f)}{r^\delta}\right) = S(r,f).$$

Lemma 2 ([5, Lemma 2.3]). *Let* f *be a meromorphic function of finite order, and* $c \in \mathbb{C}$*. Then for any small function* $a \in S(f)$ *with period* c*,*

$$m\left(r, \frac{\Delta_c f}{f-a}\right) = S(r,f).$$

Lemma 3 ([5, Lemma 2.2]). *Let* $T\colon (0,\infty) \to (0,\infty)$ *be a non-decreasing continuous function,* $s > 0$*,* $0 < \alpha < 1$*, and let* $F \subset \mathbb{R}^+$ *be the set of all* r *such that*

$$T(r) \le \alpha T(r+s).$$

If the logarithmic measure of F *is infinite,then*

$$\limsup_{r\to\infty} \frac{\log T(r)}{\log r} = \infty.$$

Lemma 4 ([8, Theorem 2.4.2]). *Let* f *be a transcendental meromorphic solution of*

$$f^n A(z,f) = B(z,f),$$

where $A(z,f)$*,* $B(z,f)$ *are differential polynomials in* f *and its derivatives with small meromorphic coefficients* a_λ*, in the sense that* $m(r,a_\lambda) = S(r,f)$ *for all* $\lambda \in I$*. If* $d(B(z,f)) \le n$*, then* $m(r, A(z,f)) = S(r,f)$*.*

Denote by $N_p\big(r, \frac{1}{f-a}\big)$ the counting function of the zeros of $f-a$, where an m-fold zero is counted m times if $m \le p$ and p times if $m > p$.

Lemma 5 ([17, Theorem 3.1]). *Let* $f_j(z)$ $(j = 1,2,3)$ *be meromorphic functions that satisfy*

$$\sum_{j=1}^{3} f_j(z) \equiv 1.$$

If $f_1(z)$ is not a constant, and

$$\sum_{j=1}^{3} N_2\left(r, \frac{1}{f_j}\right) + \sum_{j=1}^{3} \overline{N}(r, f_j) < (\lambda + o(1))T(r), \qquad r \in I,$$

where $0 \leq \lambda < 1$, $T(r) = \max_{1\leq j\leq 3} T(r, f_j)$, and I has infinite linear measure, then either $f_2(z) \equiv 1$ or $f_3(z) \equiv 1$.

LEMMA 6 ([2, Theorem 2.1]). *Let f be a non-constant meromorphic function of finite order σ, and let c be a non-zero constant. Then, for each $\varepsilon > 0$,*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Next, we introduce the auxiliary function

$$H = \left(\frac{f'}{f-1} - \frac{f'}{f}\right) - \left(\frac{g'}{g-1} - \frac{g'}{g}\right),$$

where f and g are given meromorphic functions. Using the reasoning applied in [10], we have the following lemma.

LEMMA 7. *Suppose that f and g are meromorphic functions such that $N(r, f) = N(r, g) = S(r, f)$. If $H \doteq 0$, then either*

$$2T(r, f) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + S(r, f)$$

or $f \equiv g$.

Proof. From $H = 0$, we get

$$\frac{f-1}{f} = a\frac{g-1}{g}, \tag{2.1}$$

where a is a non-zero constant. If $a = 1$, then we obtain $f \equiv g$. It remains to consider the case $a \neq 1$. It follows from (2.1) that

$$\frac{a-1}{a}\,\frac{f + \frac{1}{a-1}}{f} = \frac{1}{g}.$$

Since $N(r, f) = N(r, g) = S(r, f)$, we get $N\big(r, \frac{1}{f - \frac{1}{1-a}}\big) = S(r, f)$. Clearly, $\frac{1}{1-a} \neq 0$ and $\frac{1}{1-a} \neq 1$, so by the second main theorem, we get

$$2T(r, f) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + S(r, f). \ \blacksquare$$

3. Proof of Theorem 1. Set $F(z) = f^n(z)(f(z) - 1)f(z + c)$. Since f is a transcendental meromorphic function of finite order σ, we conclude by

Lemma 6 that

$$\begin{aligned}T(r,F) &\le T(r,f^n(z)(f(z)-1)) + T(r,f(z+c)) + S(r,f)\\ &\le (n+2)T(r,f) + O(r^{\sigma(f)-1+\varepsilon}) + S(r,f).\end{aligned}$$

Thus, $S(r,F) = o(T(r,f)) = S(r,f)$. On the other hand, by Lemma 1,

$$\begin{aligned}(n+2)T(r,f) &= T(r,f^{n+1}(f-1)) + S(r,f)\\ &= m(r,f^{n+1}(f-1)) + O(r^{\lambda(1/f)+\varepsilon}) + S(r,f)\\ &\le m\left(r,\frac{f^{n+1}(f-1)}{F}\right)\\ &\quad + m(r,F) + O(r^{\lambda(1/f)+\varepsilon}) + S(r,f)\\ &\le T(r,F) + O(r^{\lambda(1/f)+\varepsilon}) + O(r^{\sigma(f)-1+\varepsilon}) + S(r,f).\end{aligned} \tag{3.1}$$

The second main theorem yields

$$\begin{aligned}T(r,F) &\le \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-a}\right) + S(r,F)\\ &\le \overline{N}\left(r,\frac{1}{F-a}\right) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-1}\right)\\ &\quad + \overline{N}\left(r,\frac{1}{f(z+c)}\right) + O(r^{\lambda(1/f)+\varepsilon}) + S(r,f)\\ &\le \overline{N}\left(r,\frac{1}{F-a}\right) + 3T(r,f) + O(r^{\lambda(1/f)+\varepsilon}) + O(r^{\sigma(f)-1+\varepsilon}) + S(r,f).\end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2), we have

$$(n-1)T(r,f) \le \overline{N}\left(r,\frac{1}{F-a}\right) + O(r^{\lambda(1/f)+\varepsilon}) + O(r^{\sigma(f)-1+\varepsilon}) + S(r,f);$$

if $F-a$ has finitely many zeros, the above contradicts the fact that f is of order $\sigma(f)$. The conclusion follows.

4. Proof of Theorem 2. By the assumptions, we have

$$\frac{f^n(z)(f(z)-1)f(z+c)-a(z)}{g^n(z)(g(z)-1)g(z+c)-a(z)} = e^{h(z)}, \tag{4.1}$$

where $h(z)$ is a polynomial. Let

$$F_1 = \frac{f^n(z)(f(z)-1)f(z+c)}{a(z)}, \qquad F_2 = -\frac{e^{h(z)}g^n(z)(g(z)-1)g(z+c)}{a(z)},$$

$$F_3 = e^{h(z)}, \qquad T(r) = \max_{1\le j\le 3} T(r,F_j), \qquad S(r) = o(T(r)).$$

Then

$$F_1 + F_2 + F_3 = 1. \tag{4.2}$$

Next, we will estimate the counting functions of F_j $(j = 1, 2, 3)$. First,

$$\begin{aligned} N_2\left(r, \frac{1}{F_1}\right) &\leq 2N\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{f(z+c)}\right) + N\left(r, \frac{1}{f(z)-1}\right) + S(r,f) \\ &\leq \frac{2}{n}\left(nN\left(r, \frac{1}{f(z)}\right) + N\left(r, \frac{1}{f(z+c)}\right) + N\left(r, \frac{1}{f(z)-1}\right)\right) \\ &\quad + \left(1 - \frac{2}{n}\right)\left(N\left(r, \frac{1}{f(z+c)}\right) + N\left(r, \frac{1}{f(z)-1}\right)\right) + S(r,f). \end{aligned} \tag{4.3}$$

By a simple geometric observation and Lemma 3, we conclude that

$$\begin{aligned} N\left(r, \frac{1}{f(z+c)}\right) &\leq N\left(r+|c|, \frac{1}{f(z)}\right) = N\left(r, \frac{1}{f(z)}\right) + S(r,f) \\ &\leq \frac{1}{n}N\left(r, \frac{1}{f^n(z)(f(z)-1)f(z+c)}\right) + S(r,f). \end{aligned} \tag{4.4}$$

We easily obtain

$$\begin{aligned} N\left(r, \frac{1}{f^n(f(z)-1)f(z+c)}\right) &= nN\left(r, \frac{1}{f(z)}\right) \\ &\quad + N\left(r, \frac{1}{f(z+c)}\right) + N\left(r, \frac{1}{f(z)-1}\right). \end{aligned} \tag{4.5}$$

By Lemma 1, we know

$$m\left(r, \frac{1}{f(z+c)}\right) \leq m\left(r, \frac{1}{f(z)}\right) + S(r,f). \tag{4.6}$$

From (4.6), we obtain

$$\begin{aligned} (n+1)T(r,f) &= T(r, f^n(f-1)) + S(r,f) = m(r, f^n(f-1)) + S(r,f) \\ &\leq m(r, f(z)^n(f(z)-1)f(z+c)) + m\left(r, \frac{1}{f(z+c)}\right) + S(r,f) \\ &\leq T(r, F_1) + m\left(r, \frac{1}{f}\right) + S(r,f) \\ &\leq T(r, F_1) + T(r,f) + S(r,f). \end{aligned}$$

Consequently,

$$N\left(r, \frac{1}{f-1}\right) \leq T(r,f) + O(1) \leq \frac{1}{n}T(r, F_1) + S(r,f). \tag{4.7}$$

Clearly, $S(r,f)$ must be $S(r,F_1)$. Then from (4.3)–(4.7), we have

(4.8)
$$N_2\Big(r,\frac{1}{F_1}\Big)\le\frac{2}{n}N\Big(r,\frac{1}{F_1}\Big)+\Big(1-\frac{2}{n}\Big)\frac{1}{n}\Big(N\Big(r,\frac{1}{F_1}\Big)+T(r,F_1)\Big)+S(r,f)$$
$$\le\frac{4n-4}{n^2}T(r,F_1)+S(r,F_1)\le\frac{4n-4}{n^2}T(r)+S(r).$$

Similarly, we conclude that

$$N_2\Big(r,\frac{1}{F_2}\Big)\le\frac{4n-4}{n^2}T(r)+S(r). \tag{4.9}$$

Obviously F_1 is not a constant, so since $n\ge 7$, we obtain

$$\sum_{j=1}^{3}N_2\Big(r,\frac{1}{F_j}\Big)+\sum_{j=1}^{3}\overline{N}(r,F_j)<\frac{48}{49}T(r)+S(r).$$

From Lemma 5, we know that $F_2=1$ or $F_3=1$. Therefore, either $f(z)^n\cdot(f(z)-1)f(z+c)g(z)^n(g(z)-1)g(z+c)\equiv a(z)^2$ or $f^n(f-1)f(z+c)\equiv g^n(g-1)g(z+c)$. The assertion now follows as in [18, p. 407].

5. Proof of Theorem 3. If $N\big(r,\frac{1}{f-a}\big)=0$ or $N\big(r,\frac{1}{f-b}\big)=0$, then the assertion follows by Theorem C. It remains to consider the case when $N\big(r,\frac{1}{f-a}\big)\ne 0$ and $N\big(r,\frac{1}{f-b}\big)\ne 0$. Let

$$F(z)=\frac{f(z)-a(z)}{b(z)-a(z)}\quad\text{and}\quad F(z+c)=\frac{f(z+c)-a(z)}{b(z)-a(z)}. \tag{5.1}$$

Then $F(z)$ and $F(z+c)$ share 0 IM and 1 IM. Clearly, neither 0 nor 1 is a Picard value of F in this case. Moreover,

$$T(r,F)=m(r,F)+S(r,F)\le m\Big(r,\frac{F}{F(z+c)}\Big)+m(r,F(z+c))+S(r,F)$$
$$=T(r,F(z+c))+S(r,F)$$

and

$$T(r,F(z+c))=m(r,F(z+c))+S(r,F)$$
$$\le m\Big(r,\frac{F(z+c)}{F}\Big)+m(r,F)+S(r,F)=T(r,F)+S(r,F).$$

Therefore

$$T(r,F)=T(r,F(z+c))+S(r,F). \tag{5.2}$$

Denote

$$V=\frac{F'(F(z+c)-F)}{F(F-1)}. \tag{5.3}$$

From Lemma 1 and the lemma on the logarithmic derivative, we see that $m(r,V)=S(r,F)$. From (5.3), the poles of V are at the zeros and 1-points of F, and at the poles of F and $F(z+c)$. Since $F(z)$ and $F(z+c)$ share 0 and 1, and $N(r,F)=N(r,F(z+c))=S(r,F)$ by (5.1), we get $N(r,V)=S(r,F)$. Therefore, $T(r,V)=S(r,F)$.

Case 1: $V\not\equiv 0$. Then $F\not\equiv F(z+c)$. From (5.3) and Lemma 1, we obtain

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)&=N\left(r,\frac{F'}{F(F-1)}\right)+S(r,F)\\&=N\left(r,\frac{V}{F(z+c)-F}\right)+S(r,F)\\&\le T(r,F(z+c)-F)+S(r,F)\\&=m(r,F(z+c)-F)+S(r,F)\\&\le m\left(r,\frac{F(z+c)-F}{F}\right)+m(r,F)+S(r,F)\\&\le T(r,F)+S(r,F).\end{aligned}$$

According to the second main theorem and the above inequality, we get

$$T(r,F)=\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)+S(r,F). \tag{5.4}$$

Now we define

$$U=\frac{F'(z+c)(F(z+c)-F)}{F(z+c)(F(z+c)-1)}. \tag{5.5}$$

By the same argument as above, we deduce that $T(r,U)=S(r,F(z+c))=S(r,F)$. We denote by $S_{f\sim g(m,n)}(a)$ the set of those points $z\in\mathbb{C}$ such that z is an a-point of f with multiplicity m and an a-point of g with multiplicity n. Let $N_{(m,n)}\big(r,\frac{1}{f-a}\big)$ and $\overline{N}_{(m,n)}\big(r,\frac{1}{f-a}\big)$ denote the counting function and reduced counting function of f with respect to the set $S_{f\sim g(m,n)}(a)$, respectively.

For any $z_0\in S_{F(z)\sim F(z+c)(m,n)}(0)$, we have $mn\neq 0$, since 0 is not a Picard value of F. From (5.3), (5.5), and by the Taylor expansion of F and $F(z+c)$ at z_0, we obtain

$$\begin{aligned}-V(z_0)&=m\left(\frac{F'(z_0+c)}{n}-\frac{F'(z_0)}{m}\right),\\-U(z_0)&=n\left(\frac{F'(z_0+c)}{n}-\frac{F'(z_0)}{m}\right),\end{aligned}$$

and thus $nV(z_0)=mU(z_0)$.

If $nV = mU$, then we obtain

$$n\left(\frac{F'}{F-1} - \frac{F'}{F}\right) \equiv m\left(\frac{F'(z+c)}{F(z+c)-1} - \frac{F'(z+c)}{F(z+c)}\right),$$

which implies that

$$\left(\frac{F-1}{F}\right)^n \equiv d\left(\frac{F(z+c)-1}{F(z+c)}\right)^m$$

where d is a non-zero constant. If $m \neq n$ then from (5.2) we get $nT(r,F) = mT(r,F(z+c)) + S(r,F) = mT(r,F) + S(r,F)$, which is a contradiction. If $m = n$, from Lemma 7 we get

$$2T(r,F) \leq \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-1}\right) + S(r,F),$$

which contradicts (5.4).

Hence $nV \neq mU$. Therefore

$$\overline{N}_{(m,n)}\left(r,\frac{1}{F}\right) \leq N\left(r,\frac{1}{nU-mV}\right) = S(r,F).$$

Using the same reasoning, we get

$$\overline{N}_{(m,n)}\left(r,\frac{1}{F-1}\right) \leq N\left(r,\frac{1}{nU-mV}\right) = S(r,F).$$

It follows that

$$\overline{N}_{(m,n)}\left(r,\frac{1}{F}\right) + \overline{N}_{(m,n)}\left(r,\frac{1}{F-1}\right) = S(r,F). \tag{5.6}$$

From (5.4) and (5.6), we obtain

$$\begin{aligned}
T(r,F) &= \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-1}\right) + S(r,F)\\
&= \sum_{m,n}\left(\overline{N}_{(m,n)}\left(r,\frac{1}{F}\right) + \overline{N}_{(m,n)}\left(r,\frac{1}{F-1}\right)\right) + S(r,F)\\
&= \sum_{m+n\geq 5}\left(\overline{N}_{(m,n)}\left(r,\frac{1}{F}\right) + \overline{N}_{(m,n)}\left(r,\frac{1}{F-1}\right)\right) + S(r,F)\\
&\leq \frac{1}{5}\sum_{m+n\geq 5}\left(N_{(m,n)}\left(r,\frac{1}{F}\right) + N_{(m,n)}\left(r,\frac{1}{F-1}\right)\right.\\
&\quad \left. + N_{(m,n)}\left(r,\frac{1}{F(z+c)}\right) + N_{(m,n)}\left(r,\frac{1}{F(z+c)-1}\right)\right) + S(r,F)\\
&\leq \frac{4}{5}T(r,F) + S(r,F),
\end{aligned}$$

which is a contradiction.

CASE 2: $V = 0$. Then $F = F(z+c)$. Clearly, $f(z) = f(z+c)$. This completes the proof of Theorem 3.

6. Proofs of Theorem 4–6

Proof of Theorem 4. Suppose that f is a transcendental entire solution of equation (1.1) of finite order. If $\Delta_c f \equiv 0$, then $f(z)^n = Q(z)$, and the conclusion holds. If $\Delta_c f \not\equiv 0$, then rewrite (1.1) as

$$f^{n-1} f = Q(z) - P(z)\frac{(\Delta_c f)^m}{f^m} f^m.$$

Applying Lemmas 2 and 4, and invoking the assumption $n > m$, we conclude that

$$T(r,f) = m(r,f) = S(r,f),$$

a contradiction.

Proof of Theorem 5. Suppose that f is a transcendental entire solution to equation (1.1). Clearly, if $P \doteq 0$, the conclusion follows. It remains to consider the case $P \neq 0$. If $\Delta_c f \equiv 0$, then $f(z)^n = Q(z)$ and the conclusion holds. If $\Delta_c f \not\equiv 0$, then by the second fundamental theorem for three small target functions, we obtain

$$\begin{aligned} T\left(r,\frac{f^n}{P}\right) &\le \overline{N}\left(r,\frac{f^n}{P}\right) + \overline{N}\left(r,\frac{P}{f^n}\right) + \overline{N}\left(r,\frac{P}{f^n - Q}\right) + S(r,f) \\ &= \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{\Delta_c f}\right) + S(r,f) \\ &\le T(r,f) + T(r,\Delta_c f) + S(r,f). \end{aligned} \tag{6.1}$$

Moreover

$$T\left(r,\frac{f^n - Q}{P}\right) = nT(r,f) + S(r,f) = mT(r,\Delta_c f). \tag{6.2}$$

Combining (6.1) with (6.2), we get

$$nT(r,f) \le T(r,f) + \frac{n}{m}T(r,f) + S(r,f)$$

and so

$$\left(n - 1 - \frac{n}{m}\right)T(r,f) \le S(r,f),$$

which contradicts the assumption that $n > m/(m-1)$.

Proof of Theorem 6. Suppose that f is an infinite order entire solution of (1.4). Since P and $Q \neq 0$ are polynomials, they are small functions to f.

Using the second main theorem for three small target functions, we obtain

(6.3)
$$
\begin{aligned}
nT(r,f) = T(r,f^n) &\leq \overline{N}(r,f^n) + \overline{N}\left(r,\frac{1}{f^n}\right) + \overline{N}\left(r,\frac{1}{f^n-Q}\right) + S(r,f)\\
&\leq \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{\Delta_c f}\right) + S(r,f)\\
&\leq 2T(r,f) + S(r,f).
\end{aligned}
$$

Since $n \geq 3$, we get a contradiction.

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