

**Uniqueness of meromorphic functions
sharing a meromorphic function of a smaller order
with their derivatives**

by XIAO-MIN LI (Qingdao) and HONG-XUN YI (Jinan)

Abstract. We prove some uniqueness theorems for meromorphic functions and their derivatives that share a meromorphic function whose order is less than those of the above meromorphic functions. The results in this paper improve those given by G. G. Gundersen & L. Z. Yang, J. P. Wang, J. M. Chang & Y. Z. Zhu, and others. Some examples are provided to show that our results are the best possible.

1. Introduction and main results. In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [6], [9] and [19]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h , we denote by $T(r, h)$ the Nevanlinna characteristic of h and by $S(r, h)$ any quantity satisfying

$$S(r, h) = o\{T(r, h)\} \quad (r \rightarrow \infty, r \notin E).$$

Let f and g be two nonconstant meromorphic functions, and let a be a finite value. We say that f and g *share the value a CM* provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g *share a IM* provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. In addition, we say that f and g *share ∞ CM* if $1/f$ and $1/g$ share 0 CM, and we say that f and g *share ∞ IM* if $1/f$ and $1/g$ share 0 IM (see [20]).

In this paper, we also need the following definition.

DEFINITION 1.1. For a nonconstant entire function f , the *order* $\sigma(f)$, the *lower order* $\mu(f)$, and the *hyper order* $\sigma_2(f)$ are defined as

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$$\begin{aligned}\sigma(f) &= \limsup_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}, \\ \mu(f) &= \liminf_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} = \liminf_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}, \\ \sigma_2(f) &= \limsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r} = \limsup_{r \rightarrow \infty} \frac{\log \log \log M(r, f)}{\log r}\end{aligned}$$

respectively; here and in what follows, $M(r, f) = \max_{|z|=r} |f(z)|$.

In 1977, L. A. Rubel and C. C. Yang [15] proved that if an entire function f shares two distinct finite complex numbers CM with its derivative f' , then $f = f'$. What is the relation between f and f' if an entire function f shares one finite complex number a CM with its derivative f' ? In 1996, R. Brück [2] made the conjecture that if f is a nonconstant entire function satisfying $\sigma_2(f) < \infty$, where $\sigma_2(f)$ is not a positive integer, and if f and f' share one finite complex number a CM, then $f - a = c(f' - a)$ for some constant $c \neq 0$. He proved this for $a = 0$. He also proved that the conjecture is true provided that $a \neq 0$ and $N(r, 1/f') = S(r, f)$. In 2005, Al-Khaladi showed that the conjecture remains true for a nonconstant meromorphic function f such that $N(r, 1/f') = S(r, f)$ (see [1]). Recently many mathematicians have dealt with the above conjecture: A. Banerjee & P. Bhattacharjee [3], J. M. Chang & Y. Z. Zhu [4], J. Heittokangas et al. [7], I. Lahiri & A. Sarkar [8], X. M. Li & C. C. Gao [10], [11], X. M. Li & H. X. Yi [12]–[14], J. Wang & I. Laine [16], J. Wang & X. M. Li [17], Q. C. Zhang [21], J. L. Zhang & L. Z. Yang [22], [23], among others. But the conjecture is still an open question by now.

In 1998, G. G. Gundersen and L. Z. Yang proved the following result, which shows that the above conjecture is true for $a \neq 0$ provided that f satisfies the additional assumption $\sigma(f) < \infty$.

THEOREM A (see [5, Theorem 1]). *Let f be a nonconstant entire function of finite order, and let $a (\neq 0)$ be a finite complex number. If f and f' share a CM, then $f' - a = c(f - a)$ for some nonzero constant c .*

In 2004, J. P. Wang proved the following theorem, which improved Theorem A.

THEOREM B (see [18, Theorem 1]). *Let f be a nonconstant entire function of finite order, let P be a polynomial with degree $p \geq 1$, and let k be a positive integer. If $f - P$ and $f^{(k)} - P$ share 0 CM, then $f^{(k)} - P = c(f - P)$ for some complex number $c \neq 0$.*

Recently J. M. Chang and Y. Z. Zhu proved the following result, which improved Theorem B.

THEOREM C (see [4, Theorem 1]). *Let f be an entire function such that $\sigma(f) < \infty$, and let a ($\neq 0$) be an entire function such that $\sigma(a) < \sigma(f)$. If $f - a$ and $f' - a$ share 0 CM, then $f' - a = c(f - a)$ for some nonzero constant c .*

We will prove the following result, which improves Theorem C.

THEOREM 1.1. *Let f be a nonconstant entire function such that $\sigma(f) < \infty$, and let a ($\neq 0$) be an entire function such that $\sigma(a) < \sigma(f)$. If $f - a$ and $f^{(k)} - a$ share 0 CM, where k (≥ 1) is a positive integer, then $f^{(k)} - a = c(f - a)$ for some nonzero constant c .*

The following example shows that the condition “ $\sigma(a) < \sigma(f)$ ” in Theorem 1.1 is the best possible.

EXAMPLE 1.1 (see [4]). Let $f(z) = e^{2z} - (z - 1)e^z$ and $a(z) = e^{2z} - ze^z$. Then it can be verified that $f - a$ and $f' - a$ share 0 CM and $\sigma(f) = \sigma(a) = 1$. But $f' - a = e^z(f - a)$.

In the same paper [4], J. M. Chang and Y. Z. Zhu prove the following result.

THEOREM D (see [4, Theorem 2]). *Let f and a be meromorphic functions such that f and a have finitely many poles, and such that f and a have no common poles. If $f - a$ and $f' - a$ share 0 CM, and if $\sigma(a) < \sigma(f) < \infty$, then $f' - a = c(f - a)$ for some nonzero constant c .*

We will prove the following result, which improves Theorem D.

THEOREM 1.2. *Let f be a nonconstant meromorphic function, and let a ($\neq 0$) be a meromorphic function such that f and a have finitely many poles, and such that f and a have no common poles. If $\sigma(a) < \sigma(f) < \infty$, and if $f - a$ and $f^{(k)} - a$ share 0 CM, where k (≥ 1) is a positive integer, then $f^{(k)} - a = c(f - a)$ for some nonzero constant c .*

Lemma 2.1 in Section 2 implies the following result, which complements Theorem 1.2.

THEOREM 1.3. *Let f be a nonconstant rational function, and let a be a nonzero complex number. If $f - a$ and $f^{(k)} - a$ share 0 CM, where k (≥ 1) is a positive integer, then f can be expressed as*

$$f(z) = a + \frac{a(z - z_1)^k}{2k!} + \frac{A_1}{z - z_1},$$

where z_1 and A_1 ($\neq 0$) are two complex numbers.

The following example shows that the condition “ f and a have finitely many poles” in Theorem 1.2 is the best possible.

EXAMPLE 1.2 (see [5]). Let $f(z) = (2e^z + z + 1)/(e^z + 1)$. Then f is a nonconstant meromorphic function but not an entire function. Moreover, we can verify that f and f' share 1 CM, that $\mu(f) = \sigma(f) = 1$ and that f has infinitely many poles. However, $(f' - 1)/(f - 1) = -e^z/(e^z + 1)$.

J. M. Chang and Y. Z. Zhu [4] prove the following result.

THEOREM E (see [4, Theorem 3]). *Let f be a nonconstant entire function of finite order. If $1/f'$ and $1/f$ have the same fixed points with the same multiplicities, then $f = f'$.*

We will prove the following result, which improves Theorem E.

THEOREM 1.4. *Let f be a nonconstant entire function of finite order, and let R be a nonzero rational function that has at least one pole. If $f - R$ and $f^{(k)} - R$ share 0 CM, then $f = f^{(k)}$.*

In 1995, H. X. Yi and C. C. Yang posed the following question.

QUESTION 1.1 (see [20, p. 398]). Let f be a nonconstant meromorphic function, and let a be a finite nonzero complex constant. If f , $f^{(n)}$ and $f^{(m)}$ share the value a CM, where n and m ($n < m$) are distinct positive integers not all even or odd, can we infer that $f = f^{(n)}$?

Regarding Question 1.1, G. G. Gundersen and L. Z. Yang proved the following result in 1998.

THEOREM F (see [5, Theorem 2]). *Let f be a nonconstant entire function of finite order, let a ($\neq 0$) be a finite complex number, and let n be a positive integer. If a is shared by f , $f^{(n)}$ and $f^{(n+1)}$ IM, and shared by $f^{(n)}$ and $f^{(n+1)}$ CM, then $f = f'$.*

We will prove the following result, which improves Theorem F.

THEOREM 1.5. *Let f be a nonconstant meromorphic function such that f has finitely many poles and $\sigma(f) < \infty$, let a ($\neq 0$) be a finite value in the complex plane, and let n be a positive integer. If a is shared by f , $f^{(n)}$ and $f^{(n+1)}$ IM, and shared by $f^{(n)}$ and $f^{(n+1)}$ CM, then $f(z) = \gamma e^z$, where γ is a certain nonzero constant.*

2. Some lemmas

LEMMA 2.1 (see [20, Corollary of Theorem 1.23]). *Let f be a nonconstant meromorphic function, and let k (≥ 1) be a positive integer. Then*

$$T(r, f) \leq \bar{N}(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - 1}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f).$$

Let $f = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. We define by $\mu(r) = \max\{|a_n|r^n : n = 0, 1, 2, \dots\}$ the *maximum term* of f , and define by $\nu(r, f) = \max\{m : \mu(r) = |a_m|r^m\}$ the *central index* of f (see [9, p. 50]).

LEMMA 2.2 (see [9, Theorem 3.1]). *If f is an entire function of order $\sigma(f)$, then*

$$\sigma(f) = \limsup_{r \rightarrow \infty} \frac{\log \nu(r, f)}{\log r}.$$

LEMMA 2.3 (see [9, p. 5]). *Let $g : (0, \infty) \rightarrow \mathbb{R}$ and $h : (0, \infty) \rightarrow \mathbb{R}$ be increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E_1 of finite logarithmic measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(r^\alpha)$ for all $r > r_0$.*

LEMMA 2.4 (see [9, Lemma 1.1.1]). *Let $g : (0, \infty) \rightarrow \mathbb{R}$ and $h : (0, \infty) \rightarrow \mathbb{R}$ be increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

LEMMA 2.5 (see [20, Theorem 1.21]). *Suppose that f is meromorphic in the complex plane. Then $\sigma(f) = \sigma(f')$ and $\mu(f) = \mu(f')$.*

3. Proof of theorems

Proof of Theorem 1.2. From the condition $\sigma(a) < \sigma(f)$ we know that f is a transcendental meromorphic function. We discuss the following three cases.

CASE 1. Suppose that f and a are not entire functions. Then

$$(3.1) \quad a = \frac{a_1}{P_1},$$

where P_1 is a nonconstant polynomial such that P_1 and $1/a$ share 0 CM. From the condition that $f - a$ and $f^{(k)} - a$ share 0 CM we get

$$(3.2) \quad \frac{f^{(k)} - a}{f - a} = \frac{ce^{\beta_1}}{(z - \omega_1)^k \cdots (z - \omega_n)^k},$$

where c is some nonzero complex number, $\omega_1, \dots, \omega_n$ are n distinct complex numbers that are all poles of f , and β_1 is an entire function. Let

$$(3.3) \quad F = P_0 f,$$

where P_0 is a nonconstant polynomial such that P_0 and $1/f$ share 0 CM. Then F is a transcendental entire function. From (3.3) we get

$$(3.4) \quad \begin{aligned} \frac{f^{(k)} - a}{f - a} &= \frac{\frac{F^{(k)}}{F} + \frac{kR'_0}{R_0} \cdot \frac{F^{(k-1)}}{F} + \cdots + \binom{k}{j} \frac{R_0^{(j)}}{R_0} \cdot \frac{F^{(k-j)}}{F}}{1 - \frac{aP_0}{F}} \\ &+ \frac{\binom{k}{j+1} \frac{R_0^{(j+1)}}{R_0} \cdot \frac{F^{(k-j-1)}}{F} + \cdots + \binom{k}{k-2} \frac{R_0^{(k-2)}}{R_0} \cdot \frac{F^{(2)}}{F}}{1 - \frac{aP_0}{F}} \\ &+ \frac{\frac{kR_0^{(k-1)}}{R_0} \cdot \frac{F'}{F} + \frac{R_0^{(k)}}{R_0} - \frac{aP_0}{F}}{1 - \frac{aP_0}{F}}, \end{aligned}$$

where

$$(3.5) \quad R_0 = \frac{1}{P_0}.$$

From (3.5) and the definition of P_0 we get

$$(3.6) \quad \frac{R'_0}{R_0} = \frac{m_1}{z - \omega_1} + \cdots + \frac{m_n}{z - \omega_n},$$

where m_1, \dots, m_n are negative integers. By induction from (3.6) we get

$$(3.7) \quad \frac{R_0^{(j)}(z)}{R_0(z)} = \frac{\{(-1)^{j-1}(j-1)! \sum_{l=1}^n m_l\}(1 + o(1))}{z^j}$$

as $|z| \rightarrow \infty$, where j is a positive integer satisfying $1 \leq j \leq k$. Noting that F is a transcendental entire function, by Lemma 2.2 and Proposition 3.3 in [9] we know that

$$(3.8) \quad \nu(r, F) \rightarrow \infty.$$

Let

$$(3.9) \quad M(r, F) = |F(z_r)|,$$

where $z_r = re^{i\theta(r)}$ and $\theta(r) \in [0, 2\pi)$. From (3.9) and the Wiman–Valiron theory (see [9, Theorem 3.2]), we know that there exists a subset $E_j \subset (1, \infty)$ with finite logarithmic measure, i.e., $\int_{E_j} dt/t < \infty$, such that for some point $z_r = re^{i\theta(r)}$ ($\theta(r) \in [0, 2\pi)$) satisfying $|z_r| = r \notin E_j$ and $M(r, F) = |F(z_r)|$, we have

$$(3.10) \quad \frac{F^{(j)}(z_r)}{F(z_r)} = \left(\frac{\nu(r, F)}{z_r}\right)^j \{1 + o(1)\}$$

as $r \notin \bigcup_{j=1}^k E_j$ and $r \rightarrow \infty$. From (3.1) we see that a_1 is an entire function such that

$$(3.11) \quad \sigma(a_1 P_0) = \sigma(a_1) = \sigma(a).$$

From (3.3) we get $T(r, F) = T(r, f) + O(\log r)$. This together with (3.11) and the condition $\sigma(a) < \sigma(f)$ implies that

$$(3.12) \quad \sigma(a_1 P_0) < \sigma(F).$$

By (3.12), Definition 1.1 and Lemma 2.3 we can deduce that there exists an infinite sequence of points $z_{r_n} = r_n e^{i\theta(r_n)}$ satisfying $M(r_n, F) = |F(z_{r_n})|$, where $r_n \in I$, $I \subseteq \mathbb{R}^+$ is a subset with logarithmic measure $m_l(I) = \infty$, such that

$$(3.13) \quad \lim_{r_n \rightarrow \infty} \frac{\log \log M(r_n, F)}{\log r_n} = \sigma(F),$$

$$(3.14) \quad \lim_{r_n \rightarrow \infty} \frac{M(r_n, a_1 P_0)}{M(r_n, F)} = 0.$$

Next we will show that β_2 is a constant. In fact, if $k = 1$, then (3.4) can be rewritten as

$$(3.15) \quad \frac{f' - a}{f - a} = \frac{\frac{F'}{F} + \frac{R'_0}{R_0} - \frac{aP_0}{F}}{1 - \frac{aP_0}{F}}.$$

From (3.1), (3.7)–(3.10) and (3.13)–(3.15) we get

$$(3.16) \quad \frac{f'(z_{r_n}) - a(z_{r_n})}{f(z_{r_n}) - a(z_{r_n})} = \frac{\nu(r_n, F)}{z_{r_n}} \{1 + o(1)\}$$

as $r_n \in I \setminus E_1$ and $r_n \rightarrow \infty$. From (3.1) and (3.16) we get

$$(3.17) \quad |\beta_1(z_{r_n})| = |\log e^{\beta_1(z_{r_n})}| = \left| \log \frac{f^{(k)}(z_{r_n}) - a(z_{r_n})}{f(z_{r_n}) - a(z_{r_n})} \right| \\ \leq \log\{\nu(r_n, F)\} + \log r_n + O(1)$$

as $r_n \in I \setminus E_1$ and $r_n \rightarrow \infty$. From the condition $\sigma(a) < \sigma(f) < \infty$ and (3.2) we know that there exists a sufficiently large positive number r_0 and a positive number B_1 such that

$$T(r, e^{\beta_1}) \leq B_1 r^{1+\sigma(f)} \quad (r \geq r_0),$$

which implies that β_1 is a polynomial. From (3.17), Definition 1.1 and Lemma 2.2 we get

$$(3.18) \quad |\beta_1(z_{r_n})| \leq \log\{\nu(r_n, F)\} + \log r_n + O(1) \leq \{2 + \sigma(f)\} \log r_n$$

as $r_n \in I \setminus E_1$ and $r_n \rightarrow \infty$.

If $k \geq 2$, from (3.1), (3.4), (3.7)–(3.10), (3.13) and (3.14) we get

$$(3.19) \quad \frac{f^{(k)}(z_{r_n}) - a(z_{r_n})}{f(z_{r_n}) - a(z_{r_n})} \\ = \frac{\frac{F^{(k)}(z_{r_n})}{F(z_{r_n})} + \frac{kR'_0(z_{r_n})}{R_0(z_{r_n})} \cdot \frac{F^{(k-1)}(z_{r_n})}{F(z_{r_n})} + \dots + \binom{k}{j} \frac{R_0^{(j)}(z_{r_n})}{R_0(z_{r_n})} \cdot \frac{F^{(k-j)}(z_{r_n})}{F(z_{r_n})}}{1 - \frac{a(z_{r_n})P_0(z_{r_n})}{F(z_{r_n})}} \\ + \frac{\binom{k}{j+1} \frac{R_0^{(j+1)}(z_{r_n})}{R_0(z_{r_n})} \cdot \frac{F^{(k-j-1)}(z_{r_n})}{F(z_{r_n})} + \dots + \binom{k}{k-2} \frac{R_0^{(k-2)}(z_{r_n})}{R_0(z_{r_n})} \cdot \frac{F^{(2)}(z_{r_n})}{F(z_{r_n})}}{1 - \frac{a(z_{r_n})P_0(z_{r_n})}{F(z_{r_n})}} \\ + \frac{\frac{kR_0^{(k-1)}(z_{r_n})}{R_0(z_{r_n})} \cdot \frac{F'(z_{r_n})}{F(z_{r_n})} + \frac{R_0^{(k)}(z_{r_n})}{R_0(z_{r_n})} - \frac{a(z_{r_n})P_0(z_{r_n})}{F(z_{r_n})}}{1 - \frac{a(z_{r_n})P_0(z_{r_n})}{F(z_{r_n})}} \\ = \frac{N_n \sum_{j=1}^{k-1} \binom{k}{j} \sum_{j=1}^{k-1} (-1)^{j-1} (j-1)! \{\nu(r_n, F)\}^{k-j} \{1 + o(1)\}}{z_{r_n}^k} \\ + \frac{\{\nu(r_n, F)\}^k \{1 + o(1)\}}{z_{r_n}^k} = \left(\frac{\nu(r_n, F)}{z_{r_n}} \right)^k \{1 + o(1)\}$$

as $r_n \in I \setminus \bigcup_{j=1}^k E_j$ and $r_n \rightarrow \infty$, where $N_n = \sum_{l=1}^n m_l$. Next in the same manner as in the proof of (3.18) we get

$$(3.20) \quad |\beta_1(z_{r_n})| \leq k\{\log\{\nu(r_n, F)\} + \log r_n\} + O(1) \leq k\{2 + \sigma(f)\} \log r_n$$

as $r_n \in I \setminus \bigcup_{j=1}^k E_j$ and $r_n \rightarrow \infty$. From (3.18) and (3.20) we know that β_1 is a constant. Thus (3.2) can be rewritten as

$$(3.21) \quad \frac{f^{(k)} - a}{f - a} = \frac{c}{(z - \omega_1)^k \cdots (z - \omega_n)^k},$$

where c is some nonzero constant. From (3.7) we get

$$(3.22) \quad \left| \frac{R_0^{(j)}(z)}{R_0(z)} \right| \leq \frac{d_j}{|z|^j},$$

as $|z| \rightarrow \infty$, where j is a positive integer satisfying $1 \leq j \leq k$, d_j is some positive constant. We discuss the following two subcases.

SUBCASE 1.1. Suppose $k = 1$. Then (3.4) can be rewritten as (3.15). From (3.15) and (3.21) we get

$$(3.23) \quad \frac{\frac{F'}{F} + \frac{R'_0}{R_0} - \frac{aP_0}{F}}{1 - \frac{aP_0}{F}} = \frac{c}{(z - \omega_1) \cdots (z - \omega_n)}.$$

From (3.10), (3.13) (3.14), (3.23) we get

$$\begin{aligned} \left| \frac{\nu(r_n, F)}{2z_{r_n}} \right| &\leq \frac{|c|}{|z_{r_n} - \omega_1| \cdots |z_{r_n} - \omega_n|} \\ &\quad + \frac{|c|}{|z_{r_n} - \omega_1| \cdots |z_{r_n} - \omega_n|} \cdot \left| \frac{a(z_{r_n})P_0(z_{r_n})}{F(z_{r_n})} \right| \\ &\quad + \left| \frac{a(z_{r_n})P_0(z_{r_n})}{F(z_{r_n})} \right| + \left| \frac{R'_0(z_{r_n})}{R_0(z_{r_n})} \right| \end{aligned}$$

as $r_n \in I \setminus E_1$ and $r_n \rightarrow \infty$. This together with (3.6) implies

$$\begin{aligned} (3.24) \quad \nu(r_n, F) &\leq \frac{2|c||z_{r_n}|}{|z_{r_n} - \omega_1| \cdots |z_{r_n} - \omega_n|} \\ &\quad + \frac{2|c|}{|z_{r_n} - \omega_1| \cdots |z_{r_n} - \omega_n|} \cdot \left| \frac{z_{r_n} a(z_{r_n})P_0(z_{r_n})}{F(z_{r_n})} \right| \\ &\quad + 2 \left| \frac{z_{r_n} a(z_{r_n})P_0(z_{r_n})}{F(z_{r_n})} \right| + \left| \frac{2z_{r_n} R'_0(z_{r_n})}{R_0(z_{r_n})} \right| \\ &\leq 3 \left| \frac{z_{r_n} a(z_{r_n})P_0(z_{r_n})}{F(z_{r_n})} \right| + O(1) \end{aligned}$$

as $r_n \in I \setminus E_1$ and $r_n \rightarrow \infty$. From (3.1) we get

$$(3.25) \quad \sigma(za_1P_0) = \sigma(a_1) = \sigma(a).$$

From (3.25) and the condition $\sigma(a) < \sigma(F)$ we get

$$(3.26) \quad \sigma(za_1P_0) < \sigma(F).$$

From (3.1), (3.8) and (3.24) we get

$$(3.27) \quad \begin{aligned} \nu(r_n, F) &\leq 3 \left| \frac{z_{r_n} a_1(z_{r_n}) P_0(z_{r_n})}{P_1(z_{r_n}) F(z_{r_n})} \right| + O(1) \\ &\leq 4 \left| \frac{z_{r_n} a_1(z_{r_n}) P_0(z_{r_n})}{F(z_{r_n})} \right| \end{aligned}$$

as $r_n \in I \setminus E_1$ and $r_n \rightarrow \infty$. Thus

$$(3.28) \quad \nu(r_n, F) |F(z_{r_n})| \leq 4 |z_{r_n} a_1(z_{r_n}) P_0(z_{r_n})| \leq 4M(r_n, za_1P_0)$$

as $r_n \in I \setminus E_1$ and $r_n \rightarrow \infty$. From (3.13), (3.28) and Definition 1.1 we know that

$$(3.29) \quad \nu(r_n, F) e^{r_n^{\sigma(F)-\varepsilon}} \leq 4M(r_n^\alpha, za_1P_0) \leq e^{r_n^{\delta_1+\varepsilon}}$$

as $r_n \in I \setminus E_1$ and $r_n \rightarrow \infty$, where

$$(3.30) \quad \delta_1 = \sigma(za_1P_0)$$

and ε is an arbitrary positive number. From (3.8), (3.26), (3.29) and (3.30) we get a contradiction.

SUBCASE 1.2. Suppose that

$$(3.31) \quad k \geq 2.$$

From (3.4) and (3.21) we get

$$(3.32) \quad \begin{aligned} &\frac{\frac{F^{(k)}}{F} + \frac{kR'_0}{R_0} \cdot \frac{F^{(k-1)}}{F} + \dots + \binom{k}{j} \frac{R_0^{(j)}}{R_0} \cdot \frac{F^{(k-j)}}{F}}{1 - \frac{aP_0}{F}} \\ &+ \frac{\binom{k}{j+1} \frac{R_0^{(j+1)}}{R_0} \cdot \frac{F^{(k-j-1)}}{F} + \dots + \binom{k}{k-2} \frac{R_0^{(k-2)}}{R_0} \cdot \frac{F^{(2)}}{F}}{1 - \frac{aP_0}{F}} \\ &+ \frac{\frac{kR_0^{(k-1)}}{R_0} \cdot \frac{F'}{F} + \frac{R_0^{(k)}}{R_0} - \frac{aP_0}{F}}{1 - \frac{aP_0}{F}} \\ &= \frac{c}{(z - \omega_1)^k \dots (z - \omega_n)^k}. \end{aligned}$$

From (3.10), (3.13), (3.14), (3.22) and (3.32) we get

$$\begin{aligned}
 (3.33) \quad & \{\nu(r_n, F)\}^k \\
 & \leq 4 \left\{ k|d_1| \{\nu(r_n, F)\}^{k-1} + \dots + \binom{k}{j} |d_j| \{\nu(r_n, F)\}^{k-j} + \dots \right. \\
 & \quad \left. + k|d_{k-1}| \nu(r_n, F) + |d_k| + \frac{|a_1(z_{r_n})z_{r_n}^k P_0(z_{r_n})|}{|P_1(z_{r_n})F(z_{r_n})|} \right\} \\
 & \quad + \frac{|c| |z_{r_n}^k|}{|z_{r_n} - \omega_1|^k \dots |z_{r_n} - \omega_n|^k} \cdot \left\{ 1 + \frac{|a_1(z_{r_n})P_0(z_{r_n})|}{|P_1(z_{r_n})F(z_{r_n})|} \right\} \\
 & \leq 4 \left\{ k|d_1| \{\nu(r_n, F)\}^{k-1} + \dots + \binom{k}{j} |d_j| \{\nu(r_n, F)\}^{k-j} + \dots \right. \\
 & \quad \left. + k|d_{k-1}| \nu(r, F) + |d_k| + \frac{|z_{r_n}^k a_1(z_{r_n})P_0(z_{r_n})|}{|F(z_{r_n})|} \right\} \\
 & \quad + 2|c| \left\{ 1 + \frac{|a_1(z_{r_n})P_0(z_{r_n})|}{|F(z_{r_n})|} \right\}
 \end{aligned}$$

as $r_n \in I \setminus \bigcup_{j=1}^k E_j$ and $r_n \rightarrow \infty$. From (3.8) and (3.33) we get

$$(3.34) \quad \{\nu(r_n, F)\}^k \leq \frac{16|z_{r_n}^k a_1(z_{r_n})P_0(z_{r_n})|}{|F(z_{r_n})|} + O(1) \leq \frac{17|z_{r_n}^k a_1(z_{r_n})P_0(z_{r_n})|}{|F(z_{r_n})|}$$

as $r_n \in I \setminus \bigcup_{j=1}^k E_j$ and $r_n \rightarrow \infty$. From (3.34) we get

$$(3.35) \quad \{\nu(r_n, F)\}^k |F(z_{r_n})| \leq 17|z_{r_n}^k a_1(z_{r_n})P_0(z_{r_n})| \leq 17M(r_n, z^k a_1 P_0)$$

as $r_n \in I \setminus \bigcup_{j=1}^k E_j$ and $r_n \rightarrow \infty$. From (3.1) and (3.3) we get $\sigma(a_1) = \sigma(a)$ and $\sigma(F) = \sigma(f)$ respectively. This together with $\sigma(z^k a_1 P_0) = \sigma(a_1)$ and the condition $\sigma(a) < \sigma(f)$ gives

$$(3.36) \quad \delta_2 < \sigma(F),$$

where

$$(3.37) \quad \delta_2 = \sigma(z^k a_1 P_0).$$

From (3.13), (3.35), (3.37) and Definition 1.1 we get

$$(3.38) \quad \{\nu(r_n, F)\}^k e^{r_n^{\sigma(F)-\varepsilon}} \leq 17e^{r_n^{\delta_2+\varepsilon}}$$

as $r_n \in I \setminus \bigcup_{j=1}^k E_j$ and $r_n \rightarrow \infty$. From (3.8), (3.36) and (3.38) we get a contradiction.

CASE 2. Suppose that f is not an entire function and a is an entire function. Proceeding as in Case 1 we get contradictions.

CASE 3. Suppose that f is an entire function. Then from the condition that $f - a$ and $f^{(k)} - a$ share 0 CM we get

$$(3.39) \quad \frac{f^{(k)} - a}{f - a} = e^{\beta_2},$$

where β_2 is an entire function. Proceeding as in Case 1 we deduce from (3.39) that β_2 is a constant, which yields the conclusion of Theorem 1.2.

Theorem 1.2 is thus completely proved.

Proof of Theorem 1.1. By the assumptions of Theorem 1.1 we have (3.39). Next in the same manner as in Case 1 of the proof of Theorem 1.2 we get the conclusion of Theorem 1.1.

Proof of Theorem 1.4. By the assumptions of Theorem 1.4 we have

$$(3.40) \quad \frac{f^{(k)} - R}{f - R} = e^{\beta_3},$$

where β_3 is an entire function. If f is a polynomial, then it follows from (3.40) that β_3 is a constant. If $e^{\beta_3} = 1$, from (3.40) we get the conclusion of Theorem 1.4. Next we suppose that $e^{\beta_3} \neq 1$. By rewriting (3.40) we get

$$(3.41) \quad f^{(k)} - fe^{\beta_3} = R(1 - e^{\beta_3}).$$

From (3.41) and the supposition that R has at least one pole we get a contradiction. Next we suppose that f is a transcendental entire function. Proceeding as in Case 1 of the proof of Theorem 1.1 we infer from (3.41) and the condition $\sigma(f) < \infty$ that β_3 is a polynomial. If β_3 is not constant, from (3.41) we get

$$(3.42) \quad T(r, e^{\beta_3}) \leq 2T(r, f) + O\{\log T(r, f) + \log r\} \quad (r \in I \setminus E).$$

From (3.42) and Lemma 2.4 we know that there exists a sufficiently large positive number r_0 such that

$$(3.43) \quad T(r, e^{\beta_3}) \leq 2T(2r, f) + O\{\log T(2r, f) + \log r + \log 2\} \quad (r \geq r_0).$$

From (3.43) and Definition 1.1 we get

$$(3.44) \quad 1 \leq \deg(\beta_3) = \sigma(e^{\beta_3}) \leq \sigma(f).$$

From (3.44) and $\sigma(R) = 0$ we get

$$(3.45) \quad \sigma(R) < \sigma(f).$$

From (3.45) and Theorem 1.2 we find that e^{β_3} is a constant, which contradicts the above supposition. Next we suppose that β_3 is a constant. If $e^{\beta_3} \neq 1$, by rewriting (3.40) as (3.41) and using the condition that R has at least one pole we get a contradiction. Thus $e^{\beta_3} = 1$. This together with (3.40) yields the conclusion of Theorem 1.4.

Proof of Theorem 1.5. We discuss the following two cases.

CASE 1. Suppose that f , and so $f^{(n)}$, is a nonconstant rational function. Then from Theorem 1.3 and the condition that $f^{(n)}$ and $f^{(n+1)}$ share a CM

we get

$$(3.46) \quad f^{(n)}(z) = a + \frac{a(z - z_1)}{2} + \frac{A_1}{z - z_1},$$

where z_1 and $A_1 (\neq 0)$ are two complex numbers. From (3.46) we get a contradiction.

CASE 2. Suppose that f is a transcendental meromorphic function. Then from Lemma 2.5 we get

$$(3.47) \quad \sigma(f) = \sigma(f').$$

Similarly

$$(3.48) \quad \sigma(f^{(j)}) = \sigma(f^{(j+1)}) \quad (1 \leq j \leq n - 1).$$

From (3.47) and (3.48) we get

$$(3.49) \quad \sigma(f) = \sigma(f^{(n)}).$$

From (3.49) and the condition $\sigma(f) < \infty$ we get

$$(3.50) \quad \sigma(f^{(n)}) < \infty.$$

From (3.50), Theorem 1.2 and the condition that $f^{(n)}$ and $f^{(n+1)}$ share a CM we get

$$(3.51) \quad f^{(n+1)} - a = c(f^{(n)} - a),$$

where c is some nonzero constant. From (3.51) we know that f is a nonconstant entire function. This together with Theorem F yields the conclusion of Theorem 1.5.

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Xiao-Min Li
 Department of Mathematics
 Ocean University of China
 Qingdao, Shandong 266071
 People's Republic of China
 E-mail: xmli01267@gmail.com

Hong-Xun Yi
 Department of Mathematics
 Shandong University
 Jinan, Shandong 250100
 People's Republic of China
 E-mail: hxyi@sdu.edu.cn

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