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Solutions to a class of singular quasilinear elliptic equations

by LIN WEI and ZUODONG YANG (Nanjing)

Abstract. We study the existence of positive solutions to

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + q(x)u^{-\gamma} = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is $\mathbb{R}^N$ or an unbounded domain, $q(x)$ is locally Hölder continuous on Ω and $p > 1$, $\gamma > -(p-1)$.

1. Introduction. In this paper, we are concerned with the existence of positive entire solutions to quasilinear elliptic equations of the type

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + q(x)u^{-\gamma} = 0, & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a domain in $\mathbb{R}^N$ with $C^{2,\alpha}$ boundary, $q(x)$ is locally Hölder continuous on Ω and $p > 1$, $\gamma > -(p-1)$. When $\Omega = \mathbb{R}^N$, the boundary condition is omitted. By a *positive entire solution* to equation (1), we mean a positive function $u \in C^1(\mathbb{R}^N)$ which satisfies (1) at every point of $\mathbb{R}^N$ (see [10] and references therein). If $\lim_{|x|\to\infty} u(x) = 0$, we call it a *positive decaying solution.*

Equations of the above form are mathematical models occurring in the study of the p-Laplace equation, generalized reaction-diffusion theory, non-Newtonian fluid theory [2, 27], non-Newtonian filtration [19] and the turbulent flow of a gas in a porous medium [8]. In non-Newtonian fluid theory, the quantity p is a characteristic of the medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids.

In recent years, the existence and non-existence with uniqueness of positive solutions to the quasilinear eigenvalue problems

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$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda f(u) = 0 \quad \text{in } \Omega, \tag{2}$$

$$u(x) = 0 \quad \text{in } \partial\Omega, \tag{3}$$

with $\lambda > 0$, $p > 1$ on a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, have been studied by many authors (see [10–16, 29–31, 35, 37] and the references therein). When f is strictly increasing on $\mathbb{R}^+$, $f(0) = 0$, $\lim_{s\to 0^+} f(s)/s^{p-1} = 0$ and $f(s) \leq \alpha_1 + \alpha_2 s^\mu$, $0 < \mu < p-1$, $\alpha_1, \alpha_2 > 0$, it was shown in [12] that there exist at least two positive solutions to (2)–(3) when λ is sufficiently large. If $\liminf_{s\to 0^+} f(s)/s^{p-1} > 0$, $f(0) = 0$ and the monotonicity hypothesis $(f(s)/s^{p-1})' < 0$ holds for all $s > 0$, it was proved in [13] that problem (2)–(3) has a unique positive solution when λ is sufficiently large. Moreover, it was also shown in [14] that problem (2)–(3) has a unique positive large solution and at least one positive small solution when λ is large and f is nondecreasing, there exist $\alpha_1, \alpha_2 > 0$ such that $f(s) \leq \alpha_1 + \alpha_2 s^\beta$, $0 < \beta < p-1$, $\lim_{s\to 0^+} f(s)/s^{p-1} = 0$, and there exist $T, Y > 0$ with $Y \geq T$ such that

$$(f(s)/s^{p-1})' > 0 \quad \text{for } s \in (0, T)$$

and

$$(f(s)/s^{p-1})' < 0 \quad \text{for } s > Y.$$

Hai [17] considered the case when Ω is an annular domain. He obtained the existence of positive large solutions to problem (2)–(3) when λ is sufficiently small. Xuan & Chen [32] proved that the singular problem (1) has a unique positive radial solution if q is a radially symmetric and continuous function and positive on $\overline{\Omega} = B_R$ (here B_R is a ball). The existence of entire solutions to singular and non-singular problems (1) has been considered in [11, 26, 36, 33, 34].

In this paper, we consider the cases that Ω is $\mathbb{R}^N$ or an unbounded domain under new conditions. Our results complement those in [11, 26, 36, 33, 34]. For $p = 2$, the singular semilinear elliptic problems

$$\begin{cases} \Delta u + q(x)u^{-\gamma} = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

have been extensively studied when $\Omega \subset \mathbb{R}^N$ or $\Omega = \mathbb{R}^N$ (see [3–7, 18, 21–25, 38]). When $p \neq 2$, the problem becomes more complicated since certain nice properties inherent to the case $p = 2$ seem to be lost or at least difficult to verify. The main differences between the cases $p = 2$ and $p \neq 2$ can be found in [11, 15].

To state our result, we write $x = (x', x'')$ with $x' = (x_1, x_2, x_3)$. By modifying the argument in the proof of Theorem 1 in [18], we will obtain the following theorem.

THEOREM 1.1. *Let $q(x)$ be a locally Hölder continuous function on $\overline{\Omega}$, where $\Omega = \mathbb{R}^N$, $N \geq 3$ or Ω is a $C^{2,\alpha}$ smooth unbounded subdomain of $\mathbb{R}^N$.*

Assume that $q(x_0) > 0$ for some $x_0 \in \Omega$. If there exist positive constants C and $\sigma > p-1$ such that

$$0 \leq q(x) \leq \frac{C}{(1+|x'|^{\frac{p}{p-1}})^{\sigma}}$$

then the equation

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = q(x)u^{-\gamma} & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a positive solution for any $\gamma > -(p-1)$.

Theorem 1.1 is proved by constructing upper and lower solutions on bounded domains and taking a limit. First, let us fix some notation.

Since Ω is a $C^{2,\alpha}$ unbounded domain, we can choose a sequence of subdomains of Ω, denoted by Ω_m, $m = 1, 2, \ldots$, such that (1) $\bar{\Omega}_m \subset \Omega_{m+1} \subset \Omega$ for all m; (2) $\bigcup \Omega_m = \Omega$; (3) each Ω_m is a bounded $C^{2,\alpha}$ domain; (4) $\mathrm{dist}(0, \partial\Omega \setminus \partial\Omega_m) \to \infty$ as $m \to \infty$. When Ω is indeed $\mathbb{R}^N$, we can simply choose $\Omega_m = B_m(0)$, the family of balls centered at the origin with radius m.

A function $p(x)$ on Ω is said to have the property (HP) if $p(x) \geq 0$ and when m is large, the eigenvalue problem

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda q(x)|u|^{p-2}u \quad \text{on } \Omega_m(0),$$
$$u = 0 \quad \text{on } \partial\Omega_m(0),$$

has its first eigenvalue less than one.

2. Upper solutions. In this section, we will construct an upper solution to (1). For $x \in \mathbb{R}^N$, we write $x = (x', x'')$ with $x' \in \mathbb{R}^3$.

Lemma 2.1. *If there exist positive constants C and $\sigma > p-1$ such that*

$$0 \leq q(x) \leq \frac{C}{(1+|x'|^{\frac{p}{p-1}})^{\sigma}} \quad \text{on } \Omega \tag{4}$$

then the equation

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = q(x)u^{-\gamma}$$

has an upper solution $\omega(x)$ of the form

$$\omega(x) = A\{1 + (1+|x'|^{\frac{p}{p-1}})^{\frac{\alpha}{p-1}}\}$$

for some constants $A > 0$ and $\alpha < 0$.

Proof. First we consider the case of $N = 3$. Let

$$\omega_1(x) = 1 + (1+|x|^{\frac{p}{p-1}})^{\frac{\alpha}{p-1}}$$

where $-(p-1)/p < \alpha < 0$, $p-1+|\alpha| < \sigma$, $x \in \mathbb{R}^3$. Then

$$-\mathrm{div}(|\nabla\omega_1|^{p-2}\nabla\omega_1) = -(|\omega_1'|^{p-2}\omega_1')' - \frac{2}{|x|}\,|\omega_1'|^{p-2}\omega_1'.$$

We compute successively:

$$\omega_1' = (1+|x|^{\frac{p}{p-1}})^{\frac{\alpha}{p-1}-1}\,\frac{\alpha p}{(p-1)^2}\,|x|^{\frac{1}{p-1}},$$

$$|\omega_1'|^{p-2}\omega_1' = \frac{\alpha|\alpha|^{p-2}p^{p-1}}{(p-1)^{2(p-1)}}\,(1+|x|^{\frac{p}{p-1}})^{\alpha-p+1}|x|,$$

$$\begin{aligned}
&(|\omega_1'|^{p-2}\omega_1')' \\
&= \frac{p^{p-1}}{(p-1)^{2p-2}}\alpha|\alpha|^{p-2}\left[\frac{p(\alpha-p+1)}{p-1}(1+|x|^{\frac{p}{p-1}})^{\alpha-p}|x|^{\frac{p}{p-1}}+(1+|x|^{\frac{p}{p-1}})^{\alpha-p+1}\right],\\
&-(|\omega_1'|^{p-2}\omega_1')' - \frac{2}{|x|}(|\omega_1'|^{p-2}\omega_1')\\
&= \frac{p^p(\alpha-p+1)|\alpha|^{p-1}}{(p-1)^{2p-1}}(1+|x|^{\frac{p}{p-1}})^{\alpha-p}|x|^{\frac{p}{p-1}} + 3\frac{p^{p-1}|\alpha|^{p-1}}{(p-1)^{2p-2}}(1+|x|^{\frac{p}{p-1}})^{\alpha-p+1}\\
&= \frac{|\alpha|^{p-1}p^{p-1}(1+|x|^{\frac{p}{p-1}})^{\alpha}}{(p-1)^{2p-1}}\left[\frac{3(p-1)+p(\alpha-p+1)}{(1+|x|^{\frac{p}{p-1}})^{p-1}} - \frac{p(\alpha-p+1)}{(1+|x|^{\frac{p}{p-1}})^{p}}\right],
\end{aligned}$$

A direct calculation shows

$$-\operatorname{div}(|\nabla\omega_1|^{p-2}\nabla\omega_1) = V_1(x)\omega_1^{p-1} \quad \text{on } \mathbb{R}^3$$

where

$$\begin{aligned}
V_1(x) = {}& \frac{p^{p-1}|\alpha|^{p-1}(1+|x|^{\frac{p}{p-1}})^{\alpha}}{(p-1)^{2p-1}[1+(1+|x|^{\frac{p}{p-1}})^{\frac{\alpha}{p-1}}]^{p-1}}\\
&\times\left[\frac{3(p-1)+p(\alpha-p+1)}{(1+|x|^{\frac{p}{p-1}})^{p-1}} - \frac{p(\alpha-p+1)}{(1+|x|^{\frac{p}{p-1}})^{p}}\right]
\end{aligned}$$

Since $-(p-1)/p<\alpha<0$ and $V_1(x)>0$ on $\mathbb{R}^3$, there exists a constant C_1 such that

$$V_1(x) \ge \frac{C_1}{(1+|x|^{\frac{p}{p-1}})^{p-1+|\alpha|}} \quad \text{on } \mathbb{R}^3.$$

For any $N\ge 3$, we set $V(x)=V_1(x')$, $\omega(x)=A\omega_1(x')$. Then if A is large, $\omega(x)>1$ and $\gamma>-(p-1)$, for $x\in\Omega$, we have

$$\begin{aligned}
-\operatorname{div}(|\nabla\omega|^{p-2}\nabla\omega) &= V(x)A^{p-1}\omega_1^{p-1} \ge \frac{C_1}{(1+|x'|^{\frac{p}{p-1}})^{p-1+|\alpha|}}\,A^{p-1}\omega_1^{p-1}\\
&\ge \frac{C_1}{(1+|x'|^{\frac{p}{p-1}})^{p-1+|\alpha|}}\,A^{p-1}\omega_1(x)^{-\gamma} = \frac{C_1A^{p-1+\gamma}}{(1+|x'|^{\frac{p}{p-1}})^{p-1+|\alpha|}}\,\omega(x)^{-\gamma}\\
&\ge \frac{C}{(1+|x|^{\frac{p}{p-1}})^{\sigma}}\,\omega(x)^{-\gamma} \ge q(x)\omega(x)^{-\gamma}.
\end{aligned}$$

This is what we want. ■

3. Local solutions. We consider

$$(P_m) \qquad \begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = q(x)u^{-\gamma} & \text{on } \Omega_m, \\ u = 0 & \text{on } \partial\Omega_m. \end{cases}$$

We will construct solutions for (P_m) if m is large. We recall that a function $q_1(x) \geq 0$ on Ω is said to have the property (HP) if whenever m is large, the following eigenvalue problem has its first eigenvalue $\lambda_1(m)$ less than one:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda q_1(x)|u|^{p-2}u & \text{on } \Omega_m, \\ u = 0 & \text{on } \partial\Omega_m. \end{cases} \tag{5}$$

By [26, 33], we have the following lemma:

LEMMA 3.1 (Weak Comparison Principle). *Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, and let $\theta : (0,\infty) \to (0,\infty)$ be continuous and nondecreasing. If $u_1, u_2 \in W^{1,p}(\Omega)$ satisfy*

$$\int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \nabla\psi\,dx + \int_\Omega \theta(u_1)\psi\,dx \leq \int_\Omega |\nabla u_2|^{p-2}\nabla u_2 \nabla\psi\,dx + \int_\Omega \theta(u_2)\psi\,dx$$

for all nonnegative $\psi \in W_0^{1,p}(\Omega)$, then the inequality

$$u_1 \leq u_2 \quad \text{on } \partial\Omega$$

implies

$$u_1 \leq u_2 \quad \text{in } \Omega.$$

LEMMA 3.2. *If $q(x) \geq 0$ on Ω, and $q(x)$ is not identically zero on Ω, then there exists a function $q_1(x)$ having the property* (HP) *and a sequence of numbers δ_m such that*

$$q(x)t^{-\gamma} \geq q_1(x)t^{p-1} \quad \text{for } x \in \Omega_m,\ 0 < t < \delta_m. \tag{6}$$

Proof. Since $q(x) \geq 0$ and $q(x)$ is not identically zero, the first eigenvalue $\mu_1(m)$ of the following problem is positive for all $m \geq m_0$ for some m_0:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \mu q(x)|u|^{p-2}u & \text{on } \Omega_m, \\ u = 0 & \text{on } \partial\Omega_m. \end{cases} \tag{7}$$

It is well known that $\mu_1(m)$ is strictly decreasing in m. Now we set $q_1(x) = q(x)/\mu_1(m_0)$. If δ is small and $\gamma > -(p-1)$, we have $q(x)t^{-\gamma} \geq q_1(x)t^{p-1}$ on $\mathbb{R}^N$ for $0 < t < \delta$. It is clear that $q_1(x)$ has the property (HP). Thus we can choose $\delta_m = \delta$ for a small number δ. ∎

Since the function $q_1(x)$ has the property (HP), we let $\psi_m(x)$ with $\max_{\Omega_m} \psi_m(x) = 1$ be the positive first eigenfunction of the eigenvalue problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda q_1(x)|u|^{p-2}u & \text{on } \Omega_m, \\ u = 0 & \text{on } \partial\Omega_m. \end{cases} \tag{8}$$

We are ready to construct local solutions. Denote by $\omega(x)$ the upper solution constructed in Lemma 2.1.

LEMMA 3.3. *Suppose that $q(x)$ satisfies the assumptions of Theorem 1.1. For each fixed and large m, there exists a positive solution $u_m(x)$ to the problem*

$$(P_m) \qquad \begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = q(x)u^{-\gamma} & \text{on } \Omega_m, \\ u = 0 & \text{on } \partial\Omega_m. \end{cases}$$

Furthermore, $u_m(x)$ satisfies

$$(9) \qquad 0 < u_m(x) \le \omega(x) \quad \text{on } \Omega_m,$$

and for a fixed large s, when $m > s$ we have

$$(10) \qquad \frac{1}{2}\delta_s\psi_s(x) \le u_m(x) \quad \text{on } \Omega_s,$$

where δ_s is defined in Lemma 3.2.

Proof. By Lemma 3.2, there exists a function $q_1(x)$ having the property (HP) and a sequence of numbers δ_m such that

$$(11) \qquad q(x)t^{-\gamma} \ge q_1(x)t^{p-1} \quad \text{for } x \in \Omega_m,\ 0 < t < \delta_m.$$

We may assume that (recall $\omega(x)$ is an upper solution)

$$(12) \qquad \omega(x) \ge 2\delta_m \quad \text{on } \Omega_m.$$

Now let $z(t)$ be a function satisfying (1) $z(t) = 1$ if $0 < t < \frac{1}{2}\delta_m$; (2) $z(t) = 0$ if $t > \frac{3}{4}\delta_m$; (3) $z(t) \ge 0$, $z \in C^2$. For small $\varepsilon > 0$, from (11) it is easy to check that

$$(13) \qquad q(x)(t + \varepsilon z(t))^{-\gamma} \ge q_1(x)t^{p-1} \qquad \text{for } x \in \Omega_m,\ 0 < t < \frac{1}{2}\delta_m,\ 0 < \varepsilon < \frac{1}{2}\delta_m.$$

And (12) implies

$$q(x)(t + \varepsilon z(t))^{-\gamma} = q(x)t^{-\gamma} \quad \text{for } x \in \Omega_m.$$

Thus for any m, $\omega(x)$ is an upper solution of the boundary value problem

$$(P_{\varepsilon,m}) \qquad \begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = q(x)(u + \varepsilon z(t))^{-\gamma} & \text{on } \Omega_m, \\ u = 0 & \text{on } \partial\Omega_m. \end{cases}$$

Since $0 < \mu < \frac{1}{2}\delta_m$ and m is large, (8) and (11) imply

$$(14) \qquad -\operatorname{div}(|\nabla(\mu\psi_m)|^{p-2}\nabla(\mu\psi_m)) = \lambda_1(m)(\mu\psi_m)^{p-1}q_1(x) \le q_1(x)(\mu\psi_m)^{p-1} \le q(x)(\mu\psi_m)^{-\gamma}.$$

We have used the fact that $\lambda_1(m) < 1$ if m is large. Thus $\mu\psi_m(x)$ is a lower solution to $(P_{\varepsilon,m})$. Choosing μ smaller if necessary, we may assume

$$(15) \qquad \mu\psi_m(x) < \omega(x) \quad \text{on } \Omega_m.$$

Thus $(P_{\varepsilon,m})$ has a pair of upper and lower solutions $\omega(x)$ and $\mu\psi_m(x)$. It is well known that each such pair yields a solution. Thus $(P_{\varepsilon,m})$ has a solution $u_m(x,\varepsilon)$ satisfying

$$\mu\psi_m(x) \le u_m(x,\varepsilon) \le \omega(x) \quad \text{on } \Omega_m. \tag{16}$$

Let $\phi_m(x)$ be a function defined by

$$-\operatorname{div}(|\nabla\phi_m|^{p-2}\nabla\phi_m) = 1 \quad \text{on } \Omega_m, \qquad \phi_m = 0 \quad \text{on } \partial\Omega_m.$$

Then the same proof of Lemma 1.13 in [6] shows that for any $\beta > 0$ there is a constant $M(\beta,m)$ such that

$$u_m(x,\varepsilon) \le \beta + M(\beta,m)\phi_m(x) \quad \text{on } \Omega_m \text{ uniformly in } \varepsilon. \tag{17}$$

Since both bounds in (16) are independent of ε, it is easy to see that a subsequence of $u_m(x,\varepsilon)$ (with fixed m) converges to a function $u_m(x)$ in the C^1_{loc} topology. Thus $u_m(x)$ is a solution of the equation in (P_m) with $u_m(x) > 0$ on Ω_m. And (17) implies

$$0 < u_m(x) \le \beta + M(\beta,m)\phi_m(x) \quad \text{on } \Omega_m \tag{18}$$

for any $\beta > 0$. Since $\phi_m(x)$ is continuous on $\bar{\Omega}_m$ and $u_m(x) = 0$ on $\partial\Omega_m$, it follows that $u_m(x)$ is a solution of (P_m).

Now let us take care of (9) and (10). First, (9) is an immediate consequence of (16) by taking the limit (for a subsequence). For (10), we observe that we only need to prove that for a fixed and large s, if $m > s$,

$$\frac{1}{2}\delta_s\psi_s(x) \le u_m(x,\epsilon) \quad \text{on } \Omega_s. \tag{19}$$

Then (9) follows from (19) by letting $\epsilon \to 0^+$ (for a subsequence). ∎

The next lemma and its proof are similar to that of Lemma 1.13 in [19]. We show that $u_m(x)$ is uniformly controlled near a fixed point on $\partial\Omega$. Since Ω is $C^{2,\alpha}$, for $x_0 \in \partial\Omega$, there exists a ball B in $\mathbb{R}^N$ such that $B \cap \bar{\Omega} = \{x_0\}$. Then it is clear that we can find a C^3 domain Ω_0 such that $\partial\Omega \cap \partial B$ contains a neighborhood of x_0 in ∂B, $\Omega_0 \cap B$ is empty and $\Omega_0 \cap \Omega$ contains a neighborhood of x_0 in Ω. Now we choose a nonnegative function $h(x)$ on $\partial\Omega_0$ such that (1) $h(x) = \omega(x)$ on $\partial\Omega_0 \cap \Omega$; (2) $h(x) = 0$ in a neighborhood of x_0 on $\partial\Omega_0$; (3) $h \in C^3$. Let $\phi_0(x)$ be the function defined by

$$\begin{cases} -\operatorname{div}(|\nabla\phi_0(x)|^{p-2}\nabla\phi_0(x)) = 1 & \text{on } \Omega_0, \\ \phi_0(x) = h(x) & \text{on } \partial\Omega_0. \end{cases}$$

Then we have the following lemma.

Lemma 3.4. *For any $\beta > 0$, there exists a constant M depending only on β and Ω_0 such that*

$$u_m(x) \le \beta + M\phi_0(x) \quad \textit{on } \Omega_0 \cap \Omega \tag{16}$$

where $u_m(x)$ is the solution obtained in Lemma 3.3 (*when Ω is not $\mathbb{R}^N$*) *and m is large.*

Proof. Because m is large, we have $\Omega_0 \cap \Omega \subset \Omega_0 \cap \Omega_m$. Let

$$M^{p-1} = \sup\{q(x)t^{-\gamma} \mid x \in \Omega_0 \cap \bar{\Omega},\, t \geq \beta\} + 1,$$
$$\Omega_0(m,\beta) = \{x \mid x \in \Omega_0 \cap \Omega,\, u_m(x) > \beta\}.$$

Then on $\Omega_0(m,\beta)$, we get

$$\begin{aligned}(17)\qquad -\operatorname{div}(|\nabla u_m(x)|^{p-2}\nabla u_m(x)) &= q(x)u_m^{-\gamma} \leq M^{p-1}\\ &= -\operatorname{div}(|\nabla(\beta + M\phi_0)|^{p-2}\nabla(\beta + M\phi_0)).\end{aligned}$$

If $x_1 \in \partial\Omega_0(m,\beta)$, we have $u_m(x_1) = \beta$ or $x_1 \in \partial\Omega_0 \cap \Omega$. Since $\phi_0(x) = h(x) = \omega(x) \geq u_m(x)$ for $x \in \partial\Omega_0 \cap \Omega$, we see that

$$(18)\qquad u_m(x) \leq \beta + M\phi_0(x) \quad \text{on } \partial\Omega_0(m,\beta).$$

Thus (17), (18) and Lemma 3.1 imply that

$$(19)\qquad u_m(x) \leq \beta + M\phi_0(x) \quad \text{on } \Omega_0(m,\beta).$$

Since $\Omega_0 \cap \Omega = \Omega_0(m,\beta) \cup \{x \mid x \in \Omega_0 \cap \Omega,\, u_m(x) \leq \beta\}$, (19) implies (16). ■

4. Proof of Theorem 1.1. By Lemma 3.3, we can choose a subsequence of u_m such that for some function $u \in C^1(\Omega)$, $u_m(x)$ converges to $u(x)$ uniformly in the $C^1_{\mathrm{loc}}(\Omega)$ topology (for example, see Lemma 1.5 and Theorem 1.1 in [5]). Hence u satisfies

$$-\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = q(x)u^{-\gamma}(x) \quad \text{on } \Omega.$$

Also (10) implies that for any fixed and large s,

$$\frac{1}{2}\delta_s\psi_s(x) \leq u(x) \quad \text{on } \Omega_s.$$

Thus $u(x)$ is positive on Ω.

Finally, for $x_0 \in \partial\Omega$, (16) implies that

$$0 < u(x) \leq \beta + M\phi_0(x) \quad \text{on } \Omega_0.$$

Since β is arbitrary, $\phi_0(x)$ is continuous on $\bar{\Omega}_0$ and $\phi_0(x_0) = 0$, we see that $u(x)$ is continuous at x_0 and $u(x_0) = 0$. Since x_0 is an arbitrary point on $\partial\Omega$, it follows that $u(x)$ is continuous up to the boundary of Ω and $u(x) = 0$ on $\partial\Omega$.

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Lin Wei
Institute of Mathematics
School of Mathematics Science
Nanjing Normal University
Nanjing 210046, Jiangsu, China
E-mail: lynn_w8596@yahoo.com.cn

Zuodong Yang
Institute of Mathematics
School of Mathematics Science
Nanjing Normal University
Nanjing 210046, Jiangsu, China
and
College of Zhongbei
Nanjing Normal University
Nanjing 210046, Jiangsu, China
E-mail: zdyang_jin@263.net

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